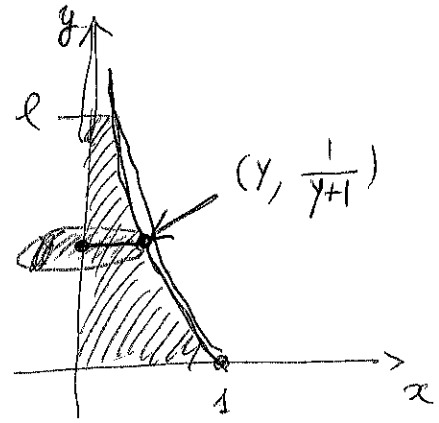
2A-(i)



$$x \geq 0$$

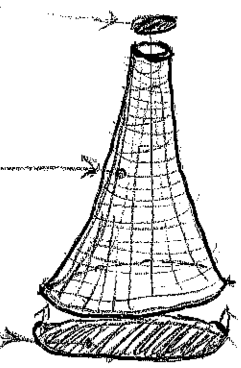
$$0 \leq y \leq \ell$$

$$y+1 \leq \frac{1}{x} \iff x \leq \frac{1}{y+1}$$

$$\mathrm{Vol}(V_\ell) = \int_0^\ell \left(\iint_{D_y} dx\,dz \right) dy = \pi \int_0^\ell \frac{1}{(y+1)^2}\, dy$$

$$= \pi \left[-\frac{1}{y+1} \right]_0^\ell = \pi \left[1 - \frac{1}{\ell+1} \right]$$

(ii) $\partial V_\ell = D_0 \cup \Sigma_\ell \cup D_\ell$



$$\phi : [0,\ell] \times [0, 2\pi] \longrightarrow \mathbb{R}^3$$

$$(u, \vartheta) \longrightarrow \begin{pmatrix} r(u)\cos\vartheta \\ u \\ r(u)\sin\vartheta \end{pmatrix} \quad \text{con} \quad r(u) := \frac{1}{u+1}$$

Pertanto

$$\mathrm{Area}(\Sigma_\ell) = 2\pi \int_0^\ell r(u) \sqrt{1 + (r'(u))^2}\, du$$

$$= 2\pi \int_0^\ell \frac{1}{u+1} \sqrt{1 + \left(\frac{1}{u+1}\right)^4}\, du$$

Cambio variabile, $t = \frac{1}{u+1}$, e ottengo

$$\text{Area}(\Sigma_\ell) = \pi \int_1^{(\ell+1)^2} \frac{\sqrt{t^2+1}}{t^2}\, dt$$

Per la giustificazione di questo passaggio, vedi versione B

$$= \pi \left[\log\left(\frac{(\ell+1)^2 + \sqrt{(\ell+1)^4+1}}{1+\sqrt{2}} \right) + \sqrt{2} - \sqrt{1+\frac{1}{(\ell+1)^4}} \right]$$

In definitiva

$$\text{Area}(\partial V_\ell) = \underbrace{\text{Area}(D_0)}_{\pi} + \text{Area}(\Sigma_\ell) + \underbrace{\text{Area}(D_\ell)}_{\pi\left(\frac{1}{\ell+1}\right)^2}$$

(iii) $\boxed{\lim_{\ell\to\infty} \text{Vol}(V_\ell) = \lim_{\ell\to\infty} \pi\left(1-\frac{1}{\ell+1}\right) = \pi}$

$$\text{Area}(\partial V_\ell) \geq \text{Area}(\Sigma_\ell) = \pi \int_1^{(\ell+1)^2} \frac{\sqrt{t^2+1}}{t^2}\, dt$$

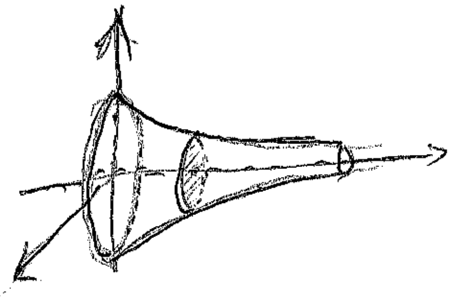
$$\geq \pi \int_1^{(\ell+1)^2} \frac{1}{t}\, dt = \pi\left(\log(\ell+1)^2 - \log(2)\right) \xrightarrow{\ell\to\infty} +\infty$$

$\boxed{\text{Quindi } \lim_{\ell\to\infty} \text{Area}(\partial V_\ell) = +\infty}$

2B - (i)

$$\mathrm{Vol}\,(V_\ell) = \int_0^\ell \left(\iint_{D_x} dy\,dz \right) dx$$

Dove D_x è la sezione di V_ℓ con un piano di ascissa x



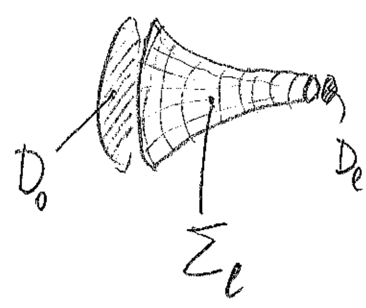
$$D_x := \left\{ (x,y,z) \in \mathbb{R}^3 : \quad y^2+z^2 = \left(\frac{2}{2x+1}\right)^2 \right\}$$

Pertanto

$$\mathrm{Vol}\,(V_\ell) = \pi \int_0^\ell \frac{4}{(2x+1)^2}\,dx = \pi \left[-\frac{2}{2x+1} \right]_0^\ell$$

$$= 2\pi \left(1 - \frac{1}{2\ell+1} \right) = \frac{4\pi\ell}{2\ell+1}$$

(ii) $\quad \partial V_\ell = D_0 \cup \Sigma_\ell \cup D_\ell$



Dove $\quad D_0 = \left\{ (x,y,z) \in \mathbb{R}^3 : \begin{array}{c} x = 0 \\ y^2+z^2 \leq 4 \end{array} \right\}$

$$D_\ell = \left\{ (x,y,z) \in \mathbb{R}^3 : \begin{array}{c} x = \ell \\ y^2+z^2 \leq \left(\frac{2}{2\ell+1}\right)^2 \end{array} \right\}$$

Mentre Σ_ℓ è la superficie laterale di V_ℓ ed è parametrizzata da

$$\phi : [0,\ell] \times [0, 2\pi] \longrightarrow R^3$$
$$(u, \vartheta) \longrightarrow \begin{pmatrix} u \\ r(u)\cos\vartheta \\ r(u)\sin\vartheta \end{pmatrix}$$

dove $r(u) := \frac{2}{2u+1}$

$$\mathrm{Area}(\Sigma_\ell) = 2\pi \int_0^\ell r(u)\sqrt{1+(r'(u))^2}\, du = 2\pi \int_0^\ell \frac{1}{u+1/2}\sqrt{1+\frac{1}{(u+1/2)^4}}\, du$$

Dopo il cambio di variabile $t = (u+\frac{1}{2})^2$ otteniamo

$$\boxed{\mathrm{Area}(\Sigma_\ell) = \pi \int_{1/4}^{(\ell+1/2)^2} \frac{\sqrt{t^2+1}}{t^2}\, dt} \quad \otimes$$

Per calcolare questo integrale conviene usare le funzioni iperboliche
$$\begin{cases} \sinh(x) := \frac{1}{2}(e^x - e^{-x}) \\ \cosh(x) := \frac{1}{2}(e^x + e^{-x}) \end{cases}$$

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$$\cosh^2(x) = 1 + \sinh^2(x) \qquad (*)$$

$$\cosh'(x) = \sinh(x); \quad \sinh'(x) = \cosh(x) \qquad (**)$$

$$\left(\frac{\cosh(x)}{\sinh(x)}\right)' = 1 - \frac{\cosh^2(x)}{\sinh^2(x)} \qquad (***)$$

$$\mathrm{arcsinh}(t) = \log(t + \sqrt{t^2+1}) - \sqrt{1+1/t^2}$$

Quindi

$$\int \frac{\sqrt{t^2+1}}{t^2}\,dt \overset{\boxed{t=\sinh(x)}}{=} \int \frac{\cosh(x)}{\sinh^2(x)} \cdot \overbrace{\cosh(x)\,dx}^{dt} = \int \frac{\cosh^2(x)}{\sinh^2(x)}\,dx =$$

per (***)

$$= x - \frac{\cosh(x)}{\sinh(x)}$$

e, ricordando che $x = \operatorname{arcsinh}(t)$ otteniamo,

$$\int \frac{\sqrt{t^2+1}}{t^2}\,dt = \log\left(t+\sqrt{t^2+1}\right) - \sqrt{1+1/t^2}$$

In definitiva,

$$\text{Area}(\partial V_\ell) = \text{Area}(D_0) + \text{Area}(\Sigma_\ell) + \text{Area}(D_\ell)$$

$$= 4\pi + \pi\left(\frac{2}{2\ell+1}\right)^2 +$$

$$+\pi\left[\log\left(\frac{(\ell+\frac{1}{2})^2+\sqrt{(\ell+\frac{1}{2})^4+1}}{1/4+\sqrt{17/16}}\right) + \sqrt{17} - \sqrt{1+(\ell+\tfrac{1}{2})^2}\right]$$

(iii)

$$\lim_{\ell\to\infty} \mathrm{Vol}(V_\ell) = \lim 2\pi\left(1-\frac{1}{2\ell+1}\right) = 2\pi$$

Per quanto riguarda l'area, osserviamo che

$$\mathrm{Area}(\partial V_\ell) \geq \mathrm{Area}(\Sigma_\ell) \underset{\text{per } \otimes}{=} \pi \int_{1/4}^{(\ell+1/2)^2} \frac{\sqrt{t^2+1}}{t^2}\,dt \geq$$

$$\geq \pi \int_{1/4}^{(\ell+1/2)^2} \frac{dt}{t} = 2\pi \log\left(\ell+\tfrac{1}{2}\right) + 2\pi\log 2 \xrightarrow{\ell\to\infty} \infty$$

Perché $\sqrt{t^2+1} \geq t$

Quindi

$$\lim_{\ell\to\infty} \mathrm{Area}(\partial V_\ell) = +\infty$$

N.B.: Per rispondere a questo punto non era necessaria la formula esplicita dell'area.