FCS
Math: Functions
Lesson 4 & 5

Massimo Caboara

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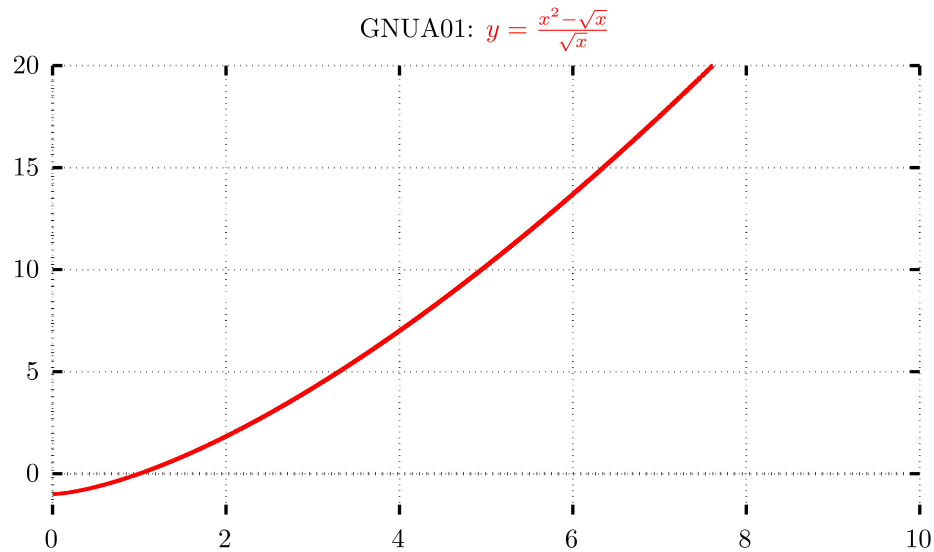
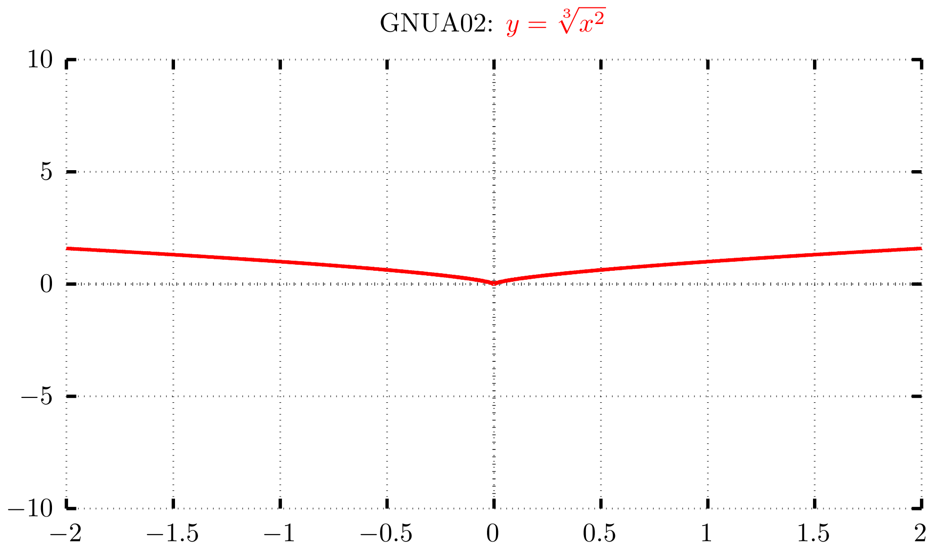
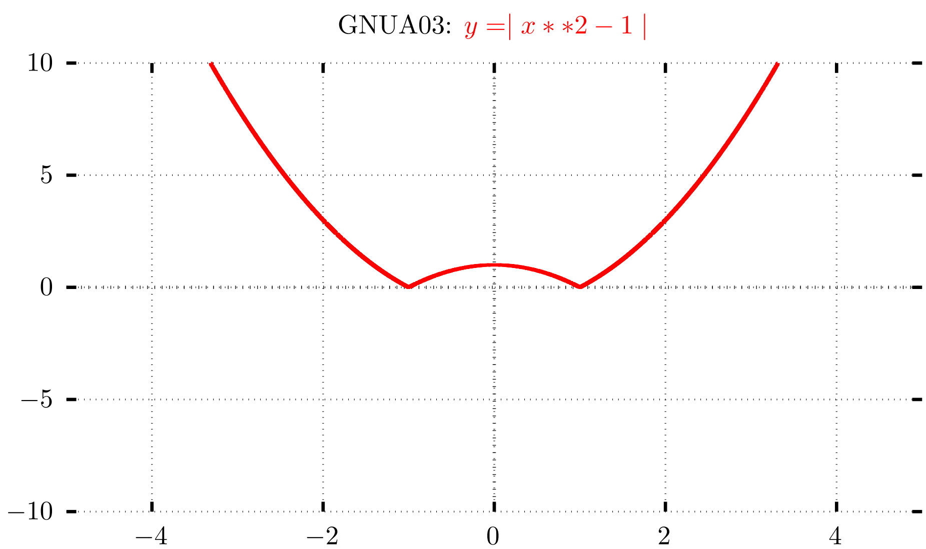
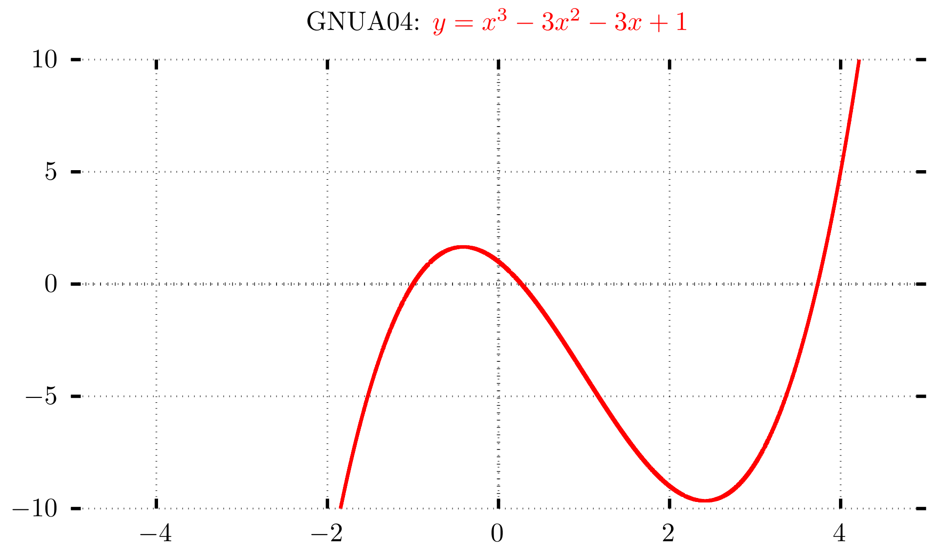
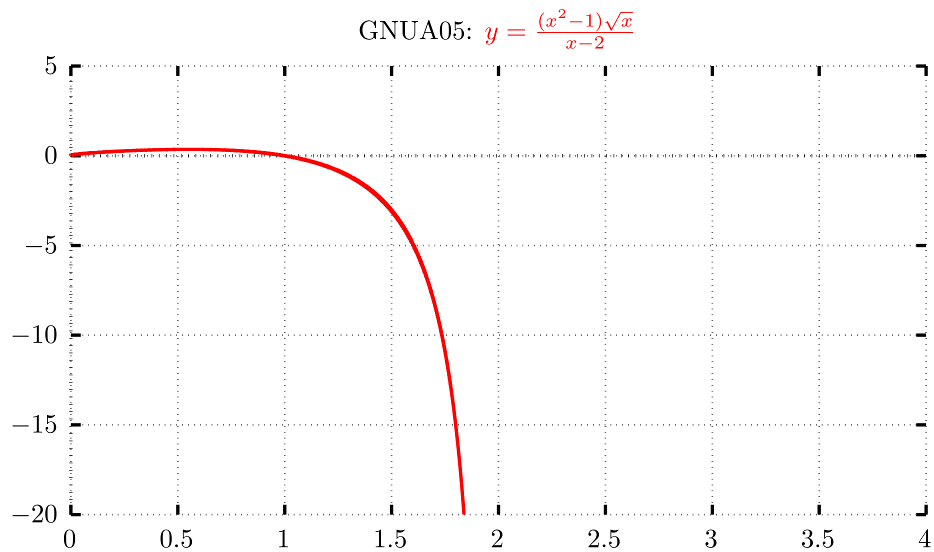
1 Lesson 3 Homework solutions




GNUA02: $y = \sqrt[3]{x^2}$
10
5
0
−5
−10
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2


GNUA03: $y =| x **2 - 1 |$
10
5
0
−5
−10
−4
−2
0
2
4





2 Lesson 4

2.1 Some definitions

Definition 1. *A function* $F : \mathbb{R} \longrightarrow \mathbb{R}$ *is*

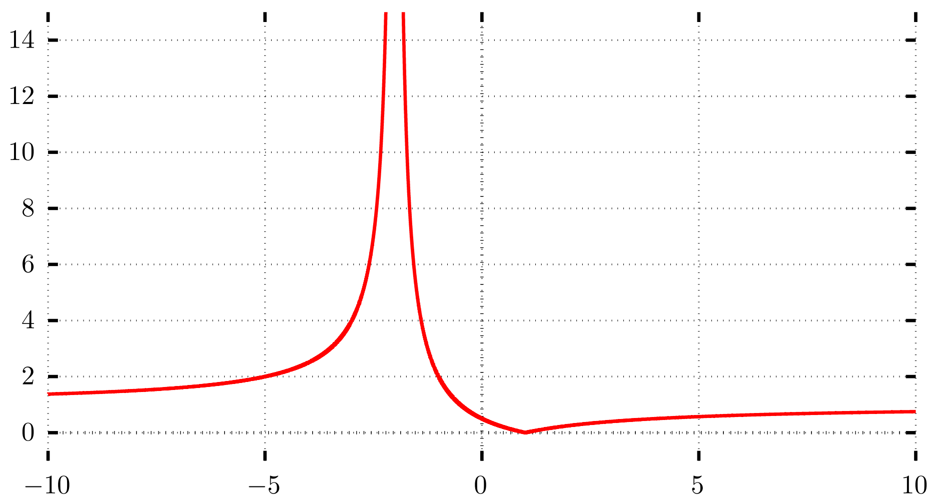
1. *EVEN if and only if for all* $x \in \mathbb{R}$ *we have* $F(x) = F(-x)$

2. *ODD if and only if for all* $x \in \mathbb{R}$ *we have* $F(x) = -F(-x)$, *or, equivalently,* $F(-x) = -F(x)$.

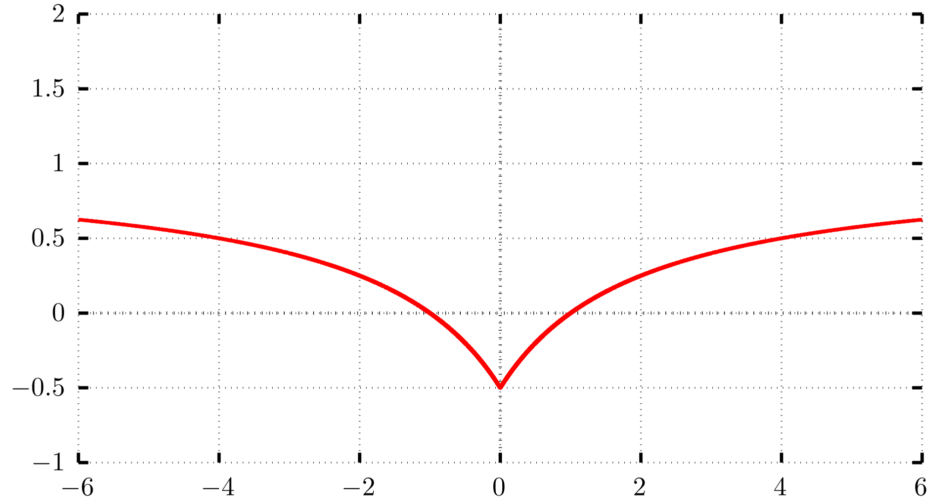
Definition 2. *A derivable function* $F : \mathbb{R} \longrightarrow \mathbb{R}$ *with derivable derivative* F' *is said the have a* positive concavity *in* $x_0 \in \mathbb{R}$ *if and only if* $F(x_0)'' > 0$. *A point* $\overline{x} \in \mathbb{R}$ *such that* $F(x_0)'' = 0$ *is called a* flex *for* F *if the sign of* F'' *changes in* $\overline{x}$.

2.2 Classwork solutions - First part

GNUA06: $y = \mid \frac{x-1}{x+2} \mid$



GNUA07: $y = \frac{|x|-1}{|x|+2}$



- If $F : \mathbb{R} \longrightarrow \mathbb{R},\ G : \mathbb{R} \longrightarrow \mathbb{R}$ are ODD, then $F \cdot G : \mathbb{R} \longrightarrow \mathbb{R}$ is EVEN, because

$$\begin{aligned} F \cdot G(x) &= F(x)G(x) \\ &= (-F(-x))(-G(-x)) \\ &= F(-x)G(-x) \\ &= F \cdot G(-x) \end{aligned}$$

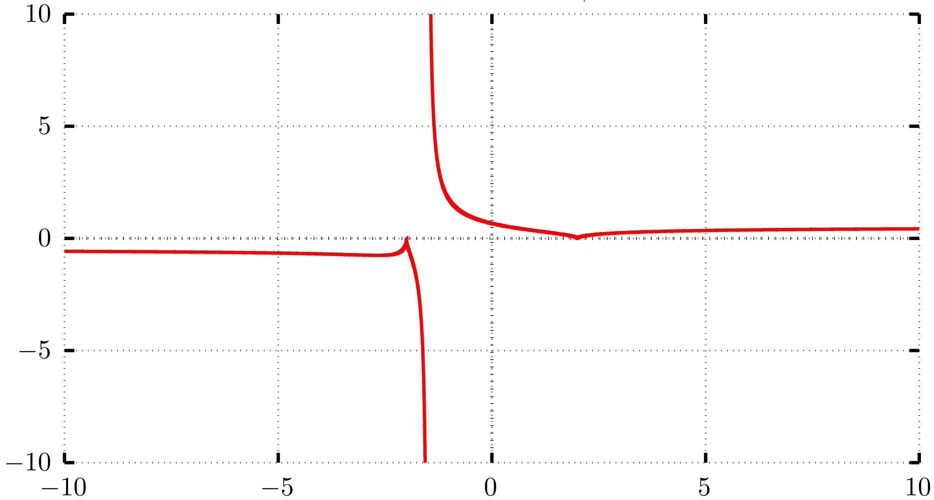
- If $F : \mathbb{R} \longrightarrow \mathbb{R},\ G : \mathbb{R} \longrightarrow \mathbb{R}$ are ODD, then $F \circ G : \mathbb{R} \longrightarrow \mathbb{R}$ is EVEN, because

$$\begin{aligned} F \circ G(x) &= F(G(x)) \\ &= F(-G(-x) \\ &= -F(G(-x) \\ &= -(F \circ G)(-x) \end{aligned}$$

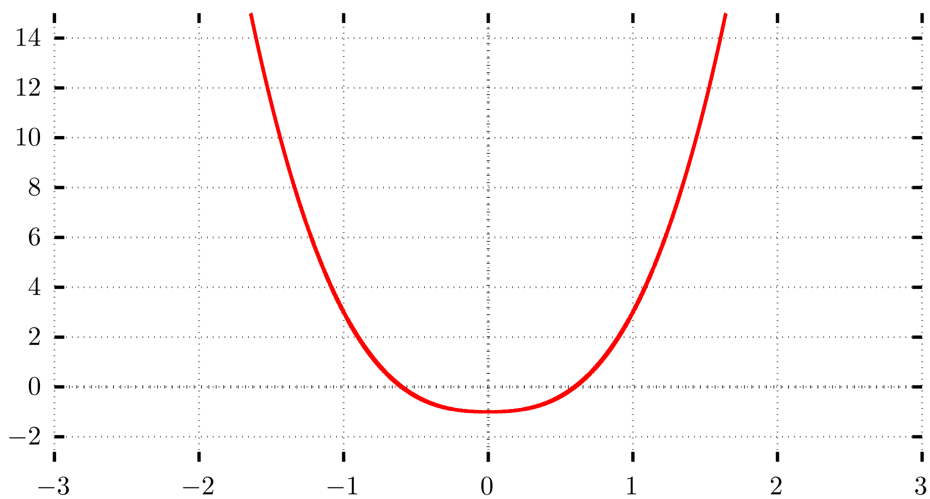
- $\left(\frac{x^2-2}{\sqrt{x}}\right)'' = \frac{3(x^2-2)}{4\sqrt{x^5}}$
- $\left(\mid \frac{x+3}{x-1} \mid\right)'' = \begin{cases} \frac{8}{x^3-3x^2+3x-1} & x \in [-3, 1) \\ -\frac{8}{x^3-3x^2+3x-1} & x \notin [-3, 1) \end{cases}$

2.3 Classwork solutions - Second part

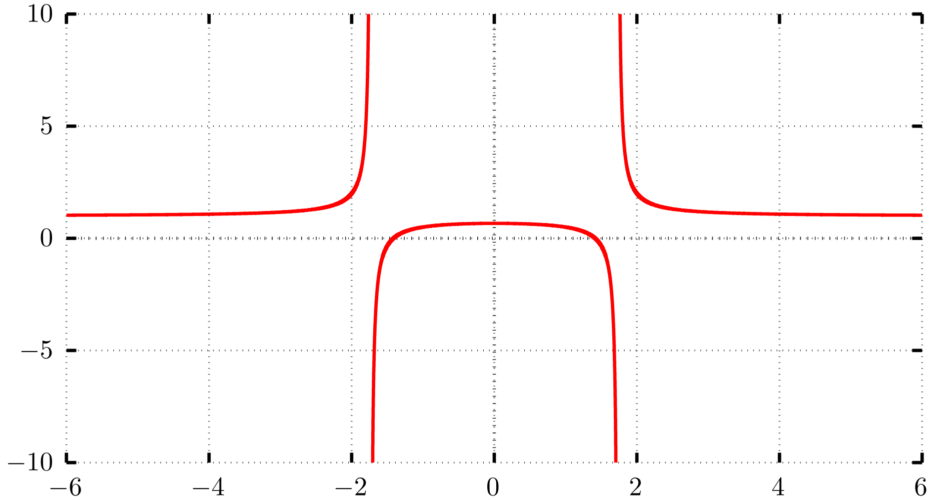
GNUA08: $y = \frac{\sqrt{|x^2-4|}}{2x+3}$



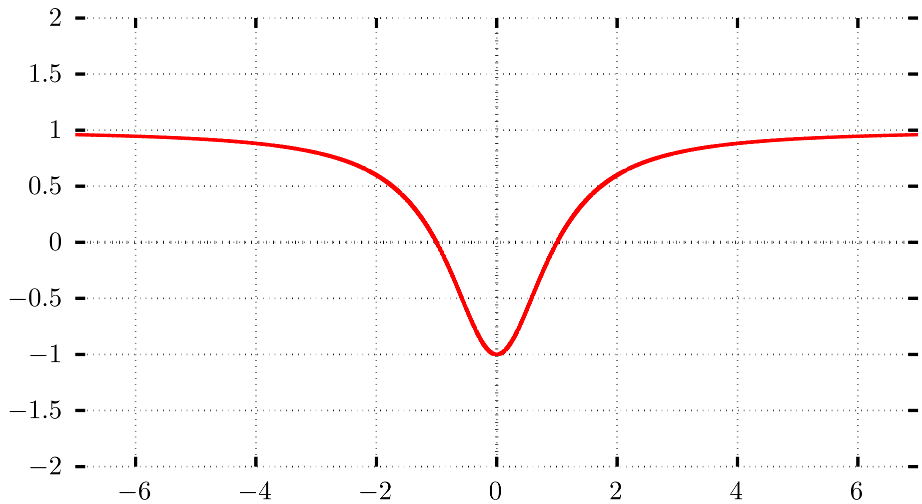
GNUA09: $y = 3 \mid x \mid^3 + x^2 - 1$



GNUA10: $y = \frac{x^2-2}{x^2-3}$



GNUA11: $y = \frac{x^3+x}{x^3-x}$



3 Lesson 5

3.1 Oblique asymptotes

Definition 3. *Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be a continous function for $x >> 0$*

$$r: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & mx+n \end{array} \qquad m \neq 0$$

an oblique line. Then we say that r is an oblique asymptote *for F, or that r is the limit of F at infinity if*

$$\lim_{x\to+\infty} F(x) - r(x) = 0$$

That is equivalent to

$$\lim_{x\to+\infty} \frac{F(x)}{x} = m \qquad \lim_{x\to+\infty} F(x) - mx = n$$

Example 1. *Determine the oblique asymptote of*

$$f: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & \sqrt{4x^2-2x} \end{array}$$

The function f is clearly continuous for $x >> 0$. We have that

$$\begin{aligned} \lim_{x\to+\infty} \frac{\sqrt{4x^2-2x}}{x} &= \lim_{x\to+\infty} \sqrt{\frac{4x^2}{x^2} - 2\frac{x}{x^2}} \\ &= \lim_{x\to+\infty} \sqrt{4+0} \\ &= 2 \end{aligned}$$

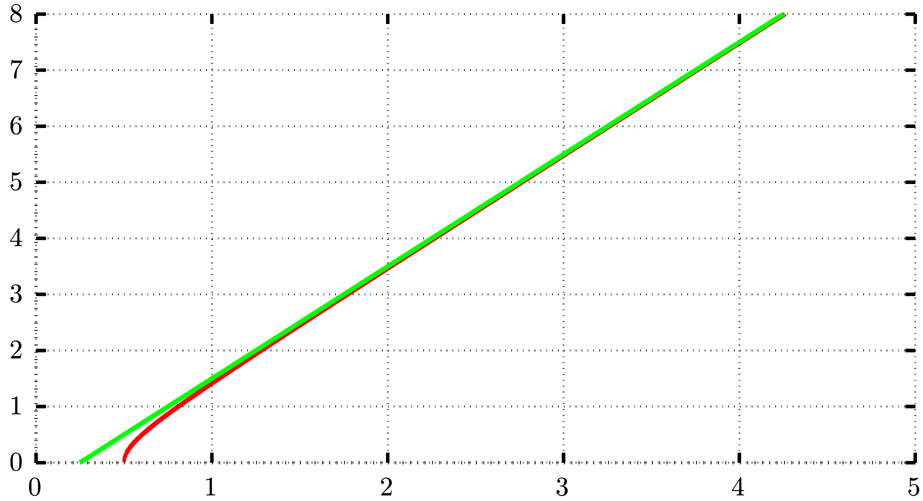
and

$$\begin{aligned} \lim_{x\to+\infty} \sqrt{4x^2-2x} - 2x &= \lim_{x\to+\infty} \frac{(\sqrt{4x^2-2x}-2x)(\sqrt{4x^2-2x}+2x)}{\sqrt{4x^2-2x}+2x} \\ &= \lim_{x\to+\infty} \frac{4x^2-2x-4x^2}{\sqrt{4x^2-2x}+2x} \\ &= \lim_{x\to+\infty} \frac{-2x}{\sqrt{4x^2-2x}+2x} \\ &= \lim_{x\to+\infty} \frac{-\frac{2x}{x}}{\sqrt{\frac{4x^2}{x^2}+\frac{2x}{x^2}}+\frac{2x}{x}} \\ &= \lim_{x\to+\infty} \frac{-2}{\sqrt{4+0}+2} \\ &= -\frac{1}{2} \end{aligned}$$

The onlique asymptote of f is

$$\begin{array}{rccc} r: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & 2x - \frac{1}{2} \end{array}$$

GNUA12: $y = \sqrt{4x^2 - 2x}$



Example 2. *Determine the oblique asymptote of*

$$\begin{array}{rccc} f: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & 3x + \frac{1}{x} \end{array}$$

The function f is clearly continuous for $x >> 0$. We have that

$$\begin{aligned} \lim_{x\to+\infty} \frac{3x + \frac{1}{x}}{x} &= \lim_{x\to+\infty} \frac{3x}{x} + \frac{1}{x^2} \\ &= \lim_{x\to+\infty} 3 + 0 = 3 \end{aligned}$$

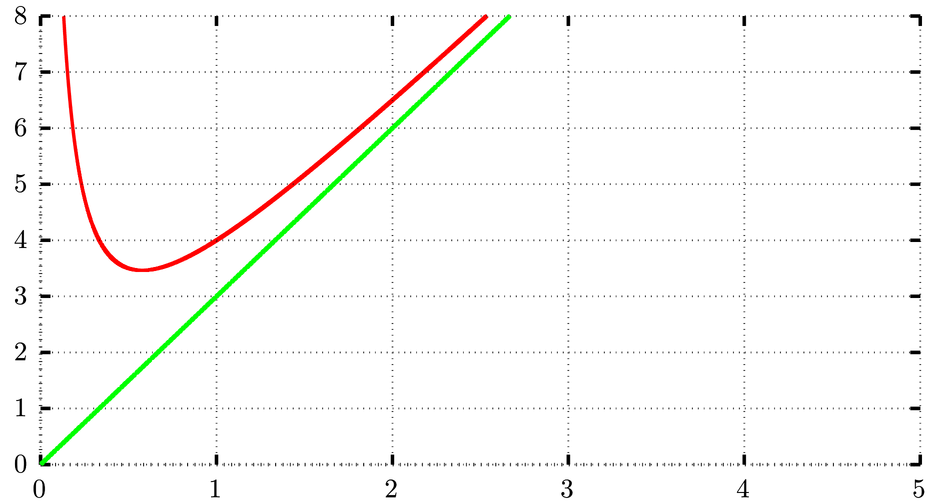
and

$$\lim_{x\to+\infty} 3x + \frac{1}{x} - 3x = \lim_{x\to+\infty} \frac{1}{x} = 0$$

The onlique asymptote of f is

$$\begin{array}{rccc} r: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & 3x \end{array}$$

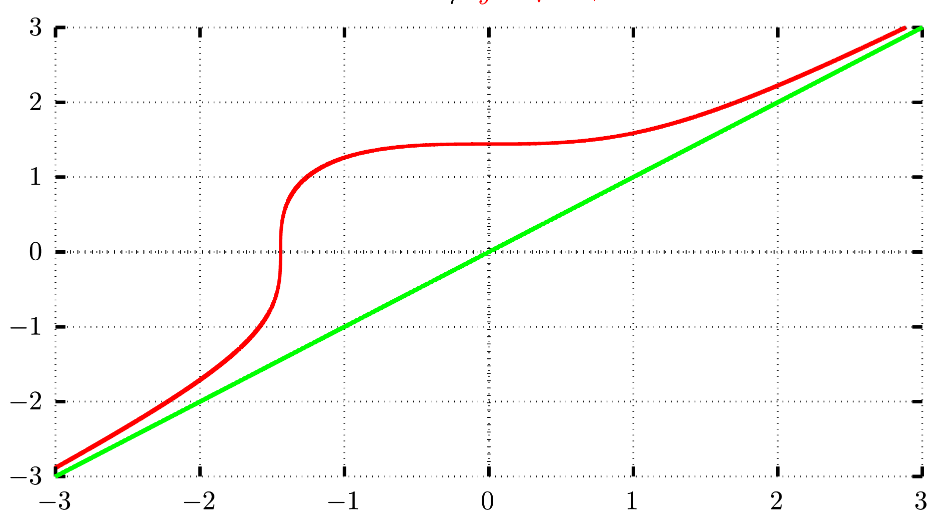
GNUA13: $y = 3x + \frac{1}{x}$



Example 3. *Draw the graph of the function*

$$\begin{array}{rccc} f: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & \sqrt[3]{x^3+3} \end{array}$$

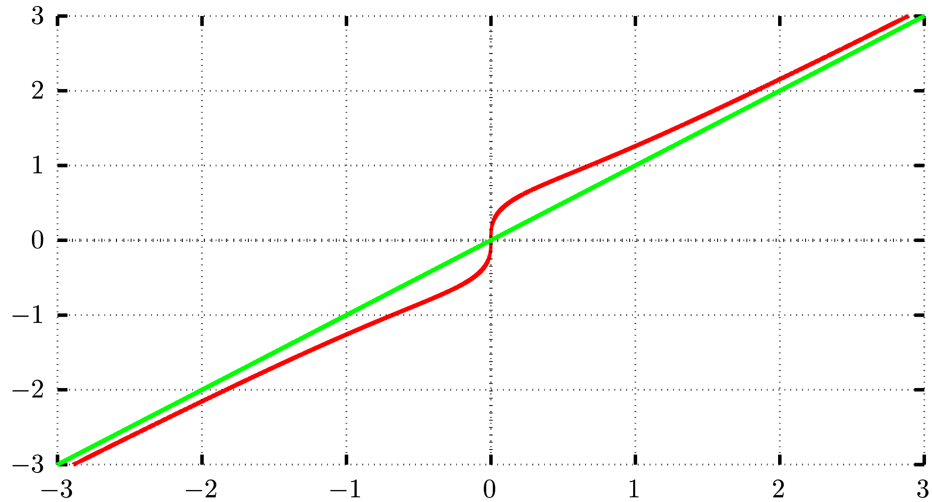
GNUA14: $y = \sqrt[3]{x^3+3}$



Example 4. *Draw the graph of the function*

$$\begin{array}{rccc} f: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & \sqrt[3]{x^3+x} \end{array}$$

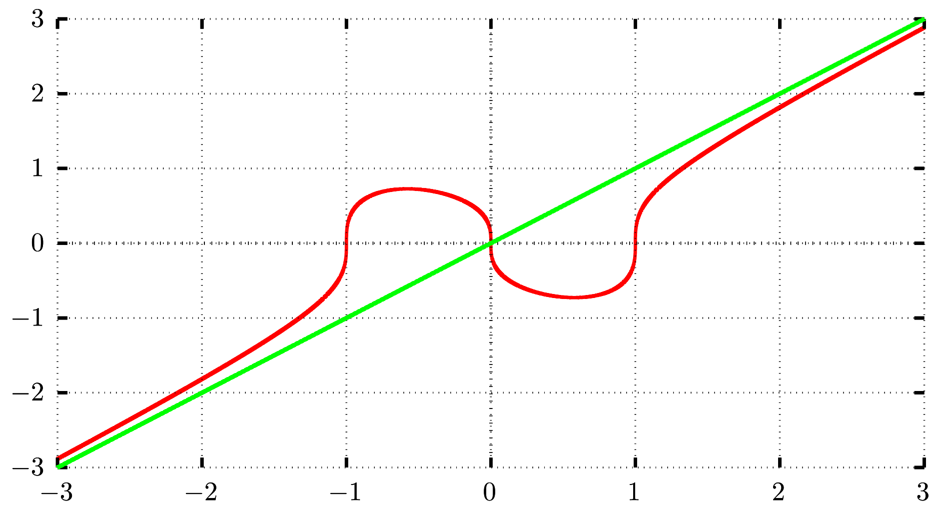
GNUA15: $y = \sqrt[3]{x^3 + x}$



Example 5. *Draw the graph of the function*

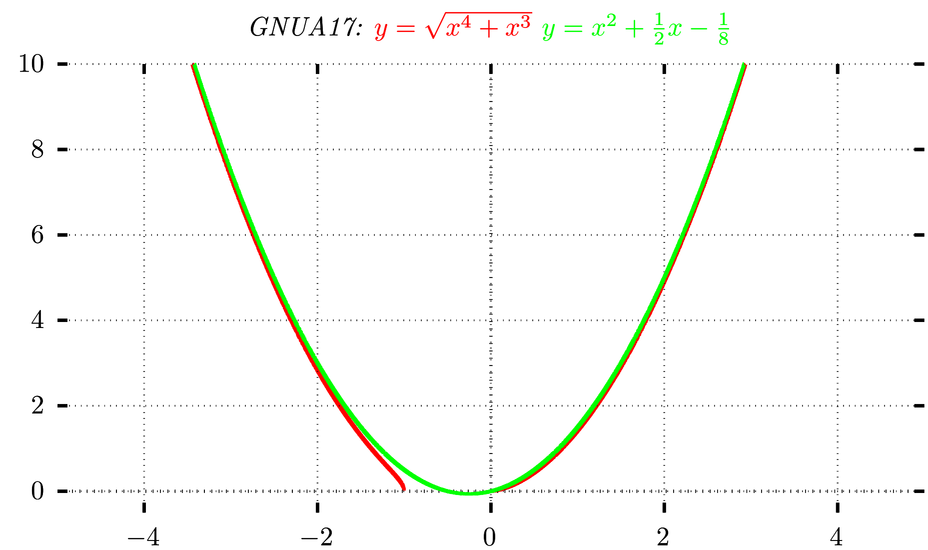
$$f: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & \sqrt[3]{x^3 - x} \end{array}$$

GNUA16: $y = \sqrt[3]{x^3 - x}$



Example 6. *Draw the graph of the function*

$$f: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & \sqrt{x^4 + x^3} \end{array}$$



Example 7. *Determine the oblique asymptote of*

$$\begin{array}{rccc} f: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & \frac{x^2-1}{x+3} \end{array}$$

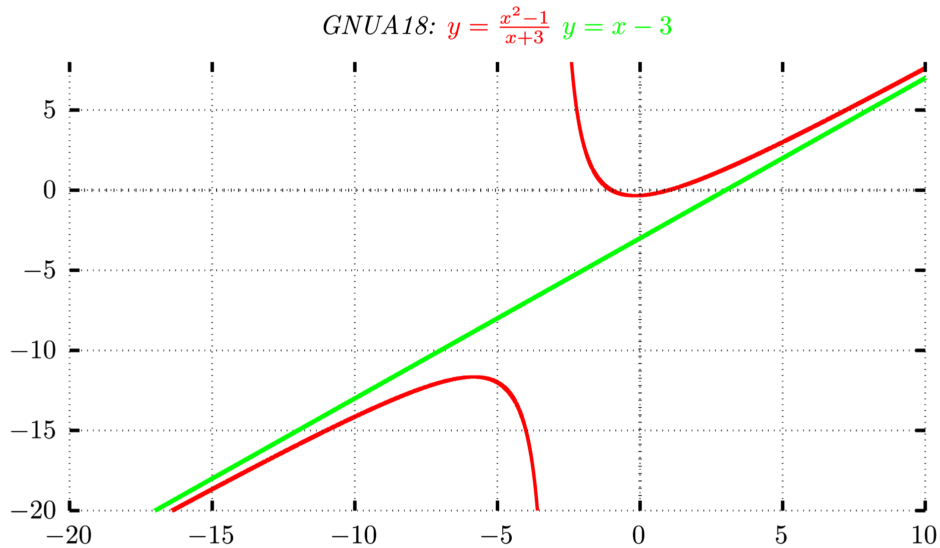
The function f is clearly continuous for $x >> 0$. Let us proceed with the polynomial division

$$\begin{array}{rr|l} x^2 & -1 & x+3 \\ \cline{3-3} -x^2-3x & & x-3 \\ \cline{1-1} -3x-1 & & \\ 3x+9 & & \\ \cline{1-2} & 8 & \end{array}$$

We have that $x^2 - 1 = (x+3)(x-3) + 8$ and so

$$\frac{x^2-1}{x+3} = \frac{(x+3)(x-3)+8}{x+3} = \frac{(x+3)(x-3)}{x+3} + \frac{8}{x+3} = (x-3) + \frac{8}{x+3}$$

It is clear that the asymptote of $\frac{x^2-1}{x+3}$ is $x - 3$



Example 8. *Determine the oblique asymptote of*

$$f: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & \frac{x^4-2x}{x^2+1} \end{array}$$

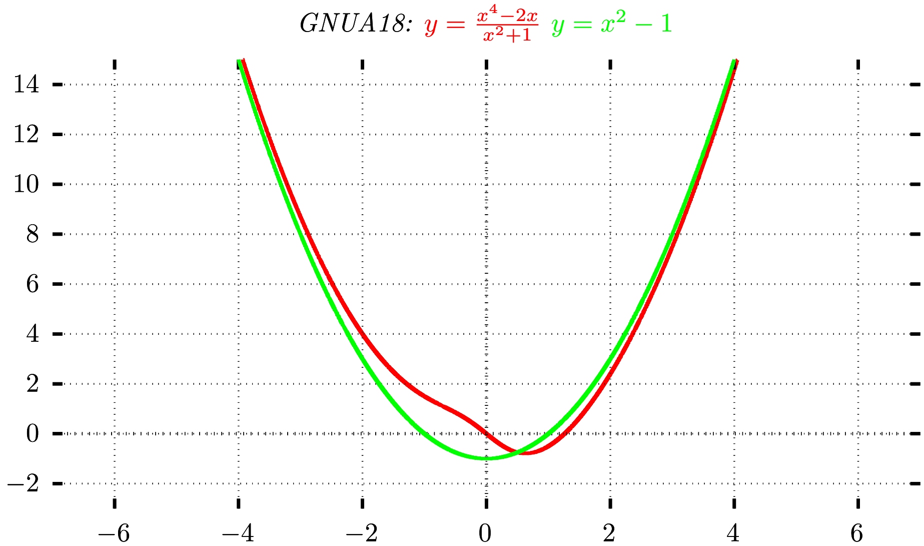
The function f *is clearly continuous for* $x >> 0$. *Let us proceed with the polynomial division*

$$\begin{array}{rr|l} x^4 & -2x & x^2+1 \\ \underline{-x^4 - x^2} & & x^2-1 \\ -x^2 & -2x & \\ \underline{x^2} & \underline{+1} & \\ & -2x+1 & \end{array}$$

We have that $x^4 - 2x = (x^2+1)(x^2-1) - 2x + 1$ *and so*

$$\frac{x^4-2x}{x^2+1} = \frac{(x^2+1)(x^2-1)-2x+1}{x^2+1} = x^2 - 1 + \frac{-2x+1}{x^2+1}$$

It is clear that the asymptote of $\frac{x^4-2x}{x^2+1}$ *is* $x^2 - 1$



GNUA18: $y = \frac{x^4-2x}{x^2+1}$ $y = x^2 - 1$
14
12
10
8
6
4
2
0
−2
−6
−4
−2
0
2
4
6