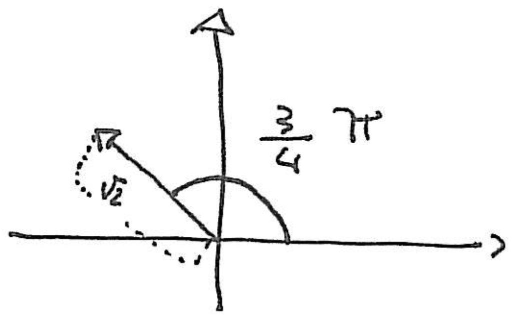
Traccia soluzioni compitino n° 1

23-2-2011

Esercizio 1

$$z_0 = \sqrt{2} \cdot e^{i\frac{3}{4}\pi}$$



modulo $= \sqrt{2}$

argomento $= \frac{3}{4}\pi$

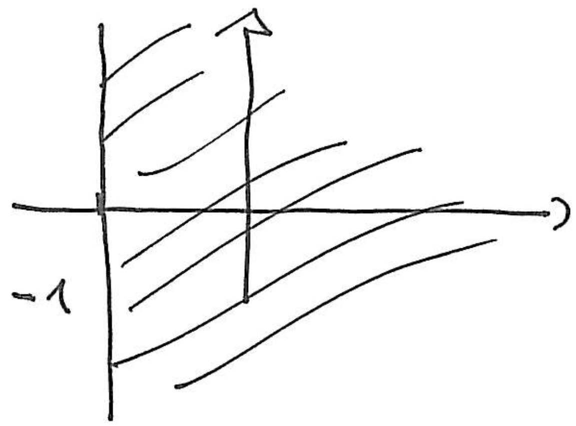
i) $z_0 = \sqrt{2} \cdot \left(\cos\left(\frac{3}{4}\pi\right) + i \operatorname{sen}\left(\frac{3}{4}\pi\right)\right) = \sqrt{2} \cdot \left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$

$= -1 + i$

ii) $z_0^{-1} = \frac{\overline{z_0}}{|z_0|^2} = \frac{-1-i}{2} = -\frac{1}{2} - i \cdot \frac{1}{2}$

iii) $\{z : Re(z) \geq Re(z_0)\} = \{z : Re(z) \geq -1\}$

$=$



iv) $w_0 = 1+i \Rightarrow z_0 \cdot w_0 = (-1+i)(1+i) = -2$

Esercizio 2

$$W = \left\langle \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle \subset \mathbb{R}^4$$

i) $\dim(W) \leq 3$

I 3 vettori sono lin. DIP.

$$\begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - 2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \dim(W) \leq 2$$

OPPURE: La matrice $(v_1\, v_2\, v_3) = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$

ha riga III = Riga IV = Riga I

quindi $\operatorname{rk}(v_1\, v_2\, v_3) = \operatorname{rk} \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}$

$\det \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \neq 0 \Rightarrow \operatorname{rk}(v_1 v_2 v_3) = 2$

CIOE' $\dim W = 2$

Base di $W = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$

ii) $\dim V = 2 = 4 - 2$

Sia $V = \langle v_1, v_2 \rangle$

cerchiamo v_1, v_2 t.c. $\det \begin{pmatrix} 1 & 0 & & \\ 1 & 1 & v_1 & v_2 \\ 1 & 0 & & \\ 1 & 0 & & \end{pmatrix} \neq 0$

ovvero $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, v_1, v_2 \right\}$ risulti essere base di $\mathbb{R}^4$

$\det \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \neq 0$ $\Rightarrow$ è suff. prendere

$v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

Esercizio 3

$$A_t = \begin{pmatrix} 1 & t & 1 \\ t & 0 & t \\ t & 0 & 1 \end{pmatrix}$$

i) $\det(A_t) = t^3 - t^2 = t^2(t-1)$

$\det(A_t) = 0 \iff t = \begin{cases} 0 \\ 1 \end{cases}$

$\Rightarrow$ Per $t \neq 0, 1$ $\det(A_t) \neq 0 \Rightarrow \begin{cases} rk(A_t) = 3 \\ \dim(\ker) = 0 \end{cases}$

Per $t = 0$:

$$A_0 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

il minore $M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ ha $\det \neq 0$

$\Rightarrow \begin{cases} rk(A_0) = 2 \\ \dim \ker = 3 - 2 = 1 \end{cases}$

Per $t = 1$:

$$A_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

il minore $M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ ha $\det \neq 0$

$\Rightarrow$ $rk(A_1) = 2$
$\dim \ker = 3 - 2 = 1$

ii) Per quanto visto in (i)

$$rk(A_t) = \begin{cases} 3 & \text{per } t \neq 0,1 \\ 2 & \text{per } t = 0,1 \end{cases}$$

$(A_t \vdots b)$ matrice 3×4 $\Rightarrow$

Per $t \neq 0, 1$ $\quad rk(A_t) = 3 = rk(A_t \vdots b)$

$\Rightarrow$ $\exists$ unica soluzione

$t = 0$

$$A_0 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad rk = 2$$

$$(A_0 \vdots b) = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 1 & 5 \end{pmatrix}$$ eliminiamo colonna $\underline{\text{II}}$

$$M = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 5 \\ 0 & 1 & 5 \end{pmatrix} \qquad \det M \neq 0$$

$\Rightarrow rk(A_0 \vdots b) = 3 > 2 = rk(A_0)$

$\Rightarrow$ Per $t = 0$ non $\exists$ soluzione

$t=1$:

$$A_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \qquad \text{rk} = 2$$

$$(A_1 \vdots b) = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 5 \\ 1 & 0 & 1 & 5 \end{pmatrix}$$

colonna I = col. III

$\Downarrow$ eliminiamo I colonna

$$\text{rk}(A_1 \vdots b) = \text{rk} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 5 \\ 0 & 1 & 5 \end{pmatrix}$$

$$\det \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 5 \\ 0 & 1 & 5 \end{pmatrix} = 0 \quad \Rightarrow \quad \text{rk}(A_1 \vdots b) = 2$$

OPPURE:

$$\begin{pmatrix} 0 \\ 5 \\ 5 \end{pmatrix} = 5 \cdot \left[\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right]$$

$\Rightarrow$ Per $t=1$ $\quad \text{rk}(A) = \text{rk}(A \vdots b) = 2$

$\Rightarrow \exists$ soluz. &

dim {soluzioni} = 3 − 2

iii) $t \neq 0, 1$ UNICA SOL.

$t = 1$ dim {SOL.} = 1

$t = 0$ nessuna SOL.

Esercizio 4

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

i) $A \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \iff A \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0_V$

$$\iff (A - Id)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\iff \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\iff \begin{cases} -x_1 + x_2 = 0 \\ x_1 - x_2 = 0 \end{cases}$$

$x_2 = t$

$x_1 = t$ SOL. : $\left\{ t \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\}$

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ii) $A \cdot B = B$

Poniamo $B = (b_1 \ \ b_2)$ 2 vettori colonna

$A \cdot B = B \iff A \cdot (b_1 \ \ b_2) = (b_1 \ \ b_2)$

SOLUZIONE: $b_1 = \begin{pmatrix} t \\ t \end{pmatrix}$

$b_2 = \begin{pmatrix} s \\ s \end{pmatrix}$ t, s PARAMETRI

$\boxed{\text{Per quanto visto in (i)}}$

CONCLUSIONE: $B = \begin{pmatrix} t & s \\ t & s \end{pmatrix}$ $t, s \in \mathbb{R}$

Esercizio 5.

$$W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : x_1 - x_2 + x_3 = 0 \right\}$$

$\dim(W) = 3 - 1 = 2$

Base di W: Poniamo $x_1 = t$

$x_3 = s$ $\longrightarrow$ $x_2 = x_1 + x_3 = s + t$

$$W = s \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Base di W: $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$

Per determinare f occorre un vettore v_3 t.c.

$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, v_3 \right\}$ Base di $\mathbb{R}^3$

Per esempio $v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\left(\det \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \neq 0 \; ! \right)$

- $f(W) \subseteq W \iff \begin{cases} f\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \in W \\ f\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \in W \end{cases}$

- $\dim \operatorname{Ker}(f) = 1 \iff \exists\, v \neq 0_V$ t.c. $f(v) = 0_V$
  e le sol. di $f(x) = 0_V$
  sono $x = t \cdot v$

$\boxed{\text{$f$ è univocamente determinata dal valore sui vettori della base}}$

$\Rightarrow$ Possiamo scegliere f tale che: $\begin{cases} 1)\; f\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \\ 2)\; f\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \\ 3)\; f\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{cases}$

$$\mathrm{Im}(f) = W$$

$$\mathrm{Ker}(f) = \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle$$

$$f\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\Leftrightarrow \quad A \cdot \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\Leftrightarrow \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cdot \left[\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}^{-1} \right]$$

<u>OPPURE</u>: Per determinare A t.c. $f = \mathcal{L}_A$ occorre determinare

$f\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} =$ I colonna di A

$f\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} =$ II colonna di A

$f\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} =$ III colonna di A

$$\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix} - \begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$\Downarrow$$

$$f\begin{pmatrix}1\\0\\0\end{pmatrix} = f\begin{pmatrix}1\\1\\0\end{pmatrix} - f\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix} - \begin{pmatrix}0\\0\\0\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix}$$

$$\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\1\end{pmatrix} - \begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$\Downarrow$$

$$f\begin{pmatrix}0\\0\\1\end{pmatrix} = f\begin{pmatrix}0\\1\\1\end{pmatrix} - f\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\1\\1\end{pmatrix}$$

CONCLUSIONE: $A = \left(f\begin{pmatrix}1\\0\\0\end{pmatrix} ; f\begin{pmatrix}0\\1\\0\end{pmatrix} ; f\begin{pmatrix}0\\0\\1\end{pmatrix} \right)$

$$A = \begin{pmatrix}1 & 0 & 0\\ 1 & 0 & 1\\ 0 & 0 & 1\end{pmatrix}$$