A NOTE ON THE HONEYCOMB OPTIMALITY AMONG PERIODIC CONVEX TILINGS

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ABSTRACT. We show that the hexagonal honeycomb is optimal among convex periodic tessellations of the plane, provided the cost functional is lower semicontinuous with respect to the Hausdorff convergence, and decreasing under Steiner symmetrization.

1. Introduction

Finding optimal planar tilings, and in particular investigating the optimality of the honeycomb structure made by a packing of regular hexagons, is a challenging and widely studied problem, both for geometric and variational functionals. For the classical De Giorgi perimeter, in the seminal paper [13] Hales proved the optimality of the honeycomb in full generality, namely among all possible coverings of the plane made by cells of equal area (see also [18]). For the Cheeger constant, the result is proved in [4, 5] (respectively with and without convexity constraint), while the cases of the first Robin Laplacian eigenvalue and of the Robin torsional rigidity, have been settled in [6] (under convexity constraint). For the first Dirichlet Laplacian eigenvalue, the optimality of the honeycomb corresponds to a conjecture formulated by Caffarelli and Lin [7], and remains, to the best of our knowledge, open.

In the recent paper [8], we proved the existence of an optimal periodic tessellation minimizing quite general nonlocal perimeter functionals defined in terms of interaction kernels, and we discussed the possible optimality of the honeycomb in the planar case, based on a symmetrization procedure. Since the description of such procedure, as it was sketched in [8, Lemma 5.1], contained some inaccuracies (though all the statements in [8] are correct), a first goal of this note is to fix rigorously its proof; a second goal is to observe that this procedure provides the optimality of the honeycomb for a much wider class of optimal partition problems.

More precisely, in this note we prove that every centrally symmetric hexagon and every parallelogram can be transformed into the regular hexagon with equal area by applying iteratively appropriate Steiner symmetrizations (see Section 2). Since the only convex polygons which tessellate the plane by translations are centrally symmetric hexagons and parallelograms, as shown in [10, 17], this implies that, for any functional which is lower semicontinuous with respect to the Hausdorff convergence of convex sets and is decreasing under Steiner symmetrization, the honeycomb is the optimal periodic convex tiling of the plane. Plenty of geometric or variational energies fit into this general framework (see Section 3); moreover, for some of them, the honeycomb turns out to be the *unique* optimal tiling.

Several challenging related questions remain open. In particular is it natural to ask whether, for all the energy functionals considered in this note, which include the first Dirichlet Laplacian eigenvalue, the honeycomb is optimal also without the periodicity and the convexity assumptions, corresponding to a generalized version of the conjecture by Caffarelli and Lin [7]. A related interesting open question concerns on the other hand the possible existence of other local minimizers for such energies.

A first step in the direction of proving the conjecture would be establishing the existence of an optimal tessellation, by analogy to what was done in [8] for nonlocal perimeters. We recall that in [9], a weaker version of this conjecture was considered in the case of anisotropic local perimeters under periodicity constraints: more precisely it is proved that the optimal periodic planar tilings are given by centrally symmetric convex hexagons (eventually degenerating to parallelograms) in the case of strictly convex anisotropy, and by nondegenerate convex hexagons when the anisotropy is also differentiable.

2. Steiner symmetrization for periodic convex tilings

Let $\mathcal{H}$ denote the family of convex polygons which tessellate the plane by translations. Recall that such a convex polygon can only be either a centrally symmetric hexagon or a parallelogram [10, 17]. The following lemma provides a constructive symmetrization procedure on the class $\mathcal{H}$.

For convenience of the reader, let us recall the classical definition of Steiner symmetrization of a set E belonging more generally to the class $\mathcal{C}$ of compact convex sets in $\mathbb{R}^2$ with nonempty interior, with respect to a straight line $\pi \subseteq \mathbb{R}^2$ passing through the origin. Denoting by v the normal to π, and setting $L_x = \{x + tv\,,\, t \in \mathbb{R}\}$ for $x \in \pi$, the Steiner symmetrization of E (see [1]) is given by

$$S_\pi(E) = \bigcup_{x\in\pi, L_x\cap E\neq\emptyset} \{x + rv,\ 2|r| \leq \mathcal{H}^1(E \cap L_x)\}. \tag{1}$$

It is straightforward to check that, if $E \in \mathcal{C}$, then $S_\pi(E) \in \mathcal{C}$, with $|S_\pi(E)| = |E|$.

Lemma 2.1. *Every $H_0 \in \mathcal{H}$ can be transformed into a regular hexagon, by using at most countable Steiner symmetrizations.*

Proof. Starting from a fixed hexagon $H_0 \in \mathcal{H}$ (possibly degenerated into a parallelogram), let us construct a sequence of centrally symmetric hexagons H_n (not degenerating into parallelograms), such that H_n converge in Hausdorff distance to a regular hexagon H^*. We proceed by steps.

Step 1: inizialization of the symmetrization procedure.

Let denote by $A_0, B_0, C_0, D_0, E_0, F_0$ the vertices of H_0, ordered in counterclockwise sense. We apply to H_0 a Steiner symmetrization with respect of the axis of a diagonal joining the vertices of two consecutive sides, say A_0E_0. By this way, H_0 is transformed into a new hexagon H_1, whose vertices $A_1, B_1, C_1, D_1, E_1, F_1$ are not aligned three by three, namely H_1 is not degenerated into a parallelogram (see Figure 1, where H_0 is represented in dotted line, and H_1 in continuous line; notice that H_1 is not degenerated into a parallelogram even if this is the case for H_0, see [8, Figure 1]). Precisely, we have: $A_1 = A_0$, $E_1 = E_0$, the triangle $A_0E_0F_0$ becomes the isosceles triangle $A_1E_1F_1$, the sides A_0B_0 and D_0E_0 are transformed into the sides A_1B_1 and D_1E_1, which are parallel to the symmetry axis, and finally the triangle $B_0C_0D_0$ becomes the isosceles triangle $B_1C_1D_1$. In particular, the pairs (A_1B_1, D_1E_1), (A_1F_1, E_1F_1) and (B_1C_1, C_1D_1) are congruent and, by symmetry with respect to the origin, also the sides A_1F_1 and C_1D_1 are congruent, as well as the sides B_1C_1 and E_1F_1.

Thus, H_1 has 4 congruent sides of length $a_1 = \overline{A_1F_1} = \overline{E_1F_1} = \overline{C_1D_1} = \overline{B_1C_1}$, and 2 congruent sides of length $b_1 = \overline{A_1B_1} = \overline{D_1E_1}$. Moreover, since Steiner symmetrization perserves the area, we have that $|H| = |H_0|$.

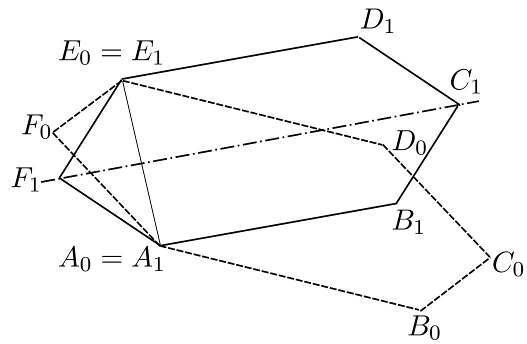

FIGURE 1.

Step 2: construction of the sequence of hexagons H_n.

We proceed by applying the same procedure as in Step 1 to H_1, by symmetrizing with respect to the axis of the diagonal connecting a side of length a_1 and a side of length b_1, say e.g. the diagonal D_1F_1. We thus obtain a new hexagon H_2, having 4 congruent sides of length a_2 and 2 congruent sides of length b_2 (see Figure 2, where H_1 is represented in dotted line, and H_2 in continuous line).

We point out that the reason for displaying this second step, is that we aim at comparing the lengths of the sides of H_1 with the lengths of the sides of H_2 (comparison that we are going to extend iteratively to the next hexagons in the construction).

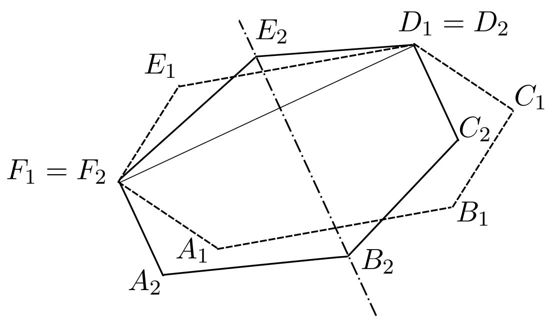

FIGURE 2.

First of all we observe that, since Steiner symmetrization decreases perimeter, we have that

$$4a_1 + 2b_1 \geq 4a_2 + 2b_2 \, .$$

Next we observe that the triangle $D_1E_1F_1$ becomes the isosceles triangle $D_2E_2F_2$, which has the same basis $D_2F_2 = D_1F_1$, and two congruent sides of length a_2. Thus, applying again the monotonicity of perimeter under Steiner symmetrization, we obtain

$$a_1 + b_1 \geq 2a_2 \, .$$

Finally we look at the vertices A_1, B_1, C_1: they are turned into the new vertices A_2, B_2, C_2, which satisfy $\overline{A_2B_2} = \overline{B_2C_2} = a_2$, and $\overline{A_2F_2} = \overline{C_2D_2} = b_2$. Since the diagonal A_1C_1 is parallel to the diagonal D_1F_1, and since we are symmetrizing with respect to the axis of D_1F_1, the side A_2F_2 is the orthogonal projection of the side A_1F_1 onto the parallel to the symmetrization axis passing through F_1. In particular, due to this fact, we see that

$$b_2 \leq a_1 \, .$$

We now continue by applying recursively the same kind of symmetrization procedure, obtaining by this way a sequence of centrally symmetric hexagons H_n, with $|H_n| = |H_0|$, each one with 2 congruent sides of length b_n, 4 congruent sides of length a_n, and such that the following conditions hold for every $n \geq 1$:

(i) $4a_{n+1} + 2b_{n+1} \leq 4a_n + 2b_n$,
(ii) $2a_{n+1} \leq a_n + b_n$,
(iii) $b_{n+1} \leq a_n$.

Step 3: convergence of the sequence.
For every $n \geq 1$, we define $c_n = \max(a_n, b_n)$ and $d_n = \min(a_n, b_n)$. Note that $0 < d_n \leq c_n$ for every $n \geq 1$. We claim that

$$\lim_n c_n = \lim_n d_n > 0 \,. \tag{2}$$

Once proved this claim, it is immediate to conclude that H_n is converging in measure (or equivalently in Hausdorff distance) to the regular hexagon H^*, with sides of length equal to any of the limits in (2). We are left to prove (2). By properties (ii),(iii) above, and by the definition of c_n, we have that

$$2a_{n+1} \leq a_n + b_n \leq 2c_n \qquad b_{n+1} \leq a_n \leq c_n. \tag{3}$$

From the above inequalities and the definition of c_{n+1}, we infer that

$$c_{n+1} \leq c_n,$$

namely the sequence c_n is monotone non increasing, so that

$$\lim_n c_n = \inf_n c_n =: c. \tag{4}$$

Notice that necessarily $c > 0$, since the area of each H_n is constantly equal to the area of H_0.
Let now $\varepsilon > 0$ be fixed, and let n_ε be such that $c_n \leq c + \varepsilon$ for $n > n_\varepsilon$.
We are going to show that

$$\max(d_n, d_{n+1}) \geq c - \varepsilon \qquad \forall n > n_\varepsilon \,. \tag{5}$$

In turn, this implies that

$$c - \varepsilon \leq \max(d_n, d_{n+1}) \leq c_n \leq c + \varepsilon \qquad \forall n > n_\varepsilon \,,$$

which ensures that $\lim d_n = c$ (and hence that our claim (2) holds true).
In order to prove (5), we consider any $n > n_\varepsilon$, and we distinguish two possible situations: either $c_n = b_n$ and $d_n = a_n$, or $c_n = a_n$ and $d_n = b_n$.

– Case $c_n = b_n$ and $d_n = a_n$. By properties (ii)-(iii) in Step 2, we have

$$2a_{n+1} \leq a_n + b_n = c_n + d_n \qquad \text{and} \qquad b_{n+1} \leq a_n = d_n \leq \frac{c_n + d_n}{2} \,.$$

Hence

$$c \leq c_{n+1} \leq \frac{c_n + d_n}{2} \leq \frac{c + \varepsilon + d_n}{2} \,,$$

yielding that $d_n \geq c - \varepsilon$.

– Case $c_n = a_n$ and $d_n = b_n$. We distinguish two further sub-cases.
If $c_{n+1} = a_{n+1}$, by property (ii), we get

$$2c \le 2c_{n+1} = 2a_{n+1} \le c_n + d_n \le c + \varepsilon + d_n \,,$$

yielding that $d_n \ge c - \varepsilon$.
If $c_{n+1} = b_{n+1}$, by properties (ii)-(iii) in Step 2, we get

$$\begin{cases} 2a_{n+2} \le a_{n+1} + b_{n+1} = c_{n+1} + d_{n+1} \\ b_{n+2} \le a_{n+1} = d_{n+1} \le \frac{c_{n+1}+d_{n+1}}{2} \,, \end{cases}$$

so that

$$c \le c_{n+2} \le \frac{c_{n+1} + d_{n+1}}{2} \le \frac{c + \varepsilon + d_{n+1}}{2} \,,$$

yielding that $d_{n+1} \ge c - \varepsilon$.

□

3. The optimality of the honeycomb

We now consider the isoperimetric problem

$$\inf \Big\{ \mathcal{F}(E) \ : \ E \in \mathcal{H} \,, \ |E| = 1 \Big\} \,, \tag{6}$$

where $\mathcal{H}$ denotes as in the previous section the class of convex polygons which tessellate the plane by translations, and $\mathcal{F}$ belongs to a broad class of cost functionals. Precisely, we assume that

$$\mathcal{F} : \mathcal{H} \to \mathbb{R} \cup \{+\infty\},$$

satisfies the following conditions:

– if H_n is a sequence in $\mathcal{H}$ converging to H in Hausdorff distance, then

$$\liminf_n \mathcal{F}(H_n) \ge \mathcal{F}(H); \tag{7}$$

– for every line π and every $H \in \mathcal{H}$, if $S_\pi(H)$ denotes the Steiner symmetrization of H with respect to the line π (see (1)), it holds

$$\mathcal{F}(S_\pi(H)) \le \mathcal{F}(H). \tag{8}$$

As a direct consequence of Lemma 2.1, we obtain:

Theorem 3.1. *For any functional $\mathcal{F}$ satisfying (7) and (8), there holds*

$$\min_{H \in \mathcal{H}} \mathcal{F}(H) = \mathcal{F}(H^*) \,,$$

where H^* is the regular hexagon with unit area. Moreover, if (8) holds as a strict inequality when $H \ne S_\pi(H)$, then H^* is the unique minimizer.

Proof. For every $H \in \mathcal{H}$, we let $H_0 = H$ and we construct the sequence H_n as in Lemma 2.1. Then by (8) we get that $\mathcal{F}(H_{n+1}) \le \mathcal{F}(H_n) \le \mathcal{F}(H)$ for every $n \ge 1$. By (7) we have $\mathcal{F}(H^*) \le \liminf_n \mathcal{F}(H_n) \le \mathcal{F}(H)$, where H^* is the regular hexagon with area 1. Moreover, we have the strict inequality $\mathcal{F}(H^*) < \mathcal{F}(H)$ in case (8) is strict when $H \ne S_\pi(H)$. □

To conclude, we point out that the class of functionals satisfying assumptions (7)-(8) includes plenty of geometric or variational energies, other than the classical De Giorgi perimeter and the Cheeger constant (for which as mentioned in the Introduction the optimality of the honeycomb is already known in a much broader sense, without the periodicity and convexity constraints). A non hexaustive list of such functionals is given below.

- Nonlocal perimeter functionals of the kind
$$\mathrm{Per}_K(E) = \int_E \int_{\mathbb{R}^2 \setminus E} K(x-y)dxdy,$$
for any kernel K which is radial, nonnegative, with $\min\{1, |h|\}K(|h|) \in L^1(\mathbb{R}^2)$, and $\liminf_{z\to 0^+} \big[K(|z|) - K(|z+x|)\big] > 0$ for $x \neq 0$, see [8]. Note that if K is strictly decreasing, then (8) holds as a strict inequality for $H \neq S_\pi(H)$.
- Riesz-type energies of the form
$$\mathcal{R}(E) = -\int_E \int_E K(x-y)dxdy,$$
for any kernel K which is radial, decreasing, and positive definite, with $\int_0^1 K(r)dr < +\infty$ (see [15] and the related papers [2,3]). Also in this case, if K is strictly decreasing, (8) holds as a strict inequality for $H \neq S_\pi(H)$.
- The logaritmic capacity:
$$\mathrm{LogCap}(E) = e^{-W(E)}, \qquad \text{where } W(E) = \inf_{\mu\in\mathcal{P}_E} \int_E \int_E \log(|x-y|)d\mu(x)d\mu(y),$$
being $\mathcal{P}_E$ the class of Borel probability measures supported in E, see [1, Theorem 6.29].
- The Riesz-α-capacity for any $\alpha \in (0,2)$:
$$\mathcal{I}_\alpha(E) = \frac{1}{W_\alpha(E)}, \qquad \text{where } W_\alpha(E) = \inf_{\mu\in\mathcal{P}_E} \int_E \int_E |x-y|^{\alpha-2}d\mu(x)d\mu(y)$$
where $\mathcal{P}_E$ is as above, see [15], [1, Theorem 6.29].
- The variational p-capacity for any $p \in (1,2)$:
$$\mathrm{Cap}_p(E) = \inf\{\int_{\mathbb{R}^2} |\nabla u|^p \ : \ u \in C_0^\infty(\mathbb{R}^2),\ u = 1 \text{ on } E\},$$
see [1, Chapter 6].
- The first Dirichlet Laplacian eigenvalue:
$$\lambda_1(E) = \inf\left\{\int_E |\nabla u|^2dx \ : \ u \in H_0^1(E),\ \int_E |u|^2dx = 1\right\}$$
see [14, Chapter 3].

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