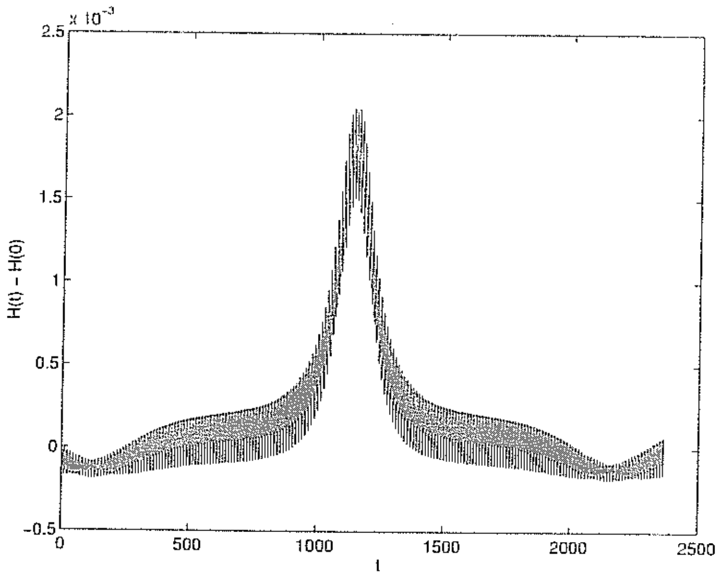

Figure 4.7: *Energy error for the Morse potential using leapfrog with* $h =$ 2.3684.

To evaluate $\mathbf{q}_n$ at any mesh point, the expression

$$\mathbf{q}_n = \frac{1}{2}(\mathbf{q}_{n-1/2} + \mathbf{q}_{n+1/2})$$

can be used.

Show that this method is explicit and second-order accurate.

(c) Integrate the Morse problem defined in the previous exercise using 1000 uniform steps h. Apply three methods: forward Euler, symplectic Euler, and leapfrog. Try the values $h = 2$, $h = 2.3684$, and $h = 2.3685$ and plot in each case the discrepancy in the Hamiltonian (which equals 0 for the exact solution). The plot for $h = 2.3684$ is given in Figure 4.7.

What are your observations? [The surprising increase in leapfrog accuracy from $h = 2.3684$ to $h = 2.3685$ relates to a phenomenon called *resonance instability*.]

[Both the symplectic Euler and the leapfrog method are *symplectic*—like the exact ODE they conserve certain volume projections for Hamiltonian systems (Section 2.5). We refer to [82, 50, 93] for much more on symplectic methods.]

4.12. The following classical example from astronomy gives a strong motivation to integrate initial value ODEs with error control.

Consider two bodies of masses $\mu = 0.012277471$ and $\hat{\mu} = 1 - \mu$ (earth and sun) in a planar motion, and a third body of negligible mass

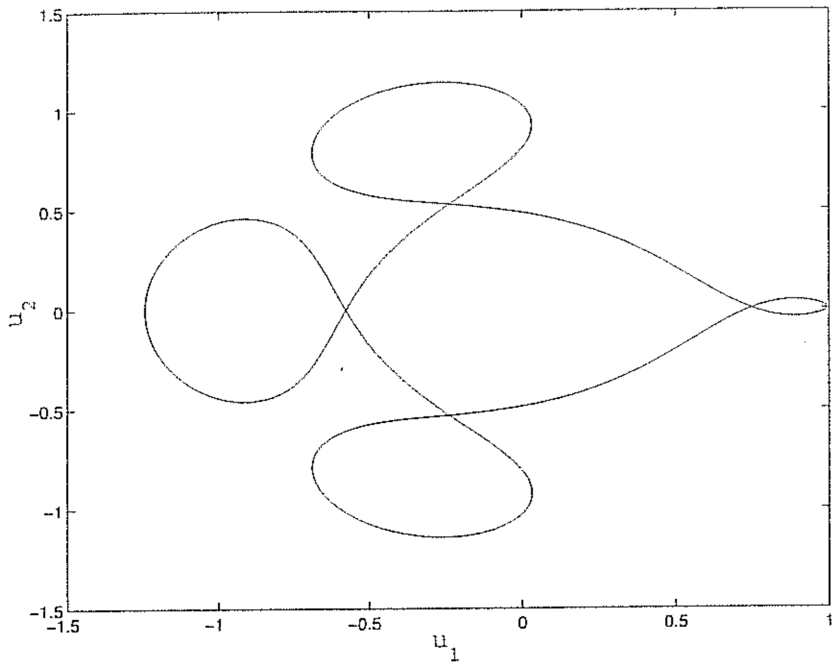

Figure 4.8: *Astronomical orbit using a Runge–Kutta* 4(5) *embedded pair method.*

(moon) moving in the same plane. The motion is governed by the equations

$$\begin{aligned} u_1'' &= u_1 + 2u_2' - \hat{\mu}\frac{u_1+\mu}{D_1} - \mu\frac{u_1-\hat{\mu}}{D_2}, \\ u_2'' &= u_2 - 2u_1' - \hat{\mu}\frac{u_2}{D_1} - \mu\frac{u_2}{D_2}, \\ D_1 &= ((u_1+\mu)^2 + u_2^2)^{3/2}, \\ D_2 &= ((u_1-\hat{\mu})^2 + u_2^2)^{3/2}. \end{aligned}$$

Starting with the initial conditions

$$u_1(0) = 0.994,\ u_2(0) = 0,\ u_1'(0) = 0,$$
$$u_2'(0) = -2.00158510637908252240537862224,$$

the solution is periodic with period < 17.1. Note that $D_1 = 0$ at $(-\mu, 0)$ and $D_2 = 0$ at $(\hat{\mu}, 0)$, so we need to be careful when the orbit passes near these singularity points.

The orbit is depicted in Figure 4.8. It was obtained using a 4(5) embedded pair with a local error tolerance $1.e-6$. This necessitated 204 time steps.

Using the classical Runge–Kutta method of order 4, integrate this problem on $[0, 17.1]$ with a *uniform* step size, using 100, 1000, 10,000,