

# THE ISOPERIMETRIC PROBLEM FOR FOUR-DIMENSIONAL PARALLELOHEDRA

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ABSTRACT. We show that the regular 24-cell uniquely minimizes surface area among four-dimensional parallelohedra of fixed volume. As a consequence, it also minimizes surface area among parallelohedra containing a fixed ball. Moreover, we establish quadratic Hausdorff stability, and we derive a second-variation formula valid across changes of Voronoi type, which identifies which root-lattice Voronoi cells are strict local minima under metric and affine perturbations.

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## 1. INTRODUCTION

A parallelohedron is a convex polytope that tiles Euclidean space  $\mathbb{R}^d$  by translations. A natural question is: among such tiles of fixed volume, which has the least surface area? We answer this question in four dimensions, showing that the unique minimizer is the regular 24-cell.

For a convex body  $P \subset \mathbb{R}^d$ ,  $d \geq 2$ , write its isoperimetric quotient as

$$\mathcal{I}_d(P) = \frac{\mathcal{H}^{d-1}(\partial P)}{\text{vol}_d(P)^{(d-1)/d}},$$

and write  $\mathcal{I} = \mathcal{I}_4$ . Throughout, the *area* of a hypersurface in  $\mathbb{R}^d$  is its  $(d-1)$ -dimensional Hausdorff measure. For a four-dimensional polytope, the area of a facet is its three-dimensional volume. The quotient  $\mathcal{I}_d$  is invariant under translations, orthogonal maps, and positive dilations. Sets related by these operations are called *similar*.

**Theorem 1.1.** *Every four-dimensional parallelohedron  $P$  satisfies*

$$\mathcal{I}(P) \geq 2^{11/4}.$$

*Equality holds if and only if  $P$  is similar to the regular 24-cell.*

For a parallelohedron  $P$  with center  $c(P)$ , let

$$\widehat{P} = \text{vol}_4(P)^{-1/4}(P - c(P)), \quad \text{def}(P) = \frac{\mathcal{I}(P)}{2^{11/4}} - 1,$$

and let  $C_{24}$  be a centered regular 24-cell of volume two. The main theorem has the following stability form.

**Corollary 1.2.** *There is a constant  $C > 0$  such that every four-dimensional parallelohedron  $P$  with  $\text{def}(P) \leq 1$  satisfies*

$$\inf_{O \in O(4)} d_H(\widehat{P}, O\widehat{C}_{24}) \leq C\sqrt{\text{def}(P)},$$

where  $d_H$  denotes Hausdorff distance.

The proof does not give an effective value of  $C$ , because the passage from small deficit to the local stability neighborhood uses compactness.

In the plane, Hales's honeycomb theorem [18] identifies the regular hexagonal tiling as minimizing perimeter density among equal-area partitions. For three-dimensional parallelohedra, Hales–Song [19] and independently Cesaroni–Novaga [7] proved that the Archimedean truncated octahedron minimizes surface area at fixed volume, settling Bezdek's conjecture [2]. Note that Lángi [20] had proved the optimality of the truncated octahedron for the mean width. In four dimensions, the regular 24-cell honeycomb is a candidate for minimizing interfacial area among general equal-volume partitions. Cesaroni–Novaga [4] discuss this question and prove the minimality under perturbations supported in sufficiently small balls. In Theorem 1.1 we establish global optimality within the class of tilings by translates of a single convex tile. The corresponding questions for nonconvex fundamental domains and general equal-volume partitions remain open.

A direct consequence concerns convex lattice fundamental domains containing a fixed ball, where both the lattice and the volume may vary.

**Corollary 1.3.** *Let  $P$  be a convex fundamental domain for a lattice in  $\mathbb{R}^4$ . If  $P$  contains a Euclidean ball of radius  $r > 0$ , then*

$$\mathcal{H}^3(\partial P) \geq 32r^3.$$

*Equality holds if and only if  $\overline{P}$  is congruent to the regular 24-cell of inradius  $r$ .*

This bound follows from Theorem 1.1 and the pyramid formula. It proves the convex case of the fixed-inradius conjecture in [5]. The analogous three-dimensional bound was proved in [8].

The global proof must control two kinds of deformation. By Delaunay's theorem, every four-dimensional parallelohedron is affinely equivalent to a lattice Voronoi cell. Changing the lattice metric changes that cell, while a further volume-preserving linear map changes its surface area and need not preserve its Euclidean Voronoi property. We therefore estimate  $\mathcal{I}(LV_H)$  for every lattice metric  $H$  in the classification and every  $L$  with  $|\det L| = 1$ . The same distinction appears in the local problem in dimension three. Cesaroni–Novaga [6] proved local minimality among lattice Voronoi cells, and Song [27] proved quadratic stability among all parallelohedra.

The four-dimensional classification reduces the global problem to three closed cones of forms, including their positive-definite boundary forms. In the principal cone, the cells are affine images of zonotopes, with generator directions indexed by the edges of  $K_5$ . Combining a determinant estimate with stationarity and the sharp three-dimensional inequality for projections, we obtain a strict gap above the 24-cell value. The two nonprincipal cones contain the  $D_4$  form, whose Voronoi cell is the regular 24-cell, as a common core. Unlike the minimizers in dimensions two and three, this cell is not a zonotope, and a nearby Voronoi cell can acquire additional facets. These facets are the main obstruction to a direct comparison with the regular core.

In either nonprincipal cone, after normalization the metric has the form

$$H = tI + \sum_{i=1}^9 x_i P_{a_i}, \quad t, x_i \geq 0, \quad P_a = aa^\top,$$

where  $a_i$  are fixed unit vectors and  $x_i$  are the *retained weights*. The facet weights satisfy

$$\sum_e w_e P_e = \text{adj } H,$$

and the same weights determine the facet areas. Convexity and this identity remove the affine variable and give

$$\mathcal{I}(LV_H) \geq 2^{11/4} \exp(F/8), \quad F = E - \theta \log 2.$$

Here  $E$  measures directional variation of the metric, and  $\theta/4$  is the fraction of volume contained in pyramids over the new facets. The term  $\theta \log 2$  records their different surface normalization. The global comparison thus reduces to the inequality  $F \geq 0$ , that is,

$$E \geq \theta \log 2.$$

We prove  $F \geq 0$  by differentiating  $(\det H)F$  in a largest retained weight. Projection perpendicular to its direction gives a configuration whose independent triples are the spanning trees of  $K_4$ . The projected weights define a product-weighted law on these trees, and uniform deletion of one edge gives the pair cofactors in the derivative. The general entropy contraction under deletion is a case of entropic independence [1]. The sharper bound needed here uses a compatibility constraint between the star probability and the three opposite-pair probabilities of a product-weighted tree law. This bound controls the derivative for a small core parameter  $t$ , and a transverse variance estimate controls it for large  $t$ . Both ranges also use explicit cofactor estimates. Integration along  $x_i(r) = \min\{x_i, r\}$  then proves  $F \geq 0$ , with strictness away from the regular core. Equality in the remaining affine comparison forces  $L$  to be orthogonal.

For stability, we return to the exact surface formula and retain the affine variable that the global comparison eliminated. The facet-weight identity determines the mixed second-order terms between the lattice metric and the affine strain without differentiating individual facet areas. The resulting expansion is uniform across changes of Voronoi type and includes the contribution of newly created facets. At the  $D_4$  metric, these facets arise from vertex contacts and contribute only at cubic order. The quadratic form therefore gives the sharp optimized coefficient  $1/20$  and controls both metric and affine distortion. A local converse to the parametrization, together with compactness, then yields Corollary 1.2 for arbitrary four-dimensional parallelohedra.

The same variation calculation applies in arbitrary dimension. Cofactor balance determines the mixed terms, and the contact codimension determines the order at which new facets enter. When the reference facet vectors have a common length, a nonfacet contact of codimension three can contribute at quadratic order, while contacts of higher codimension do not change the quadratic form. As a final application, this criterion classifies the cells of root lattices of rank at least two under nearby lattice metrics and unrestricted affine maps. The strict local minima are  $A_2$ ,  $D_d$  for  $d \geq 4$ , and  $E_6, E_7, E_8$ . Every  $A_d$  with  $d \geq 3$  and every reducible root lattice admits a decreasing variation.

For symmetric matrices we use the Frobenius pairing  $\langle A, B \rangle = \text{tr}(AB)$  and norm  $\|A\|^2 = \text{tr}(A^2)$ . We write  $\text{Sym}_0(\mathbb{R}^d)$  for the traceless symmetric matrices and  $\text{tf } A = A - (\text{tr } A)I/d$  for the traceless part. Sections 2–6 establish the global comparison. Section 7 retains affine strain in the surface formula to compute the second variation near equality, and Section 8 deduces stability of the 24-cell. Section 9 gives the root-lattice application of the same variation formula.

## 2. VORONOI CLASSIFICATION AND THE PRINCIPAL CONE

By the Venkov–McMullen theorem [28, 24], every convex translative tile is a polytope admitting a face-to-face lattice tiling. In particular, the closure of a convex lattice fundamental domain is a parallelohedron.

In Voronoi’s reduction theory [29], an  $L$ -domain consists of positive-definite forms with the same Delaunay decomposition of  $\mathbb{Z}^4$ , and is *primitive* when that decomposition is simplicial. Its closure is taken in the positive-semidefinite cone. Positive-definite boundary forms still define full-dimensional cells, while singular forms enter only as limits.

For a positive-definite form  $Q = B^T B$  on  $\mathbb{Z}^4$  define

$$\text{Vor}(Q) = \{x \in \mathbb{R}^4 : 2x^T Q z \leq z^T Q z \text{ for all } z \in \mathbb{Z}^4\}.$$

The Voronoi cell of the lattice  $B\mathbb{Z}^4$  is  $B \text{Vor}(Q)$ , of volume  $\sqrt{\det Q}$ . To use Minkowski addition within a single  $L$ -domain, we pass to the cell in dual coordinates

$$\mathcal{D}(Q) = 2Q \text{Vor}(Q) = \{y : z^T y \leq z^T Q z \text{ for all } z \in \mathbb{Z}^4\}.$$

**Lemma 2.1.** *If  $Q_1, Q_2$  lie in the closure of one  $L$ -domain, then*

$$\mathcal{D}(Q_1 + Q_2) = \mathcal{D}(Q_1) + \mathcal{D}(Q_2).$$

*If  $v \in \mathbb{Z}^4$  is primitive, then  $\mathcal{D}(vv^T) = [-v, v]$ .*

*Proof.* Within an  $L$ -domain,  $\mathcal{D}(Q)$  has a fixed normal fan and each vertex  $y$  is determined by the equations  $z_i^T y = z_i^T Q z_i$  for the facet vectors  $z_i$  active at  $y$ . They depend linearly on  $Q$ , so the support functions add. Continuity extends this to the closure. For  $Q = vv^T$ , the inequalities with  $z \perp v$  force  $y = sv$ . Primitivity makes  $v^T z$  range over  $\mathbb{Z}$ , so the remaining inequalities give  $|s| \leq 1$ .  $\square$

**Theorem 2.2.** *Every four-dimensional parallelohedron is affinely equivalent to a lattice Voronoi cell. Up to arithmetic equivalence, every positive-definite form belongs to the closure of one of three primitive  $L$ -domains, namely the principal cone (Selling’s cone) and two nonprincipal cones, called types II and III.*

*Proof.* The affine statement is Delaunay’s theorem [11, 12] (see also [17]). Table 2 of [15] lists the three primitive  $L$ -types. Section 6.2 of that paper identifies them as the principal type and types II and III, and gives the graphic and cographic descriptions used below.  $\square$

The lattice  $D_4 = \{x \in \mathbb{Z}^4 : x_1 + x_2 + x_3 + x_4 \equiv 0 \pmod{2}\}$  fixes the normalization of the sharp constant and equality case.

**Lemma 2.3.** *The Voronoi cell of  $D_4$  is the regular 24-cell*

$$C_{24} = \text{conv}\left(\{\pm e_i : 1 \leq i \leq 4\} \cup \left\{\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)\right\}\right).$$

*It has volume 2, inradius  $1/\sqrt{2}$ , and 24 regular octahedral facets, each of area  $\sqrt{2}/3$ . Consequently  $\mathcal{H}^3(\partial C_{24}) = 8\sqrt{2}$  and  $\mathcal{I}(C_{24}) = 2^{11/4}$ .*

*Proof.* The inequalities  $|y_i| + |y_j| \leq 1$  defining the  $D_4$  Voronoi cell have vertices  $\pm e_i$  and the half-sign vectors. Each root facet is a regular octahedron of circumradius  $1/\sqrt{2}$ , volume  $\sqrt{2}/3$ , and support distance  $1/\sqrt{2}$ . Summing facet volumes and applying the pyramid formula gives the stated volume, surface area, and isoperimetric quotient.  $\square$

Write  $\text{Gr}(G)$  for a graphic vector configuration, whose independent sets are the forests of  $G$ . A realization is *unimodular* if its nonzero maximal determinants have the same absolute value.

For each oriented edge  $\epsilon$  of  $K_5$ , let  $\chi_\epsilon \in \mathbb{Z}^4$  be its incidence vector with one fixed vertex coordinate deleted.

**Proposition 2.4.** *Every form in the closed principal cone has the form*

$$Q = \sum_{\epsilon \in E(K_5)} \rho_\epsilon \chi_\epsilon \chi_\epsilon^\top, \quad \rho_\epsilon \geq 0.$$

If  $Q > 0$ , its Voronoi cell is affinely equivalent to  $\sum_\epsilon \rho_\epsilon [-\chi_\epsilon, \chi_\epsilon]$ . After zero parameters are discarded, the generating system is therefore a subsystem of  $\text{Gr}(K_5)$ .

*Proof.* Use the Selling parametrization [26] and [15], followed by Lemma 2.1. Then  $\mathcal{D}(Q) = \sum_\epsilon \rho_\epsilon [-\chi_\epsilon, \chi_\epsilon]$  is an invertible linear image of the Voronoi cell.  $\square$

The next lemma excludes degenerate minimizing sequences in the proof of Theorem 2.6.

**Lemma 2.5.** *Let  $\Omega \subset \mathbb{R}^4$  be a convex body of diameter  $D$ , and write  $S(\Omega) = \mathcal{H}^3(\partial\Omega)$  and  $\omega_3 = 4\pi/3$ . Then*

$$S(\Omega)^3 \geq \frac{27\omega_3}{4} \text{vol}(\Omega)^2 D.$$

*Proof.* Choose  $x, y \in \Omega$  with  $|x - y| = D$ , set  $\xi(z) = (z - x) \cdot (y - x)/D$ , and let  $\Omega_c = \Omega \cap \{\xi = c\}$  and  $A(c) = \mathcal{H}^3(\Omega_c)$  for  $0 \leq c \leq D$ . Since  $\Omega$  lies in the balls of radius  $D$  about  $x$  and about  $y$ , the end slices  $\Omega_0, \Omega_D$  are the points  $x, y$ . By Brunn–Minkowski,  $A^{1/3}$  is concave on  $[0, D]$ , so  $A_* = \max A$  is attained at some  $c_* \in (0, D)$ . The convex hull of  $\Omega_{c_*} \cup \{x, y\}$  lies in  $\Omega$  and is the union of two cones over  $\Omega_{c_*}$  with heights  $c_*$  and  $D - c_*$ , so  $\text{vol}(\Omega) \geq A_* D/4$ . For  $0 < c < D$  the relative boundary of  $\Omega_c$  lies in  $\partial\Omega \cap \{\xi = c\}$ , and  $\xi$  is 1-Lipschitz. The coarea formula on  $\partial\Omega$  and the three-dimensional isoperimetric inequality  $\mathcal{H}^2(\partial K) \geq 3\omega_3^{1/3} \mathcal{H}^3(K)^{2/3}$  therefore give

$$\begin{aligned} S(\Omega) &\geq \int_0^D \mathcal{H}^2(\partial_{\text{rel}} \Omega_c) dc \geq 3\omega_3^{1/3} \int_0^D A(c)^{2/3} dc \\ &\geq 3\omega_3^{1/3} \frac{\text{vol}(\Omega)}{A_*^{1/3}} \geq 3\omega_3^{1/3} \text{vol}(\Omega)^{2/3} \left(\frac{D}{4}\right)^{1/3}. \end{aligned}$$

Cubing proves the claim.  $\square$

Theorem 2.6 below treats the principal cone. The generator-length estimate is the four-dimensional instance of the weighted simplex inequality of Hales–Song [19, Theorem 2.2]. We reproduce its determinant argument below. The remaining input is the sharp three-dimensional theorem [7, Theorem 1.1].

**Theorem 2.6.** *Let  $Z$  be a full-dimensional zonotopal parallelohedron which, up to translation, has a representation*

$$Z = \sum_{\epsilon \in E_0} c_\epsilon [-A\chi_\epsilon/2, A\chi_\epsilon/2], \quad E_0 \subset E(K_5), \quad c_\epsilon > 0, \quad A \in GL(4, \mathbb{R}).$$

Then

$$\mathcal{I}(Z)^4 \geq 4 \cdot 5^{1/6} (1 + 2\sqrt{3})^4 > 2048.$$

Since the hypothesis is invariant under linear maps, the bound holds for every affine image of  $Z$ . In particular, every affine image of the Voronoi cell of a positive-definite form in the closed principal cone satisfies  $\mathcal{I} > 2^{11/4}$ .

*Proof.* Enumerate  $E_0 = \{\epsilon_1, \dots, \epsilon_{n_0}\}$  and set

$$v_i = A\chi_{\epsilon_i}, \quad r_i = |v_i|, \quad n_i = v_i/r_i, \quad \ell_i = c_{\epsilon_i} r_i.$$

Then  $Z = \sum_i \ell_i [-n_i/2, n_i/2]$  and  $|n_i| = 1$ . The system  $(v_i)$  is unimodular, with nonzero maximal determinants of absolute value  $\delta = |\det A|$ . Set  $\ell_{\text{tot}} = \sum_i \ell_i$  and  $\pi_i = \ell_i/\ell_{\text{tot}}$ . Let  $\mathcal{B}$  be the bases of  $(v_i)$ , equivalently the spanning trees of the graph with edge set

$E_0$ . For  $\pi_J = \prod_{i \in J} \pi_i$ ,  $\tau(\pi) = \sum_{J \in \mathcal{B}} \pi_J$ , and  $\Sigma = \sum_i \pi_i n_i n_i^\top$ , the volume formula and Cauchy–Binet give

$$\frac{\text{vol } Z}{\ell_{\text{tot}}^4} = \sum_{J \in \mathcal{B}} \frac{\pi_J \delta}{\prod_{i \in J} r_i}, \quad \det \Sigma = \sum_{J \in \mathcal{B}} \frac{\pi_J \delta^2}{\prod_{i \in J} r_i^2}, \quad \frac{\text{vol } Z}{\ell_{\text{tot}}^4} \leq \sqrt{\tau(\pi) \det \Sigma} \leq \frac{\sqrt{\tau(\pi)}}{16}.$$

The last two inequalities are weighted Cauchy–Schwarz and AM–GM, since  $\text{tr } \Sigma = 1$ . Extend  $\pi$  by zero on missing directions. The weighted Laplacian of  $K_5$  has trace 2, so the matrix-tree theorem gives  $\tau = \frac{1}{5} \prod_{j=1}^4 \lambda_j \leq 1/80$ . Hence

$$(2.1) \quad \ell_{\text{tot}}^4 \geq 64\sqrt{5} \text{vol}(Z).$$

Fix the directions and minimize  $\mathcal{I}$  on the length simplex  $\ell_i \geq 0$ ,  $\sum_i \ell_i = 1$ . Changing positive lengths preserves the tiling property. On our simplex,  $\text{diam } Z \geq \max_i \ell_i \geq 1/n_0$ , so Lemma 2.5 gives

$$\mathcal{I}(Z) \geq \left(\frac{27\omega_3}{4}\right)^{1/3} n_0^{-1/3} \text{vol}(Z)^{-1/12} \rightarrow \infty \quad \text{as } \text{vol } Z \rightarrow 0.$$

Continuity therefore gives a full-dimensional minimizer  $Z_*$ . Its length vector lies in the relative interior of a face of the length simplex, so stationarity holds in every direction of positive length.

Projection along an active generator preserves the tiling property. To see this, quotient a unimodular basis containing that generator, observe that every independent quotient triple extends to an original basis, and combine parallel directions [23, 14]. Orthogonal projection differs from this quotient by an invertible linear map. Set  $V = \text{vol } Z_*$ ,  $S = \mathcal{H}^3(\partial Z_*)$ , and let  $V_i, S_i$  be the volume and area of  $Z_*|n_i^\perp$ . Extrusion gives  $\partial_i V = V_i$  and  $\partial_i S = S_i$  (also immediate from  $S = 2 \sum_{i < j < k} \ell_i \ell_j \ell_k |n_i \wedge n_j \wedge n_k|$ ). Euler’s identity makes the Lagrange multiplier of the degree-zero quotient vanish, so stationarity and the sharp three-dimensional theorem [7, Theorem 1.1] yield

$$S_i = \frac{3S}{4V} V_i \geq c_3 V_i^{2/3}, \quad c_3 = \frac{3(1+2\sqrt{3})}{4^{2/3}}, \quad V_i \geq \left(\frac{4c_3 V}{3S}\right)^3.$$

Multiply by  $\ell_i$  and use  $\sum_i \ell_i V_i = 4V$  to obtain  $\mathcal{I}(Z_*)^3 \geq (1+2\sqrt{3})^3 \ell_{\text{tot}}^3 / V^{1/4}$ . Raising to the power 4/3 and applying (2.1) proves the stated bound. Its strict comparison with 2048 follows from  $5^{1/6} > 13/10$ ,  $\sqrt{3} > 19/11$ , and  $(1+2\sqrt{3})^4 = 217 + 104\sqrt{3}$ . These estimates show that the left side exceeds  $113438/55 > 2048$ .  $\square$

### 3. NONPRINCIPAL GEOMETRY AND THE AFFINE COMPARISON

The two nonprincipal families share a  $D_4$  core and eight of their nine retained rays. We also use the cographic configuration  $\text{CoGr}(G)$ , whose bases are complements of spanning trees of a connected graph  $G$ . For  $v \neq 0$ , write  $P_v = vv^\top / \|v\|^2$ , and for an index set  $J$  write  $x_J = \prod_{i \in J} x_i$ , with  $x_\emptyset = 1$ . The retained rays in  $T_j = \{3j-2, 3j-1, 3j\}$  and the omitted vector  $m_j$  form the orthonormal frame  $F_j$ . The type II coordinates are

| $j$ | $a_{3j-2}$                 | $a_{3j-1}$                  | $a_{3j}$                    | $m_j$                        |
|-----|----------------------------|-----------------------------|-----------------------------|------------------------------|
| 1   | $e_1$                      | $e_2$                       | $e_3$                       | $e_4$                        |
| 2   | $\frac{1}{2}(1, 1, 1, -1)$ | $\frac{1}{2}(1, 1, -1, 1)$  | $\frac{1}{2}(1, -1, 1, 1)$  | $\frac{1}{2}(1, -1, -1, -1)$ |
| 3   | $\frac{1}{2}(1, 1, 1, 1)$  | $\frac{1}{2}(1, 1, -1, -1)$ | $\frac{1}{2}(1, -1, -1, 1)$ | $\frac{1}{2}(1, -1, 1, -1)$  |

Type III interchanges  $a_9$  and  $m_3$ . The twelve lines in the complete frames will be called the full-frame lines, with weight zero assigned to each omitted vector. Set

$$B_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}, \quad K = \sum_{i=1}^9 x_i P_{a_i}, \quad H = tI + K, \quad Q = B_0^\top H B_0,$$

where  $t, x_i \geq 0$  and  $H > 0$ . Then  $\det B_0 = 2$ . The fixed matrix  $B_0$  records the lattice basis, while  $H$  expresses the variable form in the orthonormal coordinates of the regular cell. We call  $t$  the *core weight* and  $x_1, \dots, x_9$  the *retained weights*. When all retained weights vanish,  $t$  only scales the regular cell. Write  $\mathfrak{D} = \det H$ ,  $h_e = e^\top H e$ ,  $s_j = \sum_{i \in T_j} x_i$ ,  $p_j = \prod_{i \in T_j} x_i$ ,  $\sigma = \sum_j s_j$ , and  $u_j = \sigma - s_j$ . The old normal lines are

$$\mathcal{R} = \{(e_r \pm e_s)/\sqrt{2} : 1 \leq r < s \leq 4\} \quad \text{and} \quad \mathcal{N} = \mathcal{R} \cup \{m_1, m_2, m_3\}.$$

Each element of  $\mathcal{N}$  represents an opposite facet pair, and each area below refers to one facet.

**Lemma 3.1.** *The displayed rays represent the two closed nonprincipal  $L$ -domains, with retained configurations  $\text{Gr}(K_5 - e)$  in type II and  $\text{CoGr}(K_{3,3})$  in type III. Distinct frames have squared inner products  $1/4$ , and*

$$\sum_{e \in \mathcal{R}} P_e = 3I \quad \text{and} \quad \sum_{e \in \mathcal{R}, e \perp a} P_e = 2(I - P_a)$$

for every full-frame vector  $a$ . An independent triple of full-frame lines is either a same-frame triple of Gram determinant one or has Gram determinant  $1/2$  and an old perpendicular normal. Every independent retained four-tuple has squared determinant  $1/4$ . For each old normal, the six perpendicular full-frame lines form the  $A_3$  root configuration. They have 16 independent triples, each line belongs to eight, and an orthogonal or nonorthogonal pair extends to four or three, respectively. The retained-ray symmetry orbits are  $\{1, 2, 4, 5, 7, 8\}$  and  $\{3, 6, 9\}$  in type II, and all nine rays in type III.

*Proof.* The cone and graph identifications are the explicit  $D_4$ -ray description in [15, Sections 6.2–6.3, Proposition 4]. The frame inner products and triple determinants follow from the coordinates. For a same-frame triple, the squared inner product of an independent fourth retained ray with the perpendicular unit normal is  $1/4$ . For any other independent triple, this squared inner product is  $1/2$ , so both cases give squared determinant  $1/4$ . Summing the two projectors on each coordinate pair gives the first resolution, and the second follows in the coordinate frame and then by frame symmetry. The six lines perpendicular to an old normal have coordinates  $(b_i \pm b_k)/\sqrt{2}$  in an orthonormal basis of its perpendicular space, so the remaining incidence counts are the spanning-tree counts of  $K_4$ . The type II permutations  $(12)(45)(78)$ ,  $(14)(25)(36)$ , and  $(48)(57)(69)$  are induced by orthogonal maps and give the stated orbits. In type III, row permutations and simultaneous column permutations of the  $3 \times 3$  ray array act orthogonally and transitively on the retained lines.  $\square$

We compute the facets in the dual cell, where the parameters act by Minkowski addition in fixed directions, and then apply  $H^{-1/2}$ . The cubic weights below are both the facet-volume coefficients and the coefficients in an adjugate resolution.

**Lemma 3.2.** *Up to an orthogonal map, the Voronoi cell is*

$$V_H = H^{-1/2} \mathcal{C}_H \quad \text{and} \quad \mathcal{C}_H = tC_{24} + \sum_{i=1}^9 \frac{x_i}{2} [-a_i, a_i].$$

For  $e \in \mathcal{R}$ , write  $I_e = \{i : e \perp a_i\}$  and define

$$(3.1) \quad w_e = \frac{t^3}{3} + \frac{t^2}{2} \sum_{i \in I_e} x_i + \frac{t}{2} \sum_{\substack{i < k \\ i, k \in I_e}} (1 + \mathbf{1}_{a_i \perp a_k}) x_i x_k + \frac{1}{2} \sum_{\substack{J \subset I_e, |J|=3 \\ J \text{ independent}}} x_J.$$

For a new normal set  $w_{m_j} = p_j$ , and note that  $h_{m_j} = t + u_j/4$ . Then

$$\sum_{e \in \mathcal{N}} w_e P_e = \text{adj } H \quad \text{and} \quad \sum_{e \in \mathcal{N}} w_e h_e = 4\mathfrak{D}.$$

Every positive-area facet occurs in this list, with area

$$A_e = \frac{\sqrt{2}w_e\sqrt{h_e}}{\sqrt{\mathfrak{D}}} \quad (e \in \mathcal{R}) \quad \text{and} \quad A_{m_j} = \frac{p_j\sqrt{h_{m_j}}}{\sqrt{\mathfrak{D}}},$$

and  $\text{vol}(V_H) = 2\sqrt{\mathfrak{D}}$ .

*Proof.* The vectors  $B_0^\top a_i$  are primitive integral, and the isotropic dual cell is  $2B_0^\top C_{24}$ . Lemma 2.1 therefore gives  $\mathcal{D}(\mathbf{Q}) = 2B_0^\top \mathcal{C}_H$ . Since  $\mathbf{Q}^{-1/2}B_0^\top H^{1/2}$  is orthogonal, the cell  $\frac{1}{2}\mathbf{Q}^{-1/2}\mathcal{D}(\mathbf{Q})$  is congruent to  $V_H$  and has volume  $\sqrt{\det \mathbf{Q}} = 2\sqrt{\mathfrak{D}}$ . An old normal exposes an octahedron with centered vertices  $\pm b_k/\sqrt{2}$  and tangent directions  $(b_i \pm b_k)/\sqrt{2}$ . Multilinearity under addition of tangent segments gives its area  $\sqrt{2}w_e$ . Orthogonal pairs have coefficient two, nonorthogonal pairs coefficient one, and independent triples have Gram determinant  $1/2$ . A new normal exposes a vertex of  $C_{24}$  and its three orthogonal tangent segments, giving area  $p_j$ .

To verify cofactor balance, set  $y_a = x_a + t/3$  on the twelve full-frame lines. Since  $H = \sum_a y_a P_a$ , adjugate Cauchy–Binet gives

$$\text{adj } H = \sum_{e \in \mathcal{R}} W_e P_e + \sum_{a \text{ full}} W_a P_a, \quad W_e = \frac{1}{2} \sum_{\substack{|J|=3, J \perp e \\ J \text{ independent}}} y_J, \quad W_a = \prod_{b \in F(a) \setminus \{a\}} y_b.$$

Set  $w_a = \prod_{b \in F(a) \setminus \{a\}} x_b$ , which vanishes unless  $a$  is omitted. Incidence counts above give

$$\begin{aligned} w_e - W_e &= \frac{t^3}{27} + \frac{t^2}{18} \sum_{a \perp e} x_a + \frac{t}{3} \sum_{\substack{a < b, a, b \perp e \\ a \perp b}} x_a x_b, \\ W_a - w_a &= \frac{t^3}{27} + \frac{t^2}{9} \sum_{b \in F(a) \setminus \{a\}} x_b + \frac{t}{3} \sum_{\{b, c\} \subset F(a) \setminus \{a\}} x_b x_c. \end{aligned}$$

Their projector sums agree degree by degree. The constant and linear terms use the resolution identities, while for an orthogonal pair the complementary frame vectors and perpendicular old lines are orthonormal bases of the same plane. This proves cofactor balance, and tracing against  $H$  gives its scalar form. The support distances in  $\mathcal{C}_H$  are  $h_e/\sqrt{2}$  on old normals and  $h_{m_j}$  on new ones. Under  $H^{-1/2}$  the hypersurface factor is  $\sqrt{h_e}/\sqrt{\mathfrak{D}}$ , proving the area formulas. The listed pairs exhaust the divergence identity  $\sum_F h_F A_F = 4 \text{vol}(V_H)$ , where  $F$  runs over the facets and  $h_F$  is the support distance of  $F$  from the center of  $V_H$ , so no further positive-area facet can occur. Continuity includes all zero parameters with  $H > 0$ .  $\square$

Define the marked cofactor mass and its normalization by

$$N_b = \sum_{j=1}^3 p_j h_{m_j} \quad \text{and} \quad \theta = \frac{N_b}{\mathfrak{D}}.$$

The two pyramids over the facet pair represented by  $e$  occupy the fraction  $w_e h_e/(4\mathfrak{D})$  of  $V_H$ , so new facets account for the volume fraction  $\theta/4$ .

The fixed reference normal  $e$  becomes the normal  $n_e = H^{1/2}e/\sqrt{h_e}$  of  $V_H$ . A further volume-preserving map  $L$  acts on these normals by  $L^{-\top}$ . The next lemma removes  $L$  from the comparison, leaving a determinant depending only on  $H$ .

Let  $\eta_e = 1$  on old normals and  $1/\sqrt{2}$  on new ones. The surface tensor and its normalized determinant are

$$A_f = \sum_{e \in \mathcal{N}} \frac{\eta_e w_e}{\sqrt{h_e}} P_e \quad \text{and} \quad J_s = \frac{\det A_f}{\mathfrak{D}^{5/2}}.$$

**Lemma 3.3.** *For  $|\det L| = 1$ , one has  $\mathcal{I}(LV_H) \geq 2^{11/4} J_s^{1/4}$ . At  $H = tI$ , one has  $J_s = 1$ , and equality holds precisely for orthogonal  $L$ .*

*Proof.* The facet areas and their hypersurface transformation give

$$\mathcal{I}(LV_H) = 2^{3/4} \mathfrak{D}^{-7/8} \sum_e \eta_e w_e \sqrt{h_e} \|L^{-\top} n_e\|.$$

Write  $L^{-\top} = OS$  with  $S > 0$  and  $\det S = 1$ , and set  $\hat{A}_f = H^{1/2} A_f H^{1/2}$ . Since  $\|Sn\| \geq n^\top S n$  for  $|n| = 1$ , spectral AM–GM bounds the sum below by  $\text{tr}(S\hat{A}_f) \geq 4(\mathfrak{D} \det A_f)^{1/4}$ . At  $H = tI$ , the new weights vanish,  $A_f = t^{5/2}I$ , and  $\hat{A}_f = t^{7/2}I$ . Equality in AM–GM then forces  $S = I$ , which also suffices and is equivalent to orthogonality of  $L$ .  $\square$

To estimate  $\det A_f$ , we interpolate between  $A_f$  and  $\text{adj } H$  by replacing the surface factors with variable powers. The logarithmic derivative at exponent zero yields the quantity  $F$  defined next.

For a positive-definite matrix  $A$  and a unit vector  $e$ , write

$$\text{Ent}_e(A) = e^\top A \log A e - (e^\top A e) \log(e^\top A e) \quad \text{and} \quad \text{Var}_e(A) = e^\top A^2 e - (e^\top A e)^2.$$

These nonnegative quantities extend continuously to positive semidefinite matrices. Set

$$U = \sum_e w_e \text{Ent}_e(H), \quad E = \frac{U}{\mathfrak{D}}, \quad F = E - \theta \log 2.$$

The following lemma reduces the affine comparison to the scalar inequality  $F \geq 0$ .

**Lemma 3.4.** *Set  $\lambda_e = w_e h_e / \mathfrak{D}$ . Then*

$$E = \log \mathfrak{D} - \sum_e \lambda_e \log h_e \geq 0 \quad \text{and} \quad \log J_s \geq \frac{1}{2} F.$$

*Proof.* Tracing cofactor balance against  $H \log H$  gives  $\sum_e w_e e^\top H \log H e = \mathfrak{D} \log \mathfrak{D}$ , proving the formula for  $E$ . Its nonnegativity is spectral Jensen for  $x \log x$ . Discard zero weights, set  $\kappa_e = \eta_e^{-2}$ , and define

$$\Phi_{\text{aff}}(s) = \log \det \left( \sum_e w_e (\kappa_e h_e)^s P_e \right).$$

The remaining normals span, and Cauchy–Binet makes  $\Phi_{\text{aff}}$  the logarithm of a positive sum of exponentials, hence convex. The exponent  $-1/2$  gives the surface tensor, while exponent zero gives  $\text{adj } H$ . Its inverse is  $H/\mathfrak{D}$ , so differentiating the logarithmic determinant assigns the weight  $\lambda_e$  to each normal. Cofactor balance gives

$$\begin{aligned} \Phi_{\text{aff}}(0) &= 3 \log \mathfrak{D}, \\ \Phi_{\text{aff}}(-1/2) &= \frac{5}{2} \log \mathfrak{D} + \log J_s, \\ \Phi'_{\text{aff}}(0) &= \log \mathfrak{D} - F. \end{aligned}$$

The tangent inequality at zero, evaluated at  $-1/2$ , yields the claim.  $\square$

Cofactor balance also gives  $\sum_e \lambda_e P_{n_e} = I$ . For  $L_H = \mathfrak{D}^{1/8} H^{-1/2} \in SL(4)$ , one has  $E = 2 \sum_e \lambda_e \log \|L_H n_e\|$ , the logarithmic eccentricity of this isotropic measure [3]. It remains to prove  $F \geq 0$ , which is Theorem 5.2.

#### 4. TRANSVERSE ENTROPY AND VARIANCE

Choose a retained ray  $a$  of weight  $R$ , write  $H = RP_a + G$ , and set  $B = G|_{a^\perp}$  and  $A = \det B$ . Let  $u, v$  be its companion weights, namely the other two retained weights in the frame of  $a$ , let  $s$  be the total outside its frame, and set  $h = t + s/4$ . The transverse entropy is

$$U_3 = \sum_{e \perp a} \omega_e \text{Ent}_e(B),$$

where  $\omega_e$  is the coefficient of  $R$  in  $w_e$ . We will prove the normalized bound

$$[U_3 + (A - uvh) \log 2]/A \geq (9/4) \log(4/3)$$

and the marked bound  $U_3 \geq uvh/4$ . In the proof of Theorem 5.2, with  $R$  a largest retained weight, the first bound is used when  $t \leq 2R/3$  and the second when  $t \geq 2R/3$ .

Differentiating in the weight of  $a$  turns the coefficients of the facets perpendicular to  $a$  into the pair cofactors of the rays projected to  $a^\perp$ . Together with the isotropic core, these projected rays give weights on the six-edge configuration of  $K_4$  (Proposition 4.3), so  $U_3$  becomes an entropy of the product-weighted law on its spanning trees and of the pair shadow obtained by deleting one edge.

Write  $H(p) = -\sum p \log p$  and  $D(p||q) = \sum p \log(p/q)$ , using natural logarithms and  $0 \log 0 = 0$ . The pair shadow of a law on three-edge sets deletes one of the three edges uniformly at random. The singleton shadow of a law on pairs is defined in the same way.

The product-weighted tree references below are finite projection determinantal measures in the framework of Lyons [21]. Cesaroni–Novaga [7, Sections 5.1 and 6.2] use a rank-three determinantal measure on seven cut directions, whose singleton inclusion probabilities are leverage scores. Their entropy contraction compares two product-weighted determinantal laws through these marginals. Here the six transverse generators give a law on spanning trees, and the facet cofactors require its pair shadow. We therefore need contraction from trees to pairs for an arbitrary comparison law and a product-weighted reference. This is a special case of entropic independence [1], for which we give a direct rank-two proof.

**Lemma 4.1.** *Let  $\mu_3$  be a probability law on the spanning trees of  $K_4$ , and let  $\mu_2$  be its uniform pair shadow, obtained by deleting one edge. Let  $\rho_i$  be the probability that the tree contains opposite edge pair  $i$ , and set  $p = \sum_i \rho_i$ . Then*

$$\begin{aligned} \mathcal{K}(\mu_3) &:= 3H(\mu_2) - 2H(\mu_3) + 2 \log 2 - 3 \log 3 \\ &\geq \Phi(p) := (3-p) \log 4 + \sum_i (1-\rho_i) \log(1-\rho_i) - (3-p) \log(3-p) \\ &\geq \log \frac{16}{9}. \end{aligned}$$

In particular,  $3H(\mu_2) - 2H(\mu_3) \geq \log 12$ .

*Proof.* We use an elementary rank-two entropy principle. For a law  $\nu$  on unordered pairs, set  $f_\nu(z) = \sum_{\{i,j\}} \nu_{ij} z_i z_j$  for  $z_i > 0$ , and let  $\pi$  be the law of a uniformly chosen member of the pair. If

$$f_\nu(z) \leq \left( \sum_i \pi_i z_i \right)^2,$$

then every pair law  $\mu$  with singleton shadow  $\mu_1$  satisfies

$$D(\mu||\nu) \geq 2D(\mu_1||\pi).$$

Indeed, testing the variational formula for relative entropy with  $\log z_i + \log z_j$  gives the lower bound  $2\mathbb{E}_{\mu_1} \log z - \log f_\nu(z)$ . The polynomial bound replaces the second term by  $-2 \log \sum_i \pi_i z_i$ , and optimization over  $z$  proves the claim.

We establish the deletion contraction for every positive product-weighted tree reference  $\nu_3$ . Write  $\mu_1$  for the singleton shadow of  $\mu_3$ , and  $\nu_2, \nu_1$  for the shadows of the reference. Set  $D_j = D(\mu_j||\nu_j)$ . For the reference edge weights  $y_i > 0$ , take reduced incidence vectors  $r_i$  of  $K_4$  and set  $\Sigma_y = \sum_i y_i r_i r_i^\top$  and  $\xi_i = \sqrt{y_i} \Sigma_y^{-1/2} r_i$ . The tree determinants of the  $r_i$  are  $\pm 1$ , so Cauchy–Binet identifies the generating polynomial of  $\nu_3$  as  $\det C(z)$ , where  $C(z) = \sum_i z_i \xi_i \xi_i^\top$  and  $C(\mathbf{1}) = I_3$ . Since  $C(z) + \varepsilon I_3 = C(z + \varepsilon \mathbf{1})$ , differentiating its determinant at  $\varepsilon = 0$  deletes one edge from each tree monomial. Hence the pair-shadow polynomial is  $e_2(C(z))/3$ , where  $e_2$  is the second elementary symmetric function of the eigenvalues, and the singleton mean is  $\text{tr } C(z)/3$ . The eigenvalue inequality  $e_2(C) \leq (\text{tr } C)^2/3$  and the rank-two entropy principle therefore give  $D_2 \geq 2D_1$ .

We now condition on a distinguished edge and exhibit the resulting two-dimensional law. Select an edge uniformly from the sampled tree, so its reference probability is  $\nu_1(e) = \|\xi_e\|^2/3$ . Let  $\Pi_e$  be orthogonal projection onto  $\xi_e^\perp$  and set  $\xi_i^{(e)} = \Pi_e \xi_i$  for  $i \neq e$ . In orthonormal coordinates on this plane, define

$$\mathbf{C}_e(z) = \sum_{i \neq e} z_i \xi_i^{(e)} (\xi_i^{(e)})^\top.$$

Projecting  $\sum_i \xi_i \xi_i^\top = I_3$  gives  $\mathbf{C}_e(\mathbf{1}) = I_2$ . Moreover, taking the component perpendicular to  $\xi_e$  in each remaining column gives

$$\nu_{2|e}(\{i, j\}) = \frac{\nu_3(\{e, i, j\})}{3\nu_1(e)} = \frac{\det(\xi_i^{(e)}, \xi_j^{(e)})^2}{\det(\xi_e^\perp)^2}.$$

Thus the conditional pair polynomial is exactly  $\det \mathbf{C}_e(z)$ , and its singleton mean is  $\text{tr } \mathbf{C}_e(z)/2$ . The inequality  $\det \mathbf{C}_e \leq (\text{tr } \mathbf{C}_e)^2/4$  proves the rank-two entropy principle for every such conditional law.

For  $\mu_1(e) > 0$ , let  $\mu_{2|e}$  be the analogous conditional law of the two remaining edges under  $\mu_3$ , and let  $\mu_{1|e}, \nu_{1|e}$  denote the conditional singleton shadows. The following sums run over these edges. Distinguishing a uniform edge of a tree does not change its relative entropy, so the chain rule gives

$$D_3 - D_1 = \sum_e \mu_1(e) D(\mu_{2|e} \| \nu_{2|e}) \geq 2 \sum_e \mu_1(e) D(\mu_{1|e} \| \nu_{1|e}) = 2(D_2 - D_1).$$

For the last equality, selecting one of the two remaining edges produces each ordered pair with probability  $1/6$  within a tree. The same joint law results from first taking its uniform pair shadow and then distinguishing one of that pair's two edges. Combining this inequality with  $D_2 \geq 2D_1$  yields  $2D_3 \geq 3D_2$ .

Since the contraction holds for every positive product reference, we may let the reference depend on  $\mu_3$ . Assign both edges in reference pair  $i$  the positive weight  $k_i$ , and set  $k_{\text{tot}} = \sum_i k_i$ . The reference tree partition is  $Z_k = 2(k_1 + k_2)(k_2 + k_3)(k_3 + k_1)$ . Opposite pair  $i$  has completing weight  $2(k_{\text{tot}} - k_i)$ , while every adjacent pair has completing weight  $k_{\text{tot}}$ . A path has one opposite pair and a star has none. Thus opposite pair  $i$  has mass  $\rho_i$  under  $3\mu_2$ , and the adjacent pairs have total mass  $3 - p$ . Expanding  $2D_3 - 3D_2 \geq 0$  cancels the individual edge logarithms, because deletion retains each edge with probability  $2/3$ . Consequently

$$\mathcal{K}(\mu_3) \geq \log(4Z_k) - \sum_i \rho_i \log[2(k_{\text{tot}} - k_i)] - (3 - p) \log k_{\text{tot}}.$$

The choice  $k_i/k_{\text{tot}} = (1 - p + 2\rho_i)/(3 - p)$  gives  $\mathcal{K}(\mu_3) \geq \Phi(\rho)$ . Boundary choices follow by limits of positive references. Finally, convexity gives  $\Phi \geq (3 - p) \log(4/3) \geq 2 \log(4/3)$ , since  $p \leq 1$ .  $\square$

Lemma 4.1 allows arbitrary laws on trees. A product-weighted law builds every tree probability from the same six edge weights, which forces the constraint  $\eta^2 \geq 4\rho_1\rho_2\rho_3$  between the star probability  $\eta = 1 - p$  and the opposite-pair probabilities. Minimizing  $\Phi$  under this constraint improves the constant  $\log(16/9)$  to  $\frac{9}{4} \log \frac{4}{3}$ .

**Proposition 4.2.** *Assign nonnegative edge weights  $y_e$  to  $K_4$  such that the tree partition function*

$$Z = \sum_T \prod_{e \in T} y_e,$$

*summed over the spanning trees  $T$  of  $K_4$ , is positive, and let  $\mu_3(T) = \prod_{e \in T} y_e/Z$ . For its uniform pair shadow  $\mu_2$ ,*

$$3\mathcal{H}(\mu_2) - 2\mathcal{H}(\mu_3) + 2 \log 2 - 3 \log 3 \geq \frac{9}{4} \log \frac{4}{3}.$$

*Equality holds exactly when the six edge weights are equal up to a common scale.*

*Proof.* Use  $\rho, p, \Phi$  from the preceding lemma and set  $\eta = 1 - p$ , the star probability. The product structure supplies the additional constraint

$$\eta^2 \geq 4\rho_1\rho_2\rho_3.$$

Assume first that all edge weights are positive. Let  $q_i$  be the geometric mean of the weights in opposite pair  $i$ , and let  $c_i \geq 1$  be their arithmetic mean divided by  $q_i$ . Write the total star weight as  $S_{\text{st}} = 4q_1q_2q_3c_{\text{st}}$ . Grouping the stars according to their edge in pair  $i$  and applying AM–GM to each remaining two-term sum gives  $c_{\text{st}} \geq c_i$  for every  $i$ . The total tree weight and its shape probabilities are

$$Z = 2 \sum_{i \neq j} q_i^2 q_j c_j + 4q_1q_2q_3c_{\text{st}}, \quad \rho_i = \frac{2q_i^2(q_j c_j + q_k c_k)}{Z}, \quad \eta = \frac{S_{\text{st}}}{Z}.$$

Consequently

$$c_{\text{st}}^2 Z - 2 \prod_{i < j} (q_i c_i + q_j c_j) = 2 \sum_{i \neq j} q_i^2 q_j c_j (c_{\text{st}}^2 - c_i^2) + 4q_1q_2q_3(c_{\text{st}}^3 - c_1c_2c_3) \geq 0,$$

which is precisely the claimed compatibility after division by  $Z^3/(16q_1^2q_2^2q_3^2)$ . Equality forces  $c_{\text{st}} = c_i$  for every  $i$ . Equality in the two AM–GM comparisons for any fixed pair forces balance in the other two pairs, so all three opposite pairs are internally balanced.

Zero weights follow by continuity.

If  $\eta \geq 1/4$ , convexity gives  $\Phi \geq (2 + \eta) \log(4/3) \geq (9/4) \log(4/3)$ . For  $0 \leq \eta < 1/4$ , minimize  $\Phi$  at fixed  $\eta$  subject to the product constraint. A coordinate-boundary minimum has the other two coordinates equal and satisfies

$$\Phi \geq (2 + \eta) \log 4 + (1 + \eta) \log \frac{1 + \eta}{2} - (2 + \eta) \log(2 + \eta) \geq \log 2 > \frac{9}{4} \log \frac{4}{3}.$$

The middle expression increases with  $\eta$ , since its derivative is  $\log[2(1 + \eta)/(2 + \eta)] \geq 0$ . An interior minimum must saturate the product constraint, because an inactive constraint would force the infeasible all-equal vector. Lagrange multipliers then require every coordinate to solve  $x(\log(1 - x) - \lambda) = \gamma$  for common constants  $\lambda, \gamma$ . This function is strictly concave, so at most two coordinate values occur. After relabeling, write

$$\rho_1 = \rho_2 = \frac{1}{2(1 + z)}, \quad \rho_3 = \frac{z^2}{(1 + z)^2}, \quad \eta = \frac{z}{(1 + z)^2}, \quad z > 0.$$

The resulting value of  $\Phi$  is

$$\varphi(z) = \frac{1 + 2z}{(1 + z)^2} [(z + 1) \log(z + 1) - (z + 2) \log(z + 2) + (z + 3) \log 2].$$

For  $\psi(z) = (1 + z)^3 \varphi'(z)$  one has

$$\begin{aligned} \psi(z) &= (z + 1) \log(z + 1) + (z - 1) \log(z + 2) + (1 - 3z) \log 2, \\ \psi''(z) &= \frac{1}{z + 1} + \frac{1}{z + 2} + \frac{3}{(z + 2)^2} > 0. \end{aligned}$$

Since  $\psi(0) = \psi(1) = 0$ , the unique minimum is  $\varphi(1) = (9/4) \log(4/3)$ . Equality requires  $\eta = 1/4$  and  $\rho_i = 1/4$ , which in turn force all six edge weights to be positive. The compatibility equality forces opposite-pair balance, and  $\rho_i = 1/4$  then forces the three pair weights to coincide. Thus all six original weights are equal, as claimed.  $\square$

The next proposition identifies the transverse entropy with the quantity of Proposition 4.2. Cauchy–Binet assigns every projected basis the product of its generator weights, so the transverse law is product-weighted. The companion pair has Gram determinant one instead of  $1/2$ , and this exceptional factor produces the marked term  $-uvh \log 2$ .

**Proposition 4.3.** *Choose any retained ray  $a$  of weight  $R$  and write  $H = RP_a + G > 0$ . Let  $u, v$  be the other weights in its frame, let  $s$  be the outside total, and set  $h = t + s/4$ . In transverse coordinates given by the two companion rays and the missing frame direction,*

$$B := G|_{a^\perp} = \text{diag}(u, v, 0) + W, \quad W = \sum_{i=1}^4 z_i v_i v_i^\top, \quad z_i \geq 0,$$

where

$$v_1 = \frac{1}{2}(1, 1, 1), \quad v_2 = \frac{1}{2}(1, -1, -1), \quad v_3 = \frac{1}{2}(-1, 1, -1), \quad v_4 = \frac{1}{2}(-1, -1, 1).$$

The diagonal entries of  $W$  equal  $h$ . For  $e \perp a$ , write  $w_e = R\omega_e + \zeta_e$  and set

$$A = \det B \quad \text{and} \quad U_3 = \sum_{e \perp a} \omega_e \text{Ent}_e(B).$$

Then

$$\sum_{e \perp a} \omega_e P_e = \text{adj } B \quad \text{and} \quad \frac{U_3 + (A - uvh) \log 2}{A} \geq \frac{9}{4} \log \frac{4}{3}.$$

The constant is sharp.

*Proof.* Every outside ray projects onto one of the four lines represented by  $v_i$ , so parallel projections combine into nonnegative weights  $z_i$ . The isotropic core adds  $t$  to each  $z_i$ , because  $\sum_i v_i v_i^\top = I_3$ . This also gives their common diagonal  $h = t + s/4$ . The coefficient of  $R$  in the cofactor balance of Lemma 3.2 gives the displayed adjugate resolution. The missing frame normal is the third transverse coordinate, with cofactor  $\omega_e = uv$ . The other six transverse normals are  $(e_i \pm e_j)/\sqrt{2}$ .

The two coordinate generators and  $v_1, v_2, v_3, v_4$  represent the edges 12, 34, 13, 23, 14, 24 of  $K_4$ , respectively. Indeed, their four triangle relations follow from

$$v_1 + v_2 = -v_3 - v_4 = (1, 0, 0) \quad \text{and} \quad v_1 + v_3 = -v_2 - v_4 = (0, 1, 0).$$

Every tree triple has squared determinant  $1/4$ . Every pair has Gram determinant  $1/2$ , except the distinguished opposite pair of coordinate generators, whose Gram determinant is one. Let  $y = (u, v, z_1, z_2, z_3, z_4)$ , set  $Z = \sum_{T \text{ tree}} y_T = 4A$ , and let  $\mu_3(T) = y_T/Z$ .

The pair-cofactor expansion of  $B$  is exactly the resolution  $\omega$ . Both assign weight  $uv$  to the marked normal, and the six old projectors form a basis of the space  $\text{Sym}_3$  of symmetric  $3 \times 3$  matrices, so their adjugate resolutions determine all remaining weights. For a pair  $J$  with perpendicular unit normal  $n_J$ , write  $g_J$  for its Gram determinant,  $k_J = n_J^\top B n_J$ , and  $s_J$  for the sum of its completing edge weights. Then

$$\mu_2(J) = \frac{y_J s_J}{3Z}, \quad k_J = \frac{s_J}{4g_J}, \quad \frac{g_J y_J k_J}{A} = 3\mu_2(J).$$

Tracing the adjugate resolution against  $B \log B$  gives

$$U_3 = A \log A - \sum_J g_J y_J k_J \log k_J.$$

As in the proof of Lemma 4.1, substituting the pair-shadow formula cancels the individual edge logarithms. The distinguished pair has  $3\mu_2(J) = uvh/A$ , so its exceptional Gram factor yields the exact identity

$$\frac{U_3 + (A - uvh) \log 2}{A} = 3H(\mu_2) - 2H(\mu_3) + 2 \log 2 - 3 \log 3.$$

Proposition 4.2 proves the bound, and zero edge weights follow by continuity. Equal positive weights on the six edges attain equality.  $\square$

In the range  $t \geq 2R/3$  of Theorem 5.2 we need a bound on  $U_3$  itself in units of  $uvh$ . It follows from a variance bound and the curvature of  $x \log x$ .

**Proposition 4.4.** *Let  $B = \text{diag}(u, v, 0) + W$ , where  $u, v \geq 0$  and  $W = \sum_{i=1}^4 z_i v_i v_i^\top$  with  $z_i \geq 0$  and the four transverse vectors  $v_i$  above. Write  $h$  for the common diagonal of  $W$ , and use the pair-cofactor weights  $\omega_e$ , including the marked weight  $uv$  on the third coordinate. Then*

$$V_3 := \sum_e \omega_e \text{Var}_e(B) \geq \frac{uvh}{2} \max_i \sum_j |B_{ij}| \quad \text{and} \quad U_3 := \sum_e \omega_e \text{Ent}_e(B) \geq \frac{uvh}{4}.$$

*Both constants are sharp.*

*Proof.* If  $uvh = 0$ , nonnegativity of variance and entropy proves both assertions. Otherwise normalize  $h = 1$  and write

$$W = \begin{pmatrix} 1 & r_{12} & r_{13} \\ r_{12} & 1 & r_{23} \\ r_{13} & r_{23} & 1 \end{pmatrix}, \quad \ell_u = 1 - r_{23}^2, \quad \ell_v = 1 - r_{13}^2,$$

$$f_* = -r_{12}^2 + 3r_{13}^2 + 3r_{23}^2 - 1,$$

$$L_u = 2(r_{12}^2 + r_{13}^2 - 2r_{12}r_{13}r_{23}),$$

$$L_v = 2(r_{12}^2 + r_{23}^2 - 2r_{12}r_{13}r_{23}).$$

Here  $|r_{12}|, |r_{13}|, |r_{23}| \leq 1$  and  $\ell_u, \ell_v, L_u, L_v \geq 0$ . The six old normals are  $(e_i \pm e_j)/\sqrt{2}$ . Their cofactor weights are  $\ell_{ij}/2 \pm \text{adj}(B)_{ij}$ , where

$$\ell_{12} = 1 + r_{12}^2 - r_{13}^2 - r_{23}^2,$$

$$\ell_{13} = 2v + 1 - r_{12}^2 + r_{13}^2 - r_{23}^2,$$

$$\ell_{23} = 2u + 1 - r_{12}^2 - r_{13}^2 + r_{23}^2.$$

Adding their variances and the marked contribution gives

$$2V_3 = uv(u + v) + \ell_u u^2 + \ell_v v^2 + uvf_* + uL_u + vL_v + v_*,$$

$$v_* = \frac{1}{8} \sum_{i < j} z_i z_j (z_k - z_l)^2 \geq 0,$$

where  $\{k, l\}$  complements  $\{i, j\}$ . For the last formula, the normal to  $v_i, v_j$  has cofactor weight  $z_i z_j / 2$  and variance  $(z_k - z_l)^2 / 8$  in  $W$ .

Since  $f_*$  may be negative, we absorb it into positive quadratic terms by AM–GM, row by row. The scalar inequality below handles the first row. For  $0 \leq r_{12}, r_{13} \leq 1$  and  $|r_{23}| \leq 1$ , set  $\alpha = \sqrt{1 - r_{13}^2}$  and  $\beta = \sqrt{1 - r_{23}^2}$ . Cauchy–Schwarz gives  $2\sqrt{L_u} \geq 2(r_{12} - r_{13}r_{23} + r_{13}\beta)$ . Since  $r_{12} - r_{13}^2 \geq 0$ ,  $\alpha \geq 1 - r_{13}^2$ , and  $\beta \geq 1 - r_{23}^2$ , it follows that

$$f_* - 1 - r_{12} - r_{13} + 2\sqrt{\ell_u \ell_v} + 2\sqrt{L_u} \geq \mathbf{p} := r_{13}^2 + r_{13} - 2r_{13}r_{23} + (1 - 2r_{13} + 2r_{13}^2)r_{23}^2 \geq 0.$$

The last inequality follows from the identity

$$\chi \mathbf{p} = (\chi r_{23} - r_{13})^2 + r_{13}[(1 - r_{13})^2(1 - r_{13}^2) + r_{13}^4], \quad \chi = 1 - 2r_{13} + 2r_{13}^2 > 0.$$

Coordinate sign changes permit  $r_{12}, r_{13} \geq 0$  without changing  $r_{12}r_{13}r_{23}$  or the variance expansion. AM–GM gives  $\ell_u u^2 + \ell_v v^2 \geq 2uv\sqrt{\ell_u \ell_v}$  and  $uv^2 + uL_u \geq 2uv\sqrt{L_u}$ . The scalar inequality therefore yields

$$2V_3 \geq uv(1 + u + |r_{12}| + |r_{13}|).$$

Interchanging the first two coordinates proves the second-row bound. For the third row, additionally use  $u^2v + vL_v \geq 2uv\sqrt{L_v}$ . Its scalar remainder exceeds the first-row remainder by  $|r_{12}| - |r_{23}| + 2\sqrt{L_v} \geq 0$ , because  $L_v \geq 2(|r_{12}| - |r_{23}|)^2$ . Thus  $2V_3 \geq uv(1 + |r_{13}| + |r_{23}|)$ , proving the rowwise estimate. Scaling restores  $h$ .

The largest eigenvalue of  $B$  is at most its largest absolute row sum. For  $B > 0$ , the curvature bound  $(x \log x)'' \geq 1/\lambda_{\max}(B)$  on its spectrum gives  $U_3 \geq V_3/(2\lambda_{\max}(B)) \geq uvh/4$ . Singular matrices follow by continuity. Finally, when  $W = I$  and  $u = v = \varepsilon \downarrow 0$ ,

$$2V_3 = \varepsilon^2 + 2\varepsilon^3 \quad \text{and} \quad U_3 = (1 + 2\varepsilon)[(1 + \varepsilon) \log(1 + \varepsilon) - (2 + \varepsilon) \log(1 + \varepsilon/2)] = \frac{\varepsilon^2}{4} + O(\varepsilon^3).$$

This proves sharpness of both constants.  $\square$

## 5. ENTROPY DESCENT AND COMPLETION

We prove Theorem 5.2 by showing that  $\mathfrak{D}F$  increases in every largest retained weight and integrating along the capped path  $x_i(r) = \min\{x_i, r\}$ . The derivative consists of the transverse entropy of Section 4 and a logarithmic correction, a cofactor-weighted mean over the normals not perpendicular to the chosen ray. The following lemma, proved in Section 6, bounds this correction. Its proof uses the maximality of  $R$ .

**Lemma 5.1.** *Suppose  $H = RP_a + G > 0$ , where  $a = a_i$  and  $R = x_i > 0$  is a largest retained weight, that is,  $x_k \leq R$  for  $k = 1, \dots, 9$ , and  $u, v$  are the other two retained weights in the frame of  $a$ . Set  $B = G|_{a^\perp}$ ,  $A = \det B$ , and  $h = t + s/4$ , where  $s$  is the total outside that frame. For  $e \perp a$ , write  $w_e = R\omega_e + \zeta_e$  and define*

$$\mathcal{Q} = RA - 2 \det G + \sum_{e \perp a} \zeta_e h_e.$$

*In both nonprincipal configurations,*

$$\begin{aligned} 9\mathcal{Q} &\geq \mathfrak{D} && \text{if } 0 \leq t \leq 2R/3, \\ \mathcal{Q} &\geq uvh(R + h) + \frac{5}{324}R^4 && \text{if } t \geq 2R/3. \end{aligned}$$

**Theorem 5.2.** *In either nonprincipal type, every positive-definite form satisfies  $F \geq 0$ , with equality exactly when all retained weights vanish. Consequently  $\mathcal{I}(LV_H) \geq 2^{11/4}$  for every such form and every  $|\det L| = 1$ , strictly whenever some retained weight is positive.*

*Proof.* Set  $U = \mathfrak{D}E$  and  $\mathcal{L} = U - N_b \log 2 = \mathfrak{D}F$ . Assume some retained weight is positive and choose a largest retained weight  $R = x_i > 0$ , with ray  $a = a_i$ . Let  $u, v$  be its companion weights and  $s$  the outside total. Write  $H = RP_a + G$  and, in the splitting  $\text{span}(a) \oplus a^\perp$ , write

$$G = \begin{pmatrix} h & g^\top \\ g & B \end{pmatrix}, \quad A = \det B, \quad \delta = h - g^\top B^{-1}g.$$

Thus  $\mathfrak{D} = A(R + \delta)$  and  $0 \leq \delta \leq h = t + s/4$ . Moreover,  $\delta = \min_{y \perp a} (a + y)^\top G(a + y)$ . For a nonorthogonal normal set  $\alpha_e = (e \cdot a)^2$  and  $\beta_e = e^\top G e / \alpha_e$ . Its cofactor is independent of  $R$ , and  $h_e = \alpha_e(R + \beta_e)$ . Schur completion gives  $\beta_e \geq \delta$ . Cofactor balance makes  $\varpi_e = w_e \alpha_e / A$  a probability law on those normals. For orthogonal normals write  $w_e = R\omega_e + \zeta_e$ . The transverse resolution is  $\sum_{e \perp a} \omega_e P_e = \text{adj } B$ . Indeed, differentiating the cofactor expansion selects its terms containing the chosen ray. Their normals are perpendicular to  $a$ , and deleting  $a$  leaves the pair cofactors of the projected generators.

We now differentiate in  $R$ , with the other eight retained weights and  $t$  fixed. For nonorthogonal normals,  $w'_e = 0$  and  $h'_e = \alpha_e$ , while for orthogonal normals,  $w'_e = \omega_e$  and  $h'_e = 0$ . Thus differentiation of  $U = \mathfrak{D} \log \mathfrak{D} - \sum_e w_e h_e \log h_e$  gives

$$\begin{aligned} U' &= A(\log \mathfrak{D} + 1) - \sum_{e \perp a} \omega_e h_e \log h_e - \sum_{e \not\perp a} w_e \alpha_e (\log h_e + 1) \\ &= A \log \mathfrak{D} - \sum_{e \perp a} \omega_e h_e \log h_e - A \sum_{e \not\perp a} \varpi_e \log h_e. \end{aligned}$$

The constant terms cancel because  $\sum_{e \not\perp a} w_e \alpha_e = A$ . For  $e \perp a$  we have  $h_e = e^\top B e$ , so the transverse cofactor resolution also gives

$$\sum_{e \perp a} w_e h_e = 3A \quad \text{and} \quad U_3 := \sum_{e \perp a} w_e \text{Ent}_e(B) = A \log A - \sum_{e \perp a} w_e h_e \log h_e.$$

Indeed, the spectral part of  $U_3$  is  $\text{tr}(\text{adj}(B)B \log B) = A \log A$ . Using  $\mathfrak{D} = A(R + \delta)$  and  $h_e = \alpha_e(R + \beta_e)$  on the nonorthogonal normals now yields

$$U' = U_3 - A \sum_{e \not\perp a} \varpi_e \log \frac{R + \beta_e}{R + \delta} - \sum_{e \not\perp a} w_e \alpha_e \log \alpha_e.$$

Relabel the chosen frame as the first. The old nonorthogonal normals have  $\alpha_e = 1/2$ , and the two outside new normals have  $\alpha_e = 1/4$  and weights  $p_2, p_3$ . Consequently

$$-\sum_{e \not\perp a} w_e \alpha_e \log \alpha_e = \left( A + \frac{p_2 + p_3}{4} \right) \log 2 \quad \text{and} \quad N'_b = uvh + \frac{p_2 + p_3}{4}.$$

The first term in  $N'_b$  comes from differentiating the chosen frame product  $Ruv$ , with  $h$  the value of  $H$  on its missing unit normal. The other two products are fixed, but their corresponding factors  $h_{m_j}$  have derivative  $1/4$ . Subtracting  $N'_b \log 2$  therefore cancels the outside marked contributions exactly and gives

$$\mathcal{L}'(R) = U_3 + (A - uvh) \log 2 - A \sum_{e \not\perp a} \varpi_e \log \frac{R + \beta_e}{R + \delta}.$$

To control the remaining logarithmic mean, split the identity  $\sum_e w_e h_e = 4\mathfrak{D}$  into its orthogonal and nonorthogonal parts to obtain

$$3RA + \sum_{e \perp a} \zeta_e h_e + A \sum_{e \not\perp a} \varpi_e (R + \beta_e) = 4A(R + \delta).$$

It follows that its mean excess and the complementary polynomial are

$$\begin{aligned} \Gamma &:= A \sum_{e \not\perp a} \varpi_e (\beta_e - \delta) = 3A\delta - \sum_{e \perp a} \zeta_e h_e, \\ \mathcal{Q} &= A(R + \delta) - \Gamma = RA - 2 \det G + \sum_{e \perp a} \zeta_e h_e. \end{aligned}$$

Write  $y = \mathcal{Q}/[A(R + \delta)]$ . Schur completion gives  $\Gamma \geq 0$ , and Lemma 5.1 gives  $\mathcal{Q} \geq 0$ , so  $0 \leq y \leq 1$ . The arithmetic mean of the ratios in the logarithmic correction is exactly

$$\sum_e \varpi_e \frac{R + \beta_e}{R + \delta} = 1 + \frac{\Gamma}{\mathfrak{D}} = 2 - \frac{\mathcal{Q}}{\mathfrak{D}} = 2 - y.$$

Concavity of the logarithm gives

$$\sum_e \varpi_e \log \frac{R + \beta_e}{R + \delta} \leq \log(2 - y).$$

If  $t \leq 2R/3$ , Lemma 5.1 gives  $y \geq 1/9$ . Proposition 4.3 therefore implies

$$\frac{\mathcal{L}'(R)}{A} \geq \frac{9}{4} \log \frac{4}{3} - \log \frac{17}{9} = \frac{1}{4} \log \frac{262144}{250563} > 0.$$

If  $t \geq 2R/3$ , use  $\log(2 - y) \leq \log 2 - y/2$  and Proposition 4.4 to obtain

$$\mathcal{L}'(R) \geq U_3 - uvh \log 2 + \frac{\mathcal{Q}}{2(R + \delta)} \geq uvh \left( \frac{3}{4} - \log 2 \right) + \frac{5R^4}{648(R + \delta)} > 0,$$

where the two contributions are  $U_3 \geq uvh/4$  and  $\mathcal{Q}/[2(R + \delta)] \geq uvh/2 + 5R^4/[648(R + \delta)]$ , using  $\delta \leq h$ . The positive remainder also gives strictness when either companion weight vanishes. Thus  $\partial_{x_i} \mathcal{L} > 0$  whenever  $x_i$  is a largest retained weight.

Hold  $t$  fixed and put  $R_* = \max_i x_i$  and  $x_i(r) = \min\{x_i, r\}$  for  $0 < r \leq R_*$ . On every open interval avoiding the finitely many levels  $r = x_i$ , its active coordinates are those with  $x_i > r$ . All active coordinates equal the current maximum  $r$ , including ties, so

$$\frac{d}{dr}\mathcal{L}(t, x(r)) = \sum_{i: x_i > r} \partial_{x_i}\mathcal{L}(t, x(r)) > 0, \quad 0 < r < R_*.$$

For  $t > 0$ , the path extends to the regular core at  $r = 0$ , where  $\mathcal{L} = 0$ . For  $t = 0$ , every positive level retains the original spanning support, so  $H$  remains positive definite. Set  $b_i = \mathbf{1}_{x_i > 0}$  and  $r_* = \min\{x_i : x_i > 0\}$ . Then  $x(r) = rb$  for  $0 < r < r_*$ , and  $H(0, b) > 0$ . The identity  $\sum_e w_e h_e = 4\mathfrak{D}$  cancels the scaling logarithms and gives

$$\mathcal{L}(0, x(r)) = r^4 \mathcal{L}(0, b) \longrightarrow 0, \quad \frac{d}{dr}\mathcal{L}(0, x(r)) = 4r^3 \mathcal{L}(0, b).$$

The derivative is integrable at  $r = 0$ . The path is continuous at each positive break point and smooth between them. Integrating the positive derivative from the zero limit therefore gives  $\mathcal{L} > 0$  whenever  $R_* > 0$ . At the regular core  $F = 0$ , and Lemma 3.4 gives the affine comparison.  $\square$

*Proof of Theorem 1.1.* By Theorem 2.2 and dilation invariance, it suffices to treat volume-preserving affine images of Voronoi cells in the three closed primitive cones. Theorem 2.6 gives a strict bound in the principal cone. For the nonprincipal cones, Theorem 5.2 and Lemma 3.4 give a strict bound when  $K \neq 0$ . At  $K = 0$ , Lemmas 2.3 and 3.3 identify equality precisely with orthogonal images of the regular cell. Restoring translations and dilations gives exactly its similarities.  $\square$

*Proof of Corollary 1.3.* Pass to the closure, choose  $B(x_0, r) \subset P$ , and set  $S = \mathcal{H}^3(\partial P)$ ,  $V = \text{vol}_4(P)$ . Every facet has support distance  $h_F \geq r$  from  $x_0$ . The pyramid formula therefore gives

$$4V = \sum_F h_F \mathcal{H}^3(F) \geq rS,$$

and Theorem 1.1 gives

$$S^4 \geq 2^{11} V^3 \geq 32r^3 S^3.$$

Divide by  $S^3 > 0$ . Equality forces a regular 24-cell and  $h_F = r$  for every facet, hence inradius  $r$ . Conversely, Lemma 2.3 and scaling give its area  $32r^3$ .  $\square$

**Remark 5.3.** For a convex lattice tile containing a ball of radius  $r$ , Corollary 1.3 and the pyramid formula give  $\text{vol}_4(P) \geq 8r^4$ . For lattice Voronoi cells this recovers the known optimal lattice packing density  $\pi^2/16$ , equivalently the Hermite bound  $\gamma_4 = \sqrt{2}$  [10, Chap. 1]. The 24-cell volume conjecture [25] asks for the same cellwise volume bound for Voronoi cells of arbitrary packings of balls of radius  $r$ . Since such cells need not be parallelohedra, Theorem 1.1 does not settle that conjecture.

## 6. THE MAXIMAL-WEIGHT MEAN BOUNDS

We prove the two polynomial estimates used in Lemma 5.1. The large-core estimate follows by grouping the coefficients of  $\mathcal{Q}$  into nonnegative expressions. For the small-core estimate, we first count bases at  $t = 0$  and then bound the coefficients of the positive powers of  $t$ . Throughout,  $R$  is a largest retained weight, so the other eight retained weights lie in  $[0, R]$ .

For a retained index set  $J$ , put

$$g_J = \det((a_i \cdot a_k)_{i,k \in J}), \quad g_\emptyset = 1.$$

Cauchy–Binet gives the coefficient formula

$$(6.1) \quad \det \left( tI + \sum_i x_i P_{a_i} \right) = \sum_{\substack{J \subset \{1, \dots, 9\} \\ |J| \leq 4}} t^{4-|J|} g_J x_J.$$

For  $A = \det(G|_{a^\perp})$ , the analogous formula uses subsets avoiding the selected ray and the projected Gram entries  $a_i \cdot a_k - (a_i \cdot a)(a_k \cdot a)$ , with  $t^{3-|J|}$  in place of  $t^{4-|J|}$ . The remaining term in  $\mathcal{Q}$  is obtained by setting the selected weight to zero in the facet formula for  $w_e$  and multiplying by  $h_e$ , for  $e \perp a$ . All coefficient identities below are polynomial in the ray coordinates and can be verified by direct expansion.

These formulas also show that  $\mathcal{Q}$  and  $9\mathcal{Q} - \mathfrak{D}$  are affine separately in every retained weight. For the facet terms, a weight occurring in  $w_e$  belongs to a ray perpendicular to  $e$  and therefore does not occur in  $h_e$ . We repeatedly use the following interpolation identity. If  $f$  is affine separately in  $m$  variables and  $0 \leq \xi_i \leq 1$ , then

$$(6.2) \quad f(\xi) = \sum_{\varepsilon \in \{0,1\}^m} f(\varepsilon) \prod_{i:\varepsilon_i=1} \xi_i \prod_{i:\varepsilon_i=0} (1 - \xi_i).$$

In particular, a lower bound at the vertices of the cube holds throughout the cube. All expressions being compared are homogeneous of degree four in  $(t, x)$ . When  $R > 0$ , division by  $R$  therefore reduces the cube  $[0, R]^8$  to  $[0, 1]^8$ . These polynomial inequalities hold also where  $H$  is singular. Positive definiteness is used only in Section 5.

### 6.1. Positive expansions and the large-core bound.

**Lemma 6.1.** *Choose a retained weight  $R = x_i$ , with ray  $a = a_i$ , such that  $x_k \leq R$  for every retained weight  $x_k$ . Write  $u, v$  for the other two retained weights in its frame, and let  $s$  be the sum of the six retained weights outside that frame. Set*

$$H = RP_a + G, \quad B = G|_{a^\perp}, \quad A = \det B, \quad h = t + s/4.$$

For  $e \perp a$ , write  $w_e = R\omega_e + \zeta_e$  and put

$$\mathcal{Q} = RA - 2 \det G + \sum_{e \perp a} \zeta_e h_e.$$

Then  $\mathcal{Q} \geq uvh(R + t)$ . Moreover, if  $t \geq 2R/3$  and  $R > 0$ , then

$$\mathcal{Q} \geq uvh(R + h) + \frac{5}{324} R^4.$$

*Proof.* Use the coefficient convention above and write

$$\mathcal{Q} = q_0 + tq_1 + t^2q_2 + t^3q_3.$$

The  $t^4$  coefficient cancels because  $RA$  has degree at most three in  $t$ , while  $-2 \det G$  contributes  $-2t^4$  and the six old normals perpendicular to  $a$  contribute  $2t^4$ . For each ray orbit,

$$q_3 = R - \frac{u+v}{3} \quad \text{and} \quad q_2 = R(u+v) - uv + \frac{3R - u - v}{4} s.$$

We record the lower coefficients as sums of expressions that are nonnegative under the maximality assumption. The symmetry orbits in Lemma 3.1 reduce the calculation to three representatives. We use the same outside assignment  $(y_1, \dots, y_6) = (x_4, \dots, x_9)$  in every case listed below.

| case | selected weight $R$ | companions $(u, v)$ | selected-ray orbit     |
|------|---------------------|---------------------|------------------------|
| III  | $x_1$               | $(x_2, x_3)$        | all nine rays          |
| II,1 | $x_1$               | $(x_2, x_3)$        | $\{1, 2, 4, 5, 7, 8\}$ |
| II,3 | $x_3$               | $(x_1, x_2)$        | $\{3, 6, 9\}$          |

Put  $s = \sum_{i=1}^6 y_i$  and define

$$c_R = R(u+v) - 2uv \quad \text{and} \quad \Pi(y, z; w, k) = yz\{(R-w)(u+v) + R(w+k) - 2wk\}.$$

Both quantities are nonnegative under maximality. Indeed,  $c_R = u(R-v) + v(R-u)$ , and the braces in  $\Pi$  equal  $(R-w)(u+v) + w(R-k) + k(R-w)$ . The following identities give the complete coefficients  $q_0, q_1$  for each representative. They follow from (6.1) and the facet formula, grouped so that every term is visibly nonnegative under maximality. For case III,

$$\begin{aligned} 4q_0 &= \Pi(y_2, y_3; y_1, y_4) + \Pi(y_5, y_6; y_4, y_1) \\ &\quad + (R-u)\{y_1y_2(y_4+y_5) + y_1y_4y_5 + y_2y_4y_5 + v(y_1y_2 + y_4y_5)\} \\ &\quad + (R-v)\{y_1y_3(y_4+y_6) + y_1y_4y_6 + y_3y_4y_6 + u(y_1y_3 + y_4y_6)\} \\ &\quad + c_R(y_1y_4 + y_2y_5 + y_3y_6) + Ru(y_1y_5 + y_2y_4) + Rv(y_1y_6 + y_3y_4) + Ruvs, \\ 2q_1 &= 2Ruv + \{R(u+v) - uv\}s + y_2y_3(R-y_1) + y_5y_6(R-y_4) \\ &\quad + (R-u)(y_1y_2 + y_4y_5) + (R-v)(y_1y_3 + y_4y_6) \\ &\quad + \frac{1}{2}\{(2R-u-v)(y_1y_4 + y_2y_5 + y_3y_6) \\ &\quad + (2R-u)(y_1y_5 + y_2y_4) + (2R-v)(y_1y_6 + y_3y_4)\}. \end{aligned}$$

For case II,1,

$$\begin{aligned} 4q_0 &= \Pi(y_2, y_3; y_1, y_4) + \Pi(y_4, y_6; y_5, y_2) \\ &\quad + (R-u)\{y_1y_2(y_4+y_5) + y_1y_4y_5 + y_2y_4y_5 + v(y_1y_2 + y_4y_5)\} \\ &\quad + (R-v)\{y_1y_3y_4 + y_2y_5y_6 + u(y_1y_3 + y_5y_6)\} \\ &\quad + c_R(y_1y_4 + y_2y_5) + Ru(y_1y_5 + y_2y_4 + y_3y_6) + Rv(y_2y_6 + y_3y_4) \\ &\quad + Ry_3y_6(y_2 + y_4) + Ruvs, \\ 2q_1 &= 2Ruv + \{R(u+v) - uv\}s + y_2y_3(R-y_1) + y_4y_6(R-y_5) \\ &\quad + (R-u)(y_1y_2 + y_4y_5) + (R-v)(y_1y_3 + y_5y_6) \\ &\quad + \frac{1}{2}\{(2R-u-v)(y_1y_4 + y_2y_5) \\ &\quad + (2R-u)(y_1y_5 + y_2y_4 + y_3y_6) + (2R-v)(y_2y_6 + y_3y_4)\}. \end{aligned}$$

For case II,3,

$$\begin{aligned} 4q_0 &= \Pi(y_1, y_2; y_3, 0) + \Pi(y_4, y_5; y_6, 0) \\ &\quad + (R-u)\{y_1y_3y_4 + y_2y_5y_6 + v(y_1y_3 + y_5y_6)\} \\ &\quad + (R-v)\{y_2y_3y_5 + y_1y_4y_6 + u(y_2y_3 + y_4y_6)\} \\ &\quad + c_R(y_1y_4 + y_2y_5) + Ru(y_2y_6 + y_3y_4) \\ &\quad + Rv(y_1y_6 + y_3y_5) + R(y_1y_2y_6 + y_3y_4y_5) + Ruvs, \\ 2q_1 &= 2Ruv + \{R(u+v) - uv\}s + y_1y_2(R-y_3) + y_4y_5(R-y_6) \\ &\quad + (R-u)(y_1y_3 + y_5y_6) + (R-v)(y_2y_3 + y_4y_6) \\ &\quad + \frac{1}{2}\{(2R-u-v)(y_1y_4 + y_2y_5) \\ &\quad + (2R-v)(y_1y_6 + y_3y_5) + (2R-u)(y_2y_6 + y_3y_4)\}. \end{aligned}$$

In all three cases, discarding the nonnegative terms gives

$$q_0 \geq \frac{Ruvs}{4}, \quad q_1 \geq Ru + \frac{uvs}{2}, \quad q_2 \geq uv, \quad q_3 \geq 0.$$

For  $q_1$  and  $q_2$ , use  $R(u+v) \geq 2uv$ . Comparing coefficients with  $uv(t+s/4)(R+t) = Ruvs/4 + t(Ruv + uvs/4) + t^2uv$  proves  $Q \geq uv(t+s/4)(R+t)$ .

For the stronger assertion, set  $\mathcal{T} = Q - uvh(R+h)$ . Its coefficients of  $t, t^2, t^3$  are nonnegative. Indeed, the displayed formula for  $q_1$  bounds the first by  $[R(u+v) - 2uv]s/2$ ,

while the other two are

$$R(u+v) - 2uv + (3R - u - v)s/4 \quad \text{and} \quad R - (u+v)/3.$$

The defect  $\mathcal{T}$  is therefore nondecreasing in  $t$  for fixed retained weights. By this monotonicity it suffices to take  $t = 2R/3$ , the endpoint of the small-core range. For each fixed choice of outside weights,  $\mathcal{T}$  is affine separately in  $u, v$ , so (6.2) reduces its minimum to  $u, v \in \{0, R\}$ . If  $uv = 0$ , then  $\mathcal{T} = \mathcal{Q} \geq t^3 q_3 \geq 16R^4/81$ . It remains to take  $u = v = R$ . In each outside coordinate,  $\mathcal{T}$  has second derivative  $-uv/8 \leq 0$ , so it is separately concave and attains its minimum with every outside weight in  $\{0, R\}$ . Normalize  $R = 1$ , write  $y_i \in \{0, 1\}$  for the outside weights, and set  $n = y_1 + y_2 + y_3 + y_4 + y_5 + y_6$ . Substitution in the displayed formulas gives

$$\mathcal{T} = \frac{8}{81} + \frac{n}{9} - \frac{n^2}{16} + J,$$

where

$$\begin{aligned} 12J_{\text{III}} &= 5\{y_1(y_5 + y_6) + y_4(y_2 + y_3)\} \\ &\quad + y_2y_3(10 - 7y_1 + 3y_4 - 6y_1y_4) + y_5y_6(10 - 7y_4 + 3y_1 - 6y_1y_4), \\ 12J_{\text{II},1} &= 5\{y_1y_5 + (y_2 + y_3)(y_4 + y_6)\} + 3y_3y_6(y_2 + y_4) \\ &\quad + y_2y_3(10 - 7y_1 + 3y_4 - 6y_1y_4) + y_4y_6(10 - 7y_5 + 3y_2 - 6y_5y_2), \\ J_{\text{II},3}(y_1, y_2, y_3, y_4, y_5, y_6) &= J_{\text{III}}(y_3, y_1, y_2, y_6, y_4, y_5) + \frac{1}{2}y_3y_6(y_1y_2 + y_4y_5). \end{aligned}$$

All terms are nonnegative on the binary cube because each coefficient of the form  $10 - 7y_i + 3y_j - 6y_iy_j$  has one of the four values 10, 3, 13, 0. In each of the three cases, we next prove that

$$12J \geq 5n - 12.$$

For type III set  $L = y_2 + y_3$ ,  $M = y_5 + y_6$ ,  $p_L = y_2y_3$ , and  $p_M = y_5y_6$ . If  $y_1 = y_4 = 0$  and  $n \geq 3$ , then  $p_L + p_M \geq n - 2$  gives  $12J = 10(p_L + p_M) \geq 5n - 12$ . For  $n \leq 2$ , nonnegativity suffices. If  $y_1 = y_4 = 1$ , then  $12J = 5n - 10$ . In the remaining case take  $y_1 = 1, y_4 = 0$  by symmetry. Then  $12J = 5M + 3p_L + 13p_M \geq 5M + 3(L - 1) \geq 5n - 12$ , since  $L \leq 2$  and  $M \geq n - 3$ . The second type II orbit follows from its displayed extra positive term. For the first type II orbit set  $L = y_2 + y_3$  and  $M = y_4 + y_6$ . If both are positive, then  $LM \geq L + M - 1$  and  $y_1 + y_5 - y_1y_5 \leq 1$  give  $12J \geq 5(y_1y_5 + LM) \geq 5(n - 2)$ . If one pair is empty, then  $n \leq 4$ . For  $n = 3$ , either both singled weights contribute 5 through  $y_1y_5$ , or the complete occupied pair contributes at least 3. For  $n = 4$ , both contributions occur and give  $8 = 5n - 12$ . Again  $n \leq 2$  needs only nonnegativity.

For  $0 \leq n \leq 2$  the bound on  $\mathcal{T}$  is at least  $23/324$ . For  $3 \leq n \leq 6$  it is at least

$$\frac{8}{81} - 1 + \frac{19n}{36} - \frac{n^2}{16} \geq \frac{5}{324}.$$

Both scalar quadratics are concave, so their interval endpoints establish these bounds. Restoring  $R$  proves the lemma.  $\square$

## 6.2. The zero-core count.

**Lemma 6.2.** *In either nonprincipal type, suppose the chosen ray  $a = a_i$  has weight  $R = x_i$  and every other retained weight lies in  $[0, R]$ . At zero core,  $9\mathcal{Q} \geq \mathfrak{D}$ .*

*Proof.* The assertion is immediate when  $R = 0$ , so normalize  $R = 1$ . By (6.2), it suffices to consider binary weights. Call a ray *active* when its weight is one. Let  $N$  count the active bases containing the chosen ray. Equivalently, it counts the active triples completed to a basis by that ray. Let  $M$  count the active bases omitting the chosen ray, and, for each such basis  $\mathcal{B}$ , put

$$f_{\mathcal{B}} = \#\{b \in \mathcal{B} : (\mathcal{B} \setminus \{b\}) \cup \{a\} \text{ is a basis}\}.$$

Each basis has squared determinant  $1/4$ . Its deletion normal is perpendicular to  $a$  exactly when the deleted element cannot be exchanged for  $a$ . Grouping the zero-core cofactor terms by their completed four-ray basis therefore gives

$$RA = \frac{N}{4}, \quad \det G = \frac{M}{4}, \quad \sum_{e \perp a} \zeta_e h_e = \frac{1}{4} \sum_{\mathcal{B}} (4 - f_{\mathcal{B}}), \quad \mathfrak{D} = \frac{N + M}{4}.$$

Here and below the sums over  $\mathcal{B}$  range only over active bases omitting  $a$ . Combining the four identities yields the rooted exchange formula

$$(6.3) \quad \mathcal{Z} := 4(9\mathcal{Q} - \mathfrak{D}) = 8N - \sum_{\mathcal{B}} (9f_{\mathcal{B}} - 17).$$

In both configurations  $f_{\mathcal{B}} \in \{2, 3, 4\}$ . Since every basis has the same determinant,  $\mathcal{Z}$  is a pure count.

For type III write the weights as  $(1, u, v, y_1, y_2, y_3, y_4, y_5, y_6)$ . We call  $u, y_1, v, y_4$  the *anchors* and  $y_2, y_3, y_5, y_6$  the *leaves*. The exchange polynomial is invariant under  $(u \ y_1 \ v \ y_4)(y_2 \ y_3 \ y_5 \ y_6)$  and  $(u \ v)(y_2 \ y_3)(y_5 \ y_6)$ . The reflection interchanges the second and third ambient coordinates. The quarter-turn is induced on the ray lines by the invertible map

$$\begin{pmatrix} 1 & -1/2 & -1/2 & 0 \\ 0 & -1/2 & -1/2 & -1 \\ 0 & -1/2 & -1/2 & 1 \\ 0 & 1/2 & -1/2 & 0 \end{pmatrix},$$

which fixes the chosen ray. Both maps therefore preserve the rooted basis-exchange count. Thus the occupied anchors  $u, y_1, v, y_4$  form an empty set, a singleton, an opposite pair, an adjacent pair, a triple, or the full cycle. For the four leaf bits put

$$k = y_2 + y_3 + y_5 + y_6, \quad \tau = y_2 y_3 y_5 + y_2 y_3 y_6 + y_2 y_5 y_6 + y_3 y_5 y_6, \\ M_1 = y_2 y_3 + y_5 y_6, \quad M_2 = y_2 y_5 + y_3 y_6, \quad M_3 = y_2 y_6 + y_3 y_5, \quad C = M_1 + M_2.$$

Here  $\tau = \binom{k}{3}$ , each  $M_i \leq k/2$ , and  $C \leq k$ . Also  $\tau \leq C$ , because each occupied triple contains two cycle edges and each cycle edge lies in at most two such triples. To evaluate (6.3), enumerate the active four-subsets and, for each subset omitting  $a$ , the exchanges that remain independent. The six anchor patterns listed above give the following formulas in the four leaf bits. For no anchor,  $\mathcal{Z} = 0$ . For one anchor, represented by  $u = 1$ ,  $\mathcal{Z} = 8C - \tau \geq 7C$ . For opposite anchors, represented by  $u = v = 1$ , it gives

$$\mathcal{Z} = 8k + 16M_1 - 3M_2 - M_3 - 2\tau \geq 4k.$$

For adjacent anchors, represented by  $v = y_4 = 1$ ,

$$\mathcal{Z} = 8(y_3 + y_5) + (16y_2 + 6y_6)(y_3 + y_5) - y_3 y_5 - 2\tau \geq \frac{7}{2}(y_3 + y_5).$$

Indeed,  $y_3 y_5 \leq (y_3 + y_5)/2$  and  $\tau = y_3 y_5 (y_2 + y_6) + y_2 y_6 (y_3 + y_5) \leq 2(y_3 + y_5)$ . For three anchors, represented by  $u = v = y_1 = 1$ , set  $L = y_2 + y_3$ ,  $M = y_5 + y_6$ ,  $p_L = y_2 y_3$ , and  $p_M = y_5 y_6$ . Then

$$\mathcal{Z} = 8 + 6L + 16M + 4p_L + 24p_M - 5M_2 - 2M_3 - 3(p_L M + p_M L) \\ \geq 8 + 5L + (16 - 5L)M \geq 8.$$

Here  $M_2 + M_3 = LM$ ,  $p_L \leq L/2$ , and  $L, M \leq 2$ . For all four anchors,

$$\mathcal{Z} = 13 + 12k - 4\binom{k}{2} - 3C - 4\binom{k}{3} \geq 13 + \frac{29}{3}k - \frac{2}{3}k^3 \geq 9.$$

The final cubic is concave on  $[0, 4]$  and has endpoint values 13 and 9. Each displayed formula follows by counting bases and exchanges in the formula for  $\mathcal{Z}$ , or by substituting the anchor values into the type III cofactor expansion. This proves the type III count.

For type II, we prove the slightly stronger count for every subgraph of  $K_5$  containing a distinguished edge  $v_0v_4$ . Let  $N_\star$  count spanning trees containing  $v_0v_4$ , and let  $N_j$  count spanning trees omitting it whose  $v_0-v_4$  path has length  $j$ . The exchange count  $f_B$  is exactly this path length. The required inequality is

$$8N_\star \geq N_2 + 10N_3 + 19N_4.$$

Label the remaining vertices 1, 2, 3, and write  $\alpha_i, \beta_i \in \{0, 1\}$  for their attachments to  $v_0, v_4$ . Set  $c_i = \alpha_i + \beta_i$ . For an internal forest  $J$  on these three vertices and a component  $C$ , set  $c_C = \sum_{i \in C} c_i$ . Contracting  $v_0v_4$  shows that  $J$  contributes  $N_J = \prod_C c_C$  to  $N_\star$ . For a tree omitting  $v_0v_4$ , deletion of the two endpoints leaves an internal forest  $J$ . Exactly one component  $C$  is attached once to each endpoint, and every other component is attached once to one endpoint. If the two attachments to  $C$  are at  $i, j$ , the endpoint path has length  $2 + \text{dist}_J(i, j)$ , so its coefficient  $9f_B - 17$  equals  $1 + 9 \text{dist}_J(i, j)$ . Consequently the contribution of this forest to  $N_2 + 10N_3 + 19N_4$  is

$$\sum_C \left[ \sum_{i, j \in C} \alpha_i \beta_j \{1 + 9 \text{dist}_J(i, j)\} \right] \prod_{D \neq C} c_D.$$

A singleton contributes at most  $c_i/2$ . For a two-vertex component, its numerator is

$$\alpha_i \beta_i + \alpha_j \beta_j + 10(\alpha_i \beta_j + \alpha_j \beta_i) \leq \frac{11}{2}(c_i + c_j).$$

Both inequalities count disjoint pairs of occupied attachment bits. Thus the empty internal forest has cost at most  $3N_J/2$ , and a one-edge forest has cost at most  $6N_J$ . This proves the graph inequality when the internal graph has at most one edge.

If the internal graph is the path 1 – 2 – 3, retain the attachment notation  $\alpha_i, \beta_i$ . The central attachment patterns  $(\alpha_2, \beta_2) = (0, 0)$  and  $(1, 0)$  give, after relabeling the four leaves, exactly the opposite-anchor and three-anchor polynomials already bounded above. The respective substitutions for the type III leaf tuple  $(y_2, y_3, y_5, y_6)$  are  $(\alpha_1, \alpha_3, \beta_3, \beta_1)$  and  $(\beta_1, \beta_3, \alpha_3, \alpha_1)$ . Interchanging  $v_0, v_4$  treats  $(0, 1)$ . For  $(\alpha_2, \beta_2) = (1, 1)$ , with  $k = \alpha_1 + \beta_1 + \alpha_3 + \beta_3$  and  $\tau = \binom{k}{3}$ , the forest formula gives

$$\begin{aligned} 8N_\star - N_2 - 10N_3 - 19N_4 &= 15 + 13k + 11(\alpha_1\alpha_3 + \beta_1\beta_3) \\ &\quad - 8(\alpha_1\beta_3 + \beta_1\alpha_3) - 3(\alpha_1\beta_1 + \alpha_3\beta_3) - 4\tau \\ &\geq 15 + \frac{7}{2}k. \end{aligned}$$

The negative matching terms are at most  $11k/2$ , and  $\tau \leq k$ .

It remains to treat the internal triangle. Set  $p = \sum_i \alpha_i$ ,  $q = \sum_i \beta_i$ ,  $r = \sum_i \alpha_i \beta_i$ ,  $n_{\text{att}} = p + q$ ,  $\Pi_{\text{att}} = c_1 c_2 c_3$ , and  $e_2(c) = c_1 c_2 + c_1 c_3 + c_2 c_3$ . Contraction of the endpoint path gives

$$\begin{aligned} N_\star &= \Pi_{\text{att}} + 2e_2(c) + 3n_{\text{att}}, \\ N_2 &= r(\Pi_{\text{att}}/2 + 2n_{\text{att}} - 1), \\ N_3 &= 2(pq - r) + 2(\Pi_{\text{att}} - \mathbf{1}_{p=3} - \mathbf{1}_{q=3}), \\ N_4 &= pq - r. \end{aligned}$$

For example, a path through a single internal vertex leaves completion count  $c_j c_k + 2c_j + 2c_k + 3$ , while a path through two internal vertices leaves  $c_k + 2$  choices. Using  $2e_2(c) = n_{\text{att}}^2 - n_{\text{att}} - 2r$ , the required difference becomes

$$-(12 + r/2)\Pi_{\text{att}} + 8n_{\text{att}}^2 + 16n_{\text{att}} - 39pq + (24 - 2n_{\text{att}})r + 20(\mathbf{1}_{p=3} + \mathbf{1}_{q=3}).$$

If an endpoint has degree zero, all  $N_j$  vanish. Otherwise assume  $p \geq q > 0$ . When  $p, q \leq 2$  and the attachment union has fewer than three vertices,  $\Pi_{\text{att}} = 0$  and the difference is at least  $16n_{\text{att}} - 7n_{\text{att}}^2/4 \geq 0$ . If that union has three vertices, the possibilities

$(p, q, r) = (2, 1, 0)$  and  $(2, 2, 1)$  give 30 and 27. Finally, if  $p = 3$ , then  $r = q$ ,  $\Pi_{\text{att}} = 2^q$ , and the difference is

$$140 - 35q + 6q^2 - (12 + q/2)2^q + 201_{q=3}.$$

Its values for  $q = 1, 2, 3$  are 86, 42, 1. The graph inequality follows, and applies to the type II graph  $K_5 - e$ . Multiaffine interpolation and homogeneity complete the proof in both types.  $\square$

### 6.3. Continuation through the small-core range.

**Lemma 6.3.** *Suppose that  $R = x_i > 0$  is a largest retained weight, that is,  $x_k \leq R$  for  $k = 1, \dots, 9$ . Then  $9\mathcal{Q} \geq \mathfrak{D}$  whenever  $0 \leq t \leq 2R/3$ , in both types.*

*Proof.* Normalize  $R = 1$  and set  $\mathcal{P} = 9\mathcal{Q} - \mathfrak{D}$ . Use (6.2) to take the other eight retained weights to be zero or one. Let  $n \in \{0, 1, 2\}$  count the occupied companion rays in the chosen frame, and let  $r, l \leq 3$  be the occupied sizes of the two outside rows, with  $s = r + l$ . Write

$$I = \binom{r}{2} + \binom{l}{2}, \quad J = rl, \quad N_{\text{same}} = \binom{r}{3} + \binom{l}{3}, \quad N_{\text{mix}} = l\binom{r}{2} + r\binom{l}{2}.$$

Here  $I$  and  $J$  count outside pairs within a row and across rows, and  $N_{\text{same}}$  and  $N_{\text{mix}}$  count outside triples within a row and across rows. In particular,  $I \leq s$ ,  $J \leq 3s/2$ ,  $N_{\text{same}} \leq s/3$ ,  $N_{\text{same}} \leq I/3$ ,  $N_{\text{mix}} \leq 3I$ , and  $N_{\text{same}} + N_{\text{mix}} = \binom{s}{3}$ . Pairs in one frame have Gram determinant one, and pairs in distinct frames have Gram determinant  $3/4$ . Thus the coefficients of  $t^3$  and  $t^2$  in  $\mathfrak{D}$  are, respectively,

$$1 + n + s, \quad n + uv + I + \frac{3}{4}J + \frac{3}{4}(1 + n)s.$$

Subtracting these from  $9q_3$  and  $9q_2$  gives

$$\begin{aligned} \mathcal{P}(t) &= \mathcal{P}_0 + \mathcal{P}_1 t + \mathcal{P}_2 t^2 + \mathcal{P}_3 t^3 - t^4, \\ \mathcal{P}_3 &= 8 - 4n - s, \\ \mathcal{P}_2 &= 8n - 10uv + (6 - 3n)s - I - \frac{3}{4}J, \end{aligned}$$

where  $u, v$  are the companion weights, so  $uv = 1$  precisely when  $n = 2$ . Write  $\kappa_3$  for the coefficient of  $t$  in  $\det(tI + K)$ , so  $\mathcal{P}_1 = 9q_1 - \kappa_3$ . Lemma 6.2 gives  $\mathcal{P}_0 \geq 0$ . It remains to control the positive-core terms. Every same-frame triple has Gram determinant one. An independent mixed-frame triple has Gram determinant  $1/2$ , and a dependent triple contributes zero. We count triples by their number of rays in the selected frame and bound  $\mathcal{P}_1$  accordingly.

If  $n = 0$ , let  $J_0$  count the cross-row pairs which form an independent triple with the chosen ray. The triples involving the chosen ray have total Gram contribution  $(I + J_0)/2$ , and the outside triples contribute  $N_{\text{same}} + N_{\text{mix}}/2$ . Hence

$$e_3 = \frac{I + J_0}{2} + N_{\text{same}} + \frac{1}{2}N_{\text{mix}}.$$

The coefficient packets give  $2q_1 = I - N_{\text{same}} + J_0$ , and therefore

$$\mathcal{P}_1 = 4(I + J_0) - \frac{11}{2}N_{\text{same}} - \frac{1}{2}N_{\text{mix}} \geq \frac{2}{3}I + 4J_0 \geq 0.$$

Also  $\mathcal{P}_2 \geq 0$  and  $\mathcal{P}_3 \geq 2$ , which imply  $\mathcal{P}(t) \geq \mathcal{P}_0$  on the required interval.

If  $n = 1$ , the same expansions give  $q_1 \geq s/2$ . Triples with both selected-frame rays contribute at most  $s/2$ , and those with one contribute at most  $\binom{s}{2}$ . Together with the outside triples, this gives  $\kappa_3 \leq s/2 + \binom{s}{2} + N_{\text{same}} + N_{\text{mix}}/2$ , and hence

$$\mathcal{P}_1 \geq 4s - \binom{s}{2} - N_{\text{same}} - \frac{1}{2}N_{\text{mix}} \geq \frac{s(50 - 3s - s^2)}{12} \geq 0 \quad (0 \leq s \leq 5).$$

When  $s = 6$ , all six outside weights equal one. Suppose that  $v = 0$  is the unoccupied companion weight. Each expansion of  $2q_1$  displayed in the proof of Lemma 6.1 contains, besides the term  $\{R(u+v) - uv\}s$  used for the bound  $q_1 \geq s/2$ , a summand  $(R-v)(y_i y_j + y_k y_l)$ . This summand is  $(R-v)(y_1 y_3 + y_4 y_6)$  in type III,  $(R-v)(y_1 y_3 + y_5 y_6)$  for the first type II orbit, and  $(R-v)(y_2 y_3 + y_4 y_6)$  for the other. With  $R = 1$ ,  $v = 0$ , and all  $y_i = 1$ , this summand equals two. If instead  $u = 0$ , the summand carrying the factor  $R - u$  gives the same value. Hence  $2q_1 \geq s + 2$ , that is,  $q_1 \geq 4$ , whereas the same triple count gives  $\kappa_3 \leq 29$ . Thus  $\mathcal{P}_1 \geq 7$  in that case as well. Since  $\mathcal{P}_2 \geq 8$  and  $\mathcal{P}_3 \geq -2$ , we again have  $\mathcal{P}(t) \geq \mathcal{P}_0$ .

Suppose next that  $n = 2$  and  $s \leq 5$ . Now  $q_1 \geq 1 + s/2$ . For  $s \leq 4$ , counting triples by their number of own-frame rays yields

$$\kappa_3 \leq 1 + \frac{3}{2}s + \frac{3}{2}\binom{s}{2} + N_{\text{same}} + \frac{1}{2}N_{\text{mix}}, \quad \text{hence} \quad \mathcal{P}_1 \geq 8 + \frac{s(41 - 6s - s^2)}{12} \geq 8.$$

For  $s = 5$ , the outside rows have sizes 3, 2. Orient their rays to have component  $1/2$  along the missing own-frame normal. Their other coordinates are signs divided by two, and a cross-row scalar product is negative precisely when the sign triples are opposite. The negative cross pairs therefore form a matching, of size at most two here. For a cross-row pair with positive scalar product, the Gram contributions obtained by adjoining the three selected-frame rays sum to one. For a pair with negative scalar product, they sum to  $3/2$ . These values follow by inserting the three coordinate signs in the  $3 \times 3$  Gram determinant. There are six cross-row pairs and at most two negative ones, so their total contribution is at most  $6 + 2/2 = 7$ . Counting the remaining triples gives  $\kappa_3 \leq 1 + 15/2 + 6 + 7 + 11/2 = 27$ . With  $u = v = R = 1$ , each displayed expansion of  $q_1$  contains at least four cross-row monomials with coefficient  $1/4$ , and each outside variable occurs in at most two of these monomials. Deleting one outside ray leaves at least two such pairs, so  $q_1 \geq 1 + 5/2 + 2/4 = 4$  and  $\mathcal{P}_1 \geq 9$ . Throughout  $s \leq 5$ , the pair count gives  $\mathcal{P}_2 \geq -5/2$  and  $\mathcal{P}_3 \geq -5$ . It follows that

$$\mathcal{P}(t) - \mathcal{P}_0 \geq t \left( 8 - \frac{5}{2}t - 5t^2 - t^3 \right) \geq \frac{103}{27}t \geq 0.$$

Only the fully occupied array remains. Substitution into the displayed coefficient expansions gives  $q_0 \geq 5/2$  and  $q_1 \geq 5$ . Its covariance is  $3I - \sum_j P_{m_j}$ , and the missing-normal Gram matrix has eigenvalues either  $(0, 3/2, 3/2)$  or  $(2, 1/2, 1/2)$ . Thus  $\mathfrak{D}(0) \leq 81/4$  and  $\kappa_3 \leq 81/2$ , giving

$$\mathcal{P}(t) \geq \frac{9}{4} + \frac{9}{2}t - \frac{27}{4}t^2 - 6t^3 - t^4.$$

This polynomial is concave on  $[0, 2/3]$  and takes the positive endpoint values  $9/4$  and  $89/324$ . The binary cases are complete, interpolation and homogeneity now prove the lemma.  $\square$

*Proof of Lemma 5.1.* The large-core assertion is Lemma 6.1, and the small-core assertion is Lemma 6.3.  $\square$

## 7. COFACTOR BALANCE AND SECOND VARIATION IN DIMENSION $d \geq 2$

To quantify proximity to the minimizer, we keep both the metric and the affine variables in the exact surface formula for an arbitrary lattice. Sections 8 and 9 apply the result to  $D_4$  and to root lattices. Lemma 7.1 extends cofactor balance to every lattice. It determines the mixed second-order terms, and the codimension of a nonfacet contact determines when a newly created facet contributes. Several steps follow the three-dimensional argument of Song [27], including the separation of affine strain from metric deformation, the exponential coordinates for the polar factor, the mixed second-order terms, and the comparison of coordinates with Hausdorff distance in both directions.

**7.1. Cofactor balance and the exact surface formula.** Let  $\Lambda \subset \mathbb{R}^d$  be a lattice of covolume  $\delta$ , and set

$$\mathcal{C}_H = \{x : v \cdot x \leq v^\top H v / 2 \text{ for all } v \in \Lambda\}, \quad V_H = H^{-1/2} \mathcal{C}_H$$

for  $H > 0$ . Then  $V_H = \text{Vor}(H^{1/2} \Lambda)$  has volume  $\delta \sqrt{\mathfrak{D}}$ , where  $\mathfrak{D} = \det H$ . Choose one vector  $v$  from each opposite facet pair, and let  $b_e$  be the area of its facet in  $\mathcal{C}_H$ . Define

$$\ell_e = \|v\|, \quad e = v/\ell_e, \quad w_e = \ell_e b_e / \delta.$$

Write  $P_e = ee^\top$ ,  $h_e = e^\top H e$ , and  $n_e = H^{1/2} e / \sqrt{h_e}$ . Give absent facets weight zero.

**Lemma 7.1.** *For every  $H > 0$ ,*

$$\sum_e w_e P_e = \text{adj } H.$$

*For a reference length  $\ell_0 > 0$ , set  $I_0 = 2d\delta^{1/d}/\ell_0$  and  $\eta_e = \ell_0/\ell_e$ . Then*

$$(7.1) \quad \frac{\mathcal{I}_d(LV_H)}{I_0} = \frac{\mathfrak{D}^{-1+1/(2d)}}{d} \sum_e \eta_e w_e \|L^{-\top} H^{1/2} e\|, \quad |\det L| = 1.$$

*Proof.* The facet associated with  $v$  is centrally symmetric about  $Hv/2$ . Indeed, reflection about  $H^{1/2}v/2$  interchanges the two adjacent Voronoi cells and preserves their common facet. Since  $\text{vol}_d(\mathcal{C}_H) = \delta \mathfrak{D}$ , the divergence identity gives

$$\sum_e b_e \ell_e H P_e = \int_{\partial \mathcal{C}_H} x \otimes \nu \, dS = \delta \mathfrak{D} I.$$

Multiplication by  $H^{-1}$  proves cofactor balance. Under  $H^{-1/2}$  the facet area becomes  $b_e \sqrt{h_e}/\sqrt{\mathfrak{D}}$ , and  $L$  then multiplies it by  $\|L^{-\top} n_e\|$ . Summing opposite pairs and dividing by  $(\delta \sqrt{\mathfrak{D}})^{(d-1)/d}$  gives the surface formula.  $\square$

**7.2. Facet births and the joint variation.** Suppose that all facet vectors of  $\mathcal{C}_I$  have the same length  $\ell_0$ . Since every shortest lattice vector defines a facet,  $\ell_0$  is the shortest nonzero vector length and  $I_0 = \mathcal{I}_d(V_I)$ . Write  $\mathcal{R}$  for the old normal lines and  $w_e^0 = w_e(I)$ . The variance operator on  $\text{Sym}_0(\mathbb{R}^d)$  is

$$\mathcal{A}Z = Z - \sum_{e \in \mathcal{R}} w_e^0 (e^\top Z e) P_e.$$

By cofactor balance at  $I$ , this operator is self-adjoint,  $0 \leq \mathcal{A} \leq I$ , and  $\langle Z, \mathcal{A}Z \rangle = \sum_{e \in \mathcal{R}} w_e^0 \text{Var}_e(Z)$ . Identity operators in these formulas act on  $\text{Sym}_0(\mathbb{R}^d)$ . For  $|\det L| = 1$ , write  $L^{-\top} = Oe^Y$  with  $Y = Y^\top$  and  $\text{tr } Y = 0$ . These exponential coordinates remove the orthogonal factor. The following estimates control facet weights across changes of Voronoi type.

**Lemma 7.2.** *There is a neighborhood of  $I$  in which all facet vectors belong to one finite set. For  $H = I + X$  and  $\varepsilon = \|X\|$ , the old weights satisfy  $w_e(H) = w_e^0 + O(\varepsilon)$ . If the bisector of a nonfacet lattice vector supports a face of  $\mathcal{C}_I$  of codimension  $c$ , its weight is  $O(\varepsilon^{c-1})$ . Vectors whose inequalities are strictly slack on  $\mathcal{C}_I$  remain inactive after the neighborhood is reduced. All these estimates are uniform over the adjacent Voronoi chambers.*

*Proof.* The inclusions  $(1 - \varepsilon)\mathcal{C}_I \subseteq \mathcal{C}_H \subseteq (1 + \varepsilon)\mathcal{C}_I$  give a uniform containing ball of radius  $R$ . Every facet vector satisfies  $(1 - \varepsilon)\|v\|^2/2 \leq R\|v\|$ , so only finitely many vectors need be retained. Strict slack persists for the remaining candidates.

Suppose the bisector of  $v$  supports a face  $F$  at  $I$ . For a point  $x$  of the perturbed facet,  $z = x/(1 + \varepsilon)$  belongs to  $\mathcal{C}_I$  and has slack  $O(\varepsilon)$  in the supporting equation of  $F$ . In a convex combination of the vertices representing  $z$ , the total coefficient of vertices outside  $F$  is  $O(\varepsilon)$ , since their positive slacks have a fixed positive minimum. Thus  $x$  is

within  $O(\varepsilon)$  of  $F$ . After translating  $F$  into the perturbed supporting hyperplane, its  $O(\varepsilon)$  neighborhood has  $(d-1)$ -volume  $O(\varepsilon^{c-1})$ , proving the asserted bound.

For an old facet vector  $v$ , translation by  $-Hv/2$  places the facet in  $v^\perp$ , with inequalities  $u \cdot y \leq (u^\top Hu - u^\top Hv)/2$ . Their right sides at  $I$  are positive for  $u \neq 0, v$ , since  $v$  is shortest. Their relative changes are uniformly  $O(\varepsilon)$ . Finiteness handles bounded  $u$ , and for large  $u$  the lower bound  $\|u\|^2 - \|u\|\|v\|$  controls the perturbation. The centered facet is consequently contained between dilates of its old facet with factors  $1 \pm C\varepsilon$ . Comparing their areas gives  $w_e(H) = w_e^0 + O(\varepsilon)$ .  $\square$

Give inactive candidates weight zero and define

$$\Pi(H) = \sum_{e \notin \mathcal{R}} \left(1 - \frac{\ell_0}{\ell_e}\right) w_e(H), \quad Y_*(X) = \frac{1}{2}(I + \mathcal{A})^{-1}(I - \mathcal{A})X.$$

The first quantity measures the surface contribution lost when longer lattice vectors create facets. The second is the first-order affine correction.

**Theorem 7.3.** *Suppose every nonfacet supporting lattice vector at  $I$  exposes a face of codimension at least three. Let  $H = I + X$  with  $\text{tr } X = 0$ , and let  $Y$  be the polar logarithm defined above. Uniformly as  $\rho = \|X\| + \|Y\| \rightarrow 0$ ,*

$$(7.2) \quad \log \frac{\mathcal{I}_d(LV_H)}{I_0} = \frac{\langle Y - Y_*, (I + \mathcal{A})(Y - Y_*) \rangle}{2d} + \frac{\langle X, \mathcal{A}(I + \mathcal{A})^{-1}X \rangle}{2d} - \frac{\Pi(H)}{d} + O(\rho^3),$$

where  $Y_* = Y_*(X)$ . For sufficiently small  $X$ , the minimum over all  $|\det L| = 1$  is attained, and the minimizing polar logarithm is unique. It satisfies  $Y_H = Y_*(X) + O(\|X\|^2)$  and

$$(7.3) \quad \inf_{|\det L|=1} \log \frac{\mathcal{I}_d(LV_H)}{I_0} = \frac{\langle X, \mathcal{A}(I + \mathcal{A})^{-1}X \rangle}{2d} - \frac{\Pi(H)}{d} + O(\|X\|^3).$$

If the nonfacet contacts all have codimension at least four, then  $\Pi(H) = O(\|X\|^3)$ . The remainder constants depend only on the fixed lattice and a sufficiently small neighborhood, and are independent of the Voronoi chamber.

*Proof.* Let  $R(X, Y)$  be the quotient in (7.1). We retain the mixed terms between the shape and affine variables. Their coefficients follow from cofactor balance. For a unit normal  $e$ , write  $a = e^\top X e$  and  $b = e^\top Y e$ . The factor in the surface formula has the expansion

$$\|e^Y H^{1/2} e\| = 1 + \frac{1}{2}a + b + \frac{1}{2}e^\top (XY + YX)e + e^\top Y^2 e - \frac{1}{8}a^2 - \frac{1}{2}ab - \frac{1}{2}b^2 + O(\rho^3).$$

Lemma 7.2 gives  $w_e = O(\|X\|^2)$  on new facets. Replacing their factors  $\eta_e$  by one therefore adds  $\Pi(H)/d + O(\rho^3)$  to  $R$ . For the remaining sum, cofactor balance gives

$$\begin{aligned} \sum_e w_e &= d + (1 - d/2)\|X\|^2 + O(\|X\|^3), \\ \sum_e w_e a &= -\|X\|^2 + O(\|X\|^3), \\ \sum_e w_e b &= -\langle X, Y \rangle + O(\|X\|^2\|Y\|). \end{aligned}$$

Indeed, the three sums are the traces of  $\text{adj } H$ ,  $X \text{ adj } H$ , and  $Y \text{ adj } H$ . In the quadratic terms, the old-weight estimate permits replacement by  $w_e^0$ . Multiplying by  $\mathfrak{D}^{-1+1/(2d)} = 1 + (1/2 - 1/(4d))\|X\|^2 + O(\|X\|^3)$  then gives

$$d \log R = \frac{1}{8}\langle X, (I + \mathcal{A})X \rangle + \frac{1}{2}\langle Y, (I + \mathcal{A})Y \rangle - \frac{1}{2}\langle X, (I - \mathcal{A})Y \rangle - \Pi(H) + O(\rho^3).$$

There is no linear term, and completing the square gives the asserted expansion. The facet weights enter only through cofactor balance and Lemma 7.2, so the expansion is uniform across Voronoi chambers.

We also need estimates for derivatives in the affine variable. For each fixed  $X$ , the finite set of facet weights is independent of  $Y$ , and the factors  $\|e^Y(I + X)^{1/2}e\|$  and the determinant prefactor are analytic near  $(X, Y) = (0, 0)$ . Their Taylor remainders, after zero, one, or two derivatives in  $Y$ , are uniformly  $O(\rho^3)$ ,  $O(\rho^2)$ , and  $O(\rho)$ , respectively. The substitutions used above preserve these orders. Replacing an old weight by  $w_e^0$  costs  $O(\|X\|)$  times a quadratic expression, while the new weights are  $O(\|X\|^2)$  and their constant correction  $-\Pi(H)$  has no  $Y$ -dependence. The trace identities retain the remaining linear terms exactly. Since  $R = 1 + O(\rho^2)$ , passing to  $\log R$  also preserves these derivative estimates. Consequently the remainder in the displayed formula for  $d \log R$  has  $Y$ -gradient  $O(\rho^2)$  and  $Y$ -Hessian  $O(\rho)$ , uniformly over the adjacent chambers.

It remains to show that the unrestricted affine minimum lies in this neighborhood. At  $X = 0$ , isotropy and  $\|e^Y e\| \geq e^T e^Y e$  give  $R(0, Y) \geq \text{tr}(e^Y)/d \geq 1$ , with equality only at  $Y = 0$ . For small  $X$ , the old facets give  $R(X, Y) \geq c \text{tr}(e^Y)$ . Indeed,  $\|e^Y n_e\| \geq n_e^T e^Y n_e$  and  $\sum_{\mathcal{R}} w_e \sqrt{h_e} P_{n_e} \rightarrow I$ . The determinant prefactor stays bounded away from zero. As  $\text{tr} Y = 0$ , this lower bound tends to infinity with  $\|Y\|$ , uniformly for such  $X$ . The minimum is therefore attained, and uniform convergence on bounded  $Y$ -sets shows that every minimizer tends to zero as  $X \rightarrow 0$ . In a sufficiently small convex neighborhood the derivative estimates above give uniform convergence of the Hessian in  $Y$  to  $(I + \mathcal{A})/d > 0$ . Thus  $\log R(X, \cdot)$  is strictly convex there, and since every minimizer lies there, its minimizing polar logarithm is unique. The gradient equation first yields  $Y_H = O(\|X\|)$  and then  $Y_H - Y_*(X) = O(\|X\|^2)$ . Substitution gives the formula for the affine minimum.  $\square$

**Corollary 7.4.** *Suppose the nonfacet contacts in Theorem 7.3 have codimension at least four and  $\mathcal{A} \geq \gamma I$  for some  $\gamma > 0$ . There are  $r, c, C > 0$  such that, whenever  $H = I + X$ ,  $\text{tr} X = 0$ , and  $\|X\| < r$ , every  $|\det L| = 1$  satisfies*

$$\mathfrak{d}(H, L) := \log \frac{\mathcal{I}_d(LV_H)}{I_0} \geq c(\|X\|^2 + \min\{\|Y\|^2, \|Y\|\}).$$

For a centered body  $P$ , set  $\hat{P} = \text{vol}_d(P)^{-1/d}P$ . If also  $\mathfrak{d}(H, L) < r$ , then

$$\inf_{O \in O(d)} \left\{ d_H(\widehat{LV}_H, O\widehat{V}_I) + \text{vol}_d(\widehat{LV}_H \triangle O\widehat{V}_I) \right\} \leq C\sqrt{\mathfrak{d}(H, L)},$$

where  $d_H$  denotes Hausdorff distance.

*Proof.* Under these hypotheses the quadratic form in (7.2) is positive definite in  $(X, Y)$  and dominates the cubic remainder near zero. On a fixed compact annulus in  $Y$ , the gap  $R(0, Y) > 1$  persists uniformly for small  $X$ . For large  $Y$ , the bound  $R(X, Y) \geq c_0 \text{tr}(e^Y)$  gives  $\log R(X, Y) \geq c_1 \|Y\| - C_0$ . Combining these three ranges proves the first assertion.

For small deficit, it follows that  $\|X\| + \|Y\| \leq C\sqrt{\mathfrak{d}(H, L)}$ . We pass from this coordinate estimate to Hausdorff distance by comparing support functions. Here  $L = Oe^{-Y}$ , and the inclusions in Lemma 7.2, together with  $V_H = H^{-1/2}\mathcal{C}_H$ , give

$$\|h_{\widehat{e^{-Y}V_H}} - h_{\widehat{V}_I}\|_\infty \leq C(\|X\| + \|Y\|).$$

The normalization factor is locally Lipschitz, since  $\text{vol}_d(V_H) = \delta\sqrt{\mathfrak{D}}$ , and the left side equals Hausdorff distance. Uniform inner and outer radii give the same bound for symmetric difference.  $\square$

## 8. QUADRATIC STABILITY OF THE 24-CELL

We now apply the variation formula to  $D_4$  and pass from metric and affine coordinates to arbitrary nearby parallelohedra. The root moments determine the quadratic form, and

the classified cones provide the converse from Hausdorff proximity to small coordinates. Compactness then extends the local estimate to Corollary 1.2.

**Lemma 8.1.** *For the twelve unit normal lines  $\mathcal{R}$  of  $C_{24}$  and every symmetric  $4 \times 4$  matrix  $Z$ ,*

$$\sum_{e \in \mathcal{R}} (e^\top Z e) P_e = Z + \frac{1}{2} \operatorname{tr}(Z) I, \quad \sum_{e \in \mathcal{R}} \operatorname{Var}_e(Z) = 2 \| \operatorname{tf} Z \|^2.$$

*Proof.* Summing the two normals  $(e_r \pm e_s)/\sqrt{2}$  for each pair  $r < s$  gives the first identity. Pair it with  $Z$  and use  $\sum_{e \in \mathcal{R}} P_e = 3I$  to obtain  $3 \operatorname{tr}(Z^2) - \operatorname{tr}(Z^2) - \frac{1}{2} \operatorname{tr}(Z)^2 = 2 \| \operatorname{tf} Z \|^2$ , which is the second identity.  $\square$

For  $D_4$  the variance operator is scalar, and Theorem 7.3 becomes explicit.

**Corollary 8.2.** *Let  $\Lambda = D_4$  and  $H = I + X$  with  $\operatorname{tr} X = 0$ . As  $\|X\| + \|Y\| \rightarrow 0$ ,*

$$\log \frac{\mathcal{I}(LV_H)}{2^{11/4}} = \frac{1}{20} \|X\|^2 + \frac{5}{24} \|Y - X/10\|^2 + O((\|X\| + \|Y\|)^3).$$

*Consequently,*

$$\inf_{|\det L|=1} \log \frac{\mathcal{I}(LV_H)}{2^{11/4}} = \frac{1}{20} \|X\|^2 + O(\|X\|^3), \quad Y_H = X/10 + O(\|X\|^2).$$

*In particular,  $1/20$  is the sharp optimized quadratic coefficient. The metric and geometric estimates of Corollary 7.4 hold near this metric for every volume-preserving affine realization, with the regular 24-cell as reference.*

*Proof.* The old weights are  $1/3$ , and the root moment identity in Lemma 8.1 gives  $\mathcal{A} = 2I/3$  on  $\operatorname{Sym}_0(\mathbb{R}^4)$ . Moreover,  $h_{C_I}(v) = \max(\|v\|_\infty, \|v\|_1/2)$ . For nonzero  $v \in D_4$  outside the roots, equality  $2h_{C_I}(v) = \|v\|^2$  occurs only for  $\pm 2e_i$  and vectors whose four coordinates are  $\pm 1$ , and each exposes a vertex. Thus all nonfacet contacts have codimension four. Substitution in Theorem 7.3 gives the displayed coefficients, and the positive variance gap gives the geometric estimates.  $\square$

It remains to pass from Hausdorff proximity to small parameters. For the regular 24-cell, the converse follows from the cone descriptions.

**Lemma 8.3.** *Let  $P_0 = \widehat{C}_{24}$  be the centered unit-volume regular 24-cell. For every  $\varepsilon > 0$  there is  $r > 0$  such that every centered unit-volume four-dimensional parallelohedron  $P$  with  $d_H(P, P_0) < r$  admits*

$$P = \widehat{L} \widehat{V}_H, \quad \operatorname{tr} H = 4, \quad |\det L| = 1.$$

*Here  $L^{-\top} = Oe^Y$  and  $\|H - I\| + \|Y\| < \varepsilon$ . The matrix  $H$  can be chosen in one of the two nonprincipal cones.*

*Proof.* We first note that  $C_{24}$  has no nonzero segment as a Minkowski summand. Indeed, if a polytope  $Q + [-z, z]$  has  $z \neq 0$ , exposing a vertex of its projection to  $z^\perp$  gives an edge parallel to  $z$ . Every two-face containing that edge is itself the sum of a face of  $Q$  and  $[-z, z]$ , since its exposing functional vanishes on  $z$ . Such a polygon has two parallel edges, obtained by exposing in the two directions perpendicular to  $z$  in its plane. It cannot be a triangle, whereas every two-face of  $C_{24}$  is triangular.

It suffices to prove the assertion sequentially. Let  $P_j \rightarrow P_0$  be centered unit-volume parallelohedra, and set  $\tilde{P}_j = 2^{1/4} P_j$ , so  $\tilde{P}_j \rightarrow C_{24}$ . By Theorem 2.2, pass to a subsequence belonging to one fixed cone type. In the principal case, Proposition 2.4 expresses  $\tilde{P}_j$  as a sum of at most ten centered segments. Each summand is contained in  $\tilde{P}_j$ , so its generating vector is bounded. Passing to a limit would express  $C_{24}$  as a zonotope, contrary to the preceding observation.

In either nonprincipal type, Lemma 3.2 gives

$$\tilde{P}_j = A_j \left( t_j C_{24} + \frac{1}{2} \sum_{i=1}^9 x_{ji} [-a_i, a_i] \right), \quad t_j, x_{ji} \geq 0,$$

where  $A_j$  is invertible and the ray configuration is fixed. Let  $T_j = t_j A_j$  and  $z_{ji} = x_{ji} A_j a_i$ . All summands are centered, so  $T_j C_{24}$  and each  $[-z_{ji}, z_{ji}]/2$  are contained in the uniformly bounded body  $\tilde{P}_j$ . Since  $C_{24}$  contains a ball about zero, both  $T_j$  and the  $z_{ji}$  are bounded. Along a further subsequence they converge to  $T$  and  $z_i$ , and continuity of Minkowski addition gives

$$C_{24} = TC_{24} + \frac{1}{2} \sum_{i=1}^9 [-z_i, z_i].$$

The absence of segment summands forces every  $z_i = 0$ . Thus  $TC_{24} = C_{24}$ , so  $T$  is invertible and permutes its vertices. The vertex identity  $\sum_{v \in \text{vert } C_{24}} vv^\top = 6I$  therefore gives  $TT^\top = I$ . For large  $j$ ,  $t_j > 0$ , and

$$\frac{x_{ji}}{t_j} \|T_j a_i\| = \|z_{ji}\| \rightarrow 0.$$

As  $T_j \rightarrow T \in O(4)$ , this implies  $x_{ji}/t_j \rightarrow 0$  for every  $i$ . Set

$$H_j^0 = I + \sum_i \frac{x_{ji}}{t_j} P_{a_i}, \quad q_j = \frac{\text{tr } H_j^0}{4}, \quad H_j = H_j^0 / q_j.$$

Then  $H_j \rightarrow I$ , and the same Minkowski formula gives  $\tilde{P}_j = T_j (H_j^0)^{1/2} V_{H_j^0}$ . Writing  $B_j = T_j (H_j^0)^{1/2}$  and  $L_j = |\det B_j|^{-1/4} B_j$ , we have  $L_j \rightarrow T$  and  $V_{H_j^0} = \sqrt{q_j} V_{H_j}$ . After volume normalization this is precisely  $P_j = \widehat{L_j V_{H_j}}$ . The polar logarithm  $Y_j$  tends to zero because  $L_j \rightarrow T \in O(4)$ . Thus any sequence approaching  $P_0$  has a subsequence with the asserted small parameters. A sequence violating the stated neighborhood property would give a contradiction.  $\square$

*Proof of Corollary 1.2.* We first show that small deficit forces closeness to the regular cell. Let  $P_j$  be centered parallelohedra of unit volume with  $\text{def}(P_j) \rightarrow 0$ . Lemma 2.5 bounds their diameters, so Blaschke selection gives a subsequence converging to a convex body  $P$  of volume one. For some  $r, R > 0$ , one has  $r\mathbb{B} \subset P_j \subset R\mathbb{B}$  for all sufficiently large  $j$ , where  $\mathbb{B}$  is the Euclidean unit ball. Their tiling lattices  $\Lambda_j$  have covolume one and no nonzero vector shorter than  $2r$ . Mahler's compactness theorem [22, Theorem 2] therefore gives bases  $B_j \rightarrow B$  with  $\Lambda_j = B_j \mathbb{Z}^4$  and  $|\det B| = 1$ . The translates  $P + Bz$ ,  $z \in \mathbb{Z}^4$ , have disjoint interiors, since an interior overlap would persist for large  $j$ . They also cover space, since the indices of translates containing any fixed point remain bounded. Thus  $P$  is a bounded convex translative tile and hence a polytope, so it is a parallelohedron [24]. Surface area is continuous under Hausdorff convergence of convex bodies, so  $\text{def}(P) = 0$ . Theorem 1.1 identifies  $P$  with an orthogonal image of  $\hat{C}_{24}$ .

It follows that every centered unit-volume parallelohedron with sufficiently small deficit lies in the neighborhood of Lemma 8.3 after an orthogonal map. That lemma supplies a representation by a nearby  $D_4$  metric and an affine map. Corollary 8.2 then gives

$$\inf_{O \in O(4)} d_H(\hat{P}, O\hat{C}_{24}) \leq C \sqrt{\log(1 + \text{def}(P))} \leq C \sqrt{\text{def}(P)}.$$

This proves the estimate for  $\text{def}(P) \leq \varepsilon_0$  with some  $\varepsilon_0 \in (0, 1)$ . For  $\text{def}(P) \leq 1$ , the surface area of  $\hat{P}$  is at most  $2^{15/4}$ , so Lemma 2.5 bounds its diameter and hence  $d_H(\hat{P}, \hat{C}_{24})$  by a constant  $M$ . On  $\varepsilon_0 \leq \text{def}(P) \leq 1$ , this distance is at most  $M\varepsilon_0^{-1/2} \sqrt{\text{def}(P)}$ . Enlarging  $C$  proves the assertion.  $\square$

## 9. APPLICATION TO ROOT LATTICES

The same variation criterion gives a local classification in every rank. The conclusions in this section concern nearby lattice Voronoi cells and their affine images. The passage to all nearby parallelohedra in Section 8 uses the four-dimensional classification.

A root lattice is an even lattice generated by its vectors of squared length two, called its roots. Every root lattice is an orthogonal sum of irreducible lattices of types  $A_j$ ,  $D_j$ ,  $E_6$ ,  $E_7$ ,  $E_8$ , and its Voronoi facet vectors are precisely its roots [16].

**Theorem 9.1.** *Among root lattices of rank at least two, precisely*

$$A_2, \quad D_d \ (d \geq 4), \quad E_6, \ E_7, \ E_8$$

*are strict local minimizers of the isoperimetric quotient among nearby lattice Voronoi cells and their affine images, modulo similarities. More precisely, for each lattice in this list there are  $c, r > 0$  such that*

$$\log \frac{\mathcal{I}_d(LV_H)}{\mathcal{I}_d(V_I)} \geq c \|H - I\|^2, \quad \text{tr } H = d, \quad \|H - I\| < r, \quad |\det L| = 1.$$

*Equality in the comparison with  $\mathcal{I}_d(V_I)$  requires  $H = I$  and  $L$  orthogonal. The affine and geometric stability conclusions of Corollary 7.4 also hold. Every  $A_d$  with  $d \geq 3$  admits a decreasing quadratic variation, and every reducible root lattice admits a decreasing linear variation. For the latter no additional affine map is needed.*

*Proof.* Write  $X = H - I$  with  $\text{tr } X = 0$ , and let  $h_C(v) = \max_{x \in C} v \cdot x$  denote the support function of a convex body  $C$ . When a lattice vector is used as a subscript on  $w$ , it denotes the corresponding normal line. For each positive case we will check that  $\mathcal{A}$  is positive definite and that every nonfacet contact has codimension at least four. Lemma 7.2 then gives  $\Pi(H) = O(\|X\|^3)$ , and the conclusions follow from Theorem 7.3 and Corollary 7.4. For  $A_d$ ,  $d \geq 3$ , contacts of codimension three instead contribute to the quadratic term, so their new-facet weights must be computed.

*The lattices  $D_d$ ,  $d \geq 4$ .* For  $D_d = \{z \in \mathbb{Z}^d : \sum_i z_i \text{ is even}\}$ , the root inequalities give

$$\mathcal{C}_I = \{x : |x_i| + |x_j| \leq 1 \ (i \neq j)\}, \quad h_{\mathcal{C}_I}(v) = \max\{\|v\|_\infty, \|v\|_1/2\}.$$

Its vertices are  $\pm e_i$  and the vectors  $(\varepsilon_1/2, \dots, \varepsilon_d/2)$  with  $\varepsilon_i \in \{-1, 1\}$ . A nonzero lattice vector has a supporting bisector exactly when  $\|v\|^2/2 = h_{\mathcal{C}_I}(v)$ . Equality with the  $\ell^1$  term forces every coordinate of  $v$  to have absolute value zero or one. Equality with the maximum term leaves only a single coordinate of magnitude two or two coordinates of magnitude one. Apart from the roots, the possibilities are therefore  $\pm 2e_i$  and the vectors in  $\{0, \pm 1\}^d$  with even support size  $k \geq 4$ . The former expose vertices. The latter expose  $(d - k)$ -cubes, with selected signed coordinates equal to  $1/2$  and remaining coordinates in  $[-1/2, 1/2]$ . Thus every nonfacet contact has codimension at least four.

There are  $d(d - 1)$  old normal lines, represented by  $(e_i \pm e_j)/\sqrt{2}$ ,  $i < j$ , and their projectors sum to  $(d - 1)I$ . Symmetry and cofactor balance give  $w_e^0 = 1/(d - 1)$ . Summing their directional variances yields

$$\langle X, \mathcal{A}X \rangle = \frac{d}{2(d - 1)} \|\text{diag } X\|^2 + \frac{d - 2}{d - 1} \|X - \text{diag } X\|^2.$$

Thus  $\mathcal{A}$  is positive definite. Substitution in (7.3) gives the quadratic term

$$(9.1) \quad \frac{\|\text{diag } X\|^2}{2(3d - 2)} + \frac{d - 2}{2d(2d - 3)} \|X - \text{diag } X\|^2.$$

*The exceptional lattices.* Let  $\mathbf{1}_j$  and  $\mathbf{0}_j$  denote the vectors of  $j$  ones and  $j$  zeros. Use the realizations

$$E_8 = D_8 \cup (\tfrac{1}{2}\mathbf{1}_8 + D_8), \quad E_7 = E_8 \cap \{x_7 + x_8 = 0\}, \quad E_6 = E_7 \cap \{x_6 = x_7\}.$$

Their covering-radius squares are  $4/3$ ,  $3/2$ , and  $1$  for  $E_6$ ,  $E_7$ , and  $E_8$ , respectively [9]. If the bisector of  $v$  supports a cell of covering radius  $R$ , then  $\|v\| \leq 2R$ . When equality holds, Cauchy–Schwarz forces the exposed face to be the point  $v/2$ . Since these lattices are even, this settles all nonroot contacts for  $E_8$  and all possible squared-norm-six contacts for  $E_7$ . It remains to consider squared norm four in  $E_6$  and  $E_7$ .

Set  $r = 3$  for  $E_6$ ,  $r = 2$  for  $E_7$ , and  $k = 8 - r$ . Their ambient subspaces consist of vectors  $(y_1, \dots, y_k, z\mathbf{1}_{r-1}, -z)$ . Up to permutations and even sign changes of the first  $k$  coordinates and central inversion, the squared-norm-four vectors have representatives

$$2e_1, \quad e_1 + e_2 + e_3 + e_4, \quad \sum_{i=1}^{4-r} e_i + (0_k, \mathbf{1}_{r-1}, -1), \quad \mathbf{h} = (3/2, \tfrac{1}{2}\mathbf{1}_6, -1/2).$$

Indeed, the integer norm equation allows a single coordinate of magnitude two, four of magnitude one, or  $|z| = 1$  and  $4 - r$  nonzero  $y$  coordinates. The half-integer equation allows one  $y$  coordinate of magnitude  $3/2$  and all other coordinates of magnitude  $1/2$ . The sign reductions follow from the lattice condition  $\sum_i y_i + (r - 2)z \in 2\mathbb{Z}$ . For each of the first three representatives  $u$ , the vector  $\mathbf{h} - u$  is a root and  $\mathbf{h} \cdot (\mathbf{h} - u) = 1$ . Reflection in that root sends  $\mathbf{h}$  to  $u$ , so all these vectors are equivalent to  $2e_1$  under lattice isometries.

On the supporting plane  $x_1 = 1$  for  $2e_1$ , the roots  $e_1 \pm e_j$  force  $x_j = 0$  for  $2 \leq j \leq k$ . The half-integer roots give exactly  $|z| \leq 1/r$ , and the remaining root inequalities impose no further restriction. The exposed face is therefore an edge. The least nonfacet contact codimensions in  $E_6$ ,  $E_7$ , and  $E_8$  are five, six, and eight.

The normalized exceptional roots are spherical four-designs [13]. Their equal co-factor weights have total mass  $d$ , so for traceless  $X$  the fourth-moment identity is  $\sum_e w_e^0 (e^\top X e)^2 = 2\|X\|^2/(d + 2)$ . Hence  $\mathcal{A} = dI/(d + 2)$ , and the minimized logarithmic quotient equals

$$(9.2) \quad \frac{\|X\|^2}{4(d + 1)} + O(\|X\|^3).$$

The lattices  $A_d$ . Set  $m = d + 1$ , let  $\mathcal{P} = \mathbf{1}_m^\perp \subset \mathbb{R}^m$ , and realize  $A_d = \mathbb{Z}^m \cap \mathcal{P}$ . The root inequalities give

$$\mathcal{C}_I = \{x \in \mathcal{P} : \max_i x_i - \min_i x_i \leq 1\}, \quad h_{\mathcal{C}_I}(v) = \|v\|_1/2 \quad (v \in A_d).$$

A supporting lattice vector must satisfy  $\|v\|^2 = \|v\|_1$ . It therefore has  $k$  coordinates equal to one,  $k$  equal to minus one, and all others zero. The exposed face has dimension  $m - 2k$ . When  $d = 2$ , only  $k = 1$  is possible, so there are no nonfacet contacts. The regular hexagonal root set gives  $\mathcal{A} = I/2$ , and (7.3) has quadratic coefficient  $1/12$ . This proves the remaining positive case.

Suppose  $d \geq 3$ . Instability for  $A_3$  is already known [6]. Only  $k = 2$  can give a quadratic new-facet weight, since  $k \geq 3$  gives contact codimension  $2k - 1 \geq 5$ . Choose the symmetric matrix  $X(t)$  with nonzero entries

$$X_{12} = X_{21} = X_{34} = X_{43} = -t, \quad X_{13} = X_{31} = X_{24} = X_{42} = t.$$

It annihilates  $\mathbf{1}_m$  and will be regarded as an operator on  $\mathcal{P}$ . Then  $\text{diag } X = 0$ ,  $\|X\|^2 = 8t^2$ , and  $I + X > 0$  for  $|t| < 1/2$ . The  $m(m - 1)/2$  old unit normal lines are  $(e_i - e_j)/\sqrt{2}$ ,  $i < j$ , and their projectors sum to  $mI/2$  on  $\mathcal{P}$ . Thus  $w_e^0 = 2/m$ , and summing their fourth moments gives  $\mathcal{A}X = dX/(d + 1)$ . The quadratic term before the facet penalty is consequently  $4t^2/(2d + 1)$ . The map  $L_t = \exp[-X(t)/(2(2d + 1))]$  gives the optimal affine correction to this order.

We compute the new weights within this zero-diagonal family. For four distinct indices set  $a = X_{ij} + X_{kl}$ ,  $b = X_{ik} + X_{jl}$ , and  $c = X_{il} + X_{jk}$ . For  $v = e_i + e_j - e_k - e_l$  one has

$$(9.3) \quad w_v(I + X) = \frac{4}{m}(b - a)_+(c - a)_+ + O(\|X\|^3), \quad s_+ = \max\{s, 0\}.$$

The old exposed face is parametrized by a unit cube in its  $m - 4$  free coordinates. For a point  $x = (x_1, \dots, x_m)$  in its supporting plane, put  $\beta = (x_k + x_l)/2$  and, for  $s \notin \{i, j, k, l\}$ , set  $\xi_s = x_s - \beta$ . At the reference face,  $x_i = x_j = \beta + 1$ ,  $x_k = x_l = \beta$ , and  $\xi \in [0, 1]^{m-4}$ . The condition  $\sum_s x_s = 0$  determines  $\beta$ . The tangent Gram matrix in these coordinates is  $I - \mathbf{1}_{m-4} \mathbf{1}_{m-4}^\top / m$ , of determinant  $4/m$ , so the old face has volume  $2/\sqrt{m}$ . Because  $\text{diag } X = 0$ , the new supporting plane is  $x_i + x_j - x_k - x_l = 2 + a - b - c$  and the four cross-root inequalities are  $x_p - x_q \leq 1 - X_{pq}$  for  $p \in \{i, j\}$  and  $q \in \{k, l\}$ . Set  $u = x_i - x_j$ ,  $w = x_k - x_l$ ,  $z_1 = u - w$ , and  $z_2 = u + w$ . These inequalities give interval widths  $2(c - a)$  and  $2(b - a)$  in  $z_1, z_2$ , while Euclidean transverse area is  $dz_1 dz_2 / 4$ . The area is therefore  $(b - a)_+(c - a)_+$ . Multiplying by the old face volume and by  $\|v\|/\text{covol}(A_d) = 2/\sqrt{m}$  gives the claimed coefficient. It remains to show that the error in (9.3) is uniform, including directions in which one of the two factors vanishes. Put  $\varepsilon = \|X\|$  and retain the finite candidate set from Lemma 7.2. The four cross-root inequalities give the transverse rectangle just computed, independently of  $\xi$ . Its area is  $O(\varepsilon^2)$  and all its transverse coordinates are  $O(\varepsilon)$ . The tangent directions in  $\xi$  are perpendicular to the two transverse directions, and their volume factor remains  $2/\sqrt{m}$ . For every point of this rectangle, the root inequalities between a free coordinate and the selected high and low coordinates force  $\xi \in [-C\varepsilon, 1 + C\varepsilon]^{m-4}$  for a fixed  $C$ . Conversely, apart from the four cross roots and the chosen new vector, every candidate inequality has positive slack in the relative interior of the old cube. Its reference slack is an affine function of  $\xi$ , nonnegative on the cube and positive in its interior. For a finite family of such functions, their slack on  $[C\varepsilon, 1 - C\varepsilon]^{m-4}$  dominates all the  $O(\varepsilon)$  changes of the support numbers and transverse coordinates, after  $C$  is increased. Thus this smaller cube is admissible in every transverse fiber. The admissible fiber volume differs from the old face volume by  $O(\varepsilon)$ , uniformly over the rectangle. Multiplication by its  $O(\varepsilon^2)$  area gives the  $O(\varepsilon^3)$  remainder in (9.3). When  $m = 4$ , there are no free coordinates and all remaining inequalities are uniformly slack. The same argument uses the zero-dimensional fiber volume one. If either rectangle width is nonpositive, the four cross-root inequalities permit no positive-area facet, so the positive-part formula remains valid.

For  $t > 0$ , sum these gap products over the three partitions of each four-element index set. The set  $\{1, 2, 3, 4\}$  contributes  $8t^2$ . For a set containing three of these indices and one other index, the three pair sums are  $-t, 0, t$  in some order, giving  $2t^2$ . There are  $4(m - 4)$  such sets. For a set containing two of these indices, a nonzero contribution occurs only for the pairs  $\{1, 2\}$  and  $\{3, 4\}$ , each giving  $t^2$  for every choice of the two remaining indices. The total is therefore

$$[8 + 8(m - 4) + (m - 4)(m - 5)]t^2 = (m^2 - m - 4)t^2.$$

Sets containing at most one of the four indices contribute zero. The new vectors of this order have length two, so their surface factor is  $1/\sqrt{2}$ . Theorem 7.3 gives

$$\log \frac{\mathcal{I}_d(L_t V_{I+X(t)})}{\mathcal{I}_d(V_I)} = \left\{ \frac{4}{2d+1} - 4 \left( 1 - \frac{1}{\sqrt{2}} \right) \left( 1 - \frac{4}{d(d+1)} \right) \right\} t^2 + O(t^3).$$

The coefficient is negative at  $d = 3$  and decreases with  $d$ , proving descent for all sufficiently small  $t > 0$ .

*Reducible lattices.* Write  $\Lambda = \Lambda_1 \perp \Lambda_2$  with both factors nonzero, grouping factors if necessary. Choose roots  $u \in \Lambda_1$  and  $v \in \Lambda_2$ , and set  $X = -(uv^\top + vu^\top)/4$  and  $H = I + tX$ . Then  $\text{tr } X = 0$ ,  $u^\top X v = -1$ , and all old inequalities of  $\mathcal{C}_H$  are unchanged. At  $I$ , the bisector of  $u + v$  supports the product ridge  $F_u \times F_v$ . Choose a compact patch  $K$  of positive  $(d - 2)$ -dimensional measure in its relative interior. The only tight nonzero lattice inequalities there are those for  $u$ ,  $v$ , and  $u + v$ . Equality for a vector with components in both factors requires equality in each factor separately. All other candidate inequalities

have uniform positive slack on  $K$ . For small  $t > 0$ , the points

$$z - \frac{a}{2}u - \frac{b}{2}v, \quad z \in K, \quad a, b \geq 0, \quad a + b = t,$$

therefore lie on the new facet, whose supporting equation is  $(u + v) \cdot x = 2 - t$ . The transverse segment has length  $t$  and is perpendicular to  $K$ . Hence  $w_{u+v}(H) \geq c_0 t$  for some  $c_0 > 0$ .

Lemma 7.2 gives  $w_e(H) = O(t)$  for every new facet. Expanding the exact surface formula and using cofactor balance yields

$$\frac{\mathcal{I}_d(V_H)}{\mathcal{I}_d(V_I)} = 1 - \frac{1}{d} \sum_{e \text{ new}} (1 - \eta_e) w_e + O(t^2), \quad \eta_e = \frac{\sqrt{2}}{\|v_e\|}.$$

Indeed, replacing every  $\eta_e$  by one gives  $1 + O(t^2)$ , since  $\mathfrak{D} = 1 + O(t^2)$ ,  $\sum_e w_e = \text{tr}(\text{adj } H) = d + O(t^2)$ , and  $\sum_e w_e e^\top X e = \text{tr}(X \text{adj } H) = O(t)$ . All terms in the correction are nonnegative, while the term for  $u + v$  is at least  $(1 - 1/\sqrt{2})c_0 t$ . This proves the asserted linear descent and completes the classification.  $\square$

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#### REFERENCES

- [1] N. Anari, V. Jain, F. Koehler, H. T. Pham, and T.-D. Vuong, *Entropic independence I: Modified log-Sobolev inequalities for fractionally log-concave distributions and high-temperature Ising models*, preprint, 2021, arXiv:2106.04105.
- [2] K. Bezdek, *Sphere packings revisited*, European J. Combin. **27** (2006), no. 6, 864–883. doi:10.1016/j.ejc.2005.05.001
- [3] K. J. Böröczky, E. Lutwak, D. Yang, and G. Zhang, *Affine images of isotropic measures*, J. Differential Geom. **99** (2015), no. 3, 407–442. doi:10.4310/jdg/1424880981
- [4] A. Cesaroni and M. Novaga, *Minimal periodic foams with equal cells*, in *Anisotropic Isoperimetric Problems and Related Topics*, Springer INdAM Ser. **62**, Springer, Singapore, 2024, 15–24. doi:10.1007/978-981-97-6984-1\_2
- [5] A. Cesaroni and M. Novaga, *Minimal periodic foams with fixed inradius*, Mathematika **71** (2025), no. 2, Paper No. e70020. doi:10.1112/mtk.70020
- [6] A. Cesaroni and M. Novaga, *Local minimality of the truncated octahedron for the isoperimetric problem on parallelohedra*, Boll. Unione Mat. Ital. (2026). doi:10.1007/s40574-026-00542-6
- [7] A. Cesaroni and M. Novaga, *The truncated octahedron minimizes surface area among parallelohedra of equal volume*, preprint, 2026, arXiv:2609.02384.
- [8] A. Cesaroni and M. Novaga, *The rhombic dodecahedral conjecture*, preprint, 2026, <https://cvgmt.sns.it/paper/7990/>
- [9] J. H. Conway and N. J. A. Sloane, *Voronoi regions of lattices, second moments of polytopes, and quantization*, IEEE Trans. Inform. Theory **28** (1982), no. 2, 211–226. doi:10.1109/TIT.1982.1056483
- [10] J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices and Groups*, third ed., Grundlehren der Mathematischen Wissenschaften 290, Springer-Verlag, New York, 1999. doi:10.1007/978-1-4757-6568-7
- [11] B. Delaunay, *Sur la partition régulière de l'espace à 4 dimensions. Première partie*, Bull. Acad. Sci. URSS, VII série, no. 1 (1929), 79–110.
- [12] B. Delaunay, *Sur la partition régulière de l'espace à 4 dimensions. Deuxième partie*, Bull. Acad. Sci. URSS, VII série, no. 2 (1929), 147–164.
- [13] P. de la Harpe and C. Pache, *Spherical designs and finite group representations (some results of E. Bannai)*, European J. Combin. **25** (2004), no. 2, 213–227. doi:10.1016/S0195-6698(03)00101-X
- [14] M. Deza and V. P. Grishukhin, *Voronoi's conjecture and space tiling zonotopes*, Mathematika **51** (2004), nos. 1–2, 1–10. doi:10.1112/S0025579300015461

- [15] M. Dutour Sikirić and V. P. Grishukhin, *The decomposition of the hypermetric cone into  $L$ -domains*, European J. Combin. **30** (2009), no. 4, 853–865. doi:10.1016/j.ejc.2008.08.003
- [16] M. Dutour Sikirić, D. A. Madore, P. Moustrou, and F. Vallentin, *Coloring the Voronoi tessellation of lattices*, J. London Math. Soc. **104** (2021), no. 3, 1135–1171. doi:10.1112/jlms.12456
- [17] A. I. Garber and A. N. Magazinov, *On Voronoi’s conjecture for four- and five-dimensional parallelehedra*, Russian Math. Surveys **77** (2022), no. 1, 174–176. doi:10.1070/RM10020
- [18] T. C. Hales, *The honeycomb conjecture*, Discrete Comput. Geom. **25** (2001), no. 1, 1–22. doi:10.1007/s004540010071
- [19] T. Hales and L. Song, *The Truncated Octahedral Conjecture*, preprint, 2026, arXiv:2609.00997.
- [20] Z. Lángi, *An isoperimetric problem for three-dimensional parallelohedra*, Pacific J. Math. **316** (2022), no. 1, 169–181. doi:10.2140/pjm.2022.316.169
- [21] R. Lyons, *Determinantal probability measures*, Publ. Math. Inst. Hautes Études Sci. **98** (2003), 167–212. doi:10.1007/s10240-003-0016-0
- [22] K. Mahler, *On lattice points in  $n$ -dimensional star bodies. I. Existence theorems*, Proc. Roy. Soc. London Ser. A **187** (1946), no. 1009, 151–187. doi:10.1098/rspa.1946.0072
- [23] P. McMullen, *Space tiling zonotopes*, Mathematika **22** (1975), no. 2, 202–211. doi:10.1112/S0025579300006082
- [24] P. McMullen, *Convex bodies which tile space by translation*, Mathematika **27** (1980), no. 1, 113–121. doi:10.1112/S0025579300010007
- [25] O. R. Musin, *Towards a proof of the 24-cell conjecture*, Acta Math. Hungar. **155** (2018), no. 1, 184–199. doi:10.1007/s10474-018-0828-5
- [26] E. Selling, *Ueber die binären und ternären quadratischen Formen*, J. Reine Angew. Math. **77** (1874), 143–229. doi:10.1515/crll.1874.77.143
- [27] L. Song, *Local quadratic isoperimetric stability of the truncated octahedron among parallelohedra*, preprint, 2026, arXiv:2609.03093.
- [28] B. A. Venkov, *On a class of Euclidean polytopes*, Vestnik Leningrad Univ. **9** (1954), no. 2, 11–31. (Russian)
- [29] G. Voronoi, *Nouvelles applications des paramètres continus à la théorie des formes quadratiques. Deuxième mémoire: Recherches sur les paralléloèdres primitifs*, J. Reine Angew. Math. **134** (1908), 198–287. doi:10.1515/crll.1908.134.198

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