FCS
Math: Functions

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1 Sets

A quick review of the definitions for sets and set operations.

Definition 1. *Given A, B sets,*

- $A \subseteq B \iff$ *for all* $a \in A$ *we have* $a \in B$.
- $A = B \iff A \subseteq B$ *and* $A \supseteq B$.

It is easy to see that

- $A \not\subseteq B \iff$ *there is* $a \in A,\ a \notin B$.
- $A \neq B \iff$ (*there is* $a \in A,\ a \notin B$) *or* (*there is* $b \in B,\ b \notin A$).

We can write the previous definition using the symbols $\forall$ for "for all" and $\exists$ for "there exists":

Definition 2. *Given A, B sets,*

- $A \subseteq B \iff \forall\ a \in A$ *we have* $a \in B$.
- $A = B \iff A \subseteq B$ *and* $A \supseteq B$.

It is easy to see that

- $A \not\subseteq B \iff \exists\ a \in A,\ a \notin B$.
- $A \neq B \iff (\exists\ a \in A,\ a \notin B)$ *or* $(\exists\ b \in B,\ b \notin A)$.

Definition 3. *Given A, B subsets of a set U we have that*

- $A \cup B = \{c \in U \ \mid\ c \in A \text{ or } c \in B\}$
- $A \cap B = \{c \in U \ \mid\ c \in A \text{ and } c \in B\}$
- $A - B = \{c \in U \ \mid\ c \in A \text{ and } c \notin B\}$

Definition 4. *Given A, B sets $A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}$ is the* Cartesian product *of A, B.*

Example 1.

1. $\{1,2,3\} \times \{7,11\} = \{(1,7),(1,11),(2,7),(2,11),(3,7),(3,11)\}$
2. $\mathbb{N} \times \{1,3\} = \{(0,1),(1,1),(2,1),(3,1),\ldots,(0,3),(1,3),(2,3),(3,3),\ldots\}$
3. $\mathbb{R} \times \mathbb{R} = \{(x,y) \mid x,y \in \mathbb{R}\} = \mathbb{R}^2$ *the real plane.*

2 Exercises

Exercise 1. *Given the sets*

$$A = \{1,3,5,7,14\} \text{ and } B = \{-2,3,4,8\}$$

Describe $A \cup B$, $A \cap B$, $A - B$, $A \times B$.

Solution: The first solutions are trivial,

$$\begin{aligned} A \times B &= \{(1,-2),(1,3),(1,4),(1,8),(3,-2),(3,3),(3,4),(3,8),(5,-2),(5,3),(5,4),\\ &\quad (5,8),(7,-2),(7,3),(7,4),(7,8),(14,-2),(14,3),(14,4),(14,8)\} \end{aligned}$$

Exercise 2. *Given the sets*

$$A = \{a \in \mathbb{N} \mid a \text{ is a multiple of } 12\} \text{ and } B = \{k \in \mathbb{N} \mid k \text{ is a multiple of } 15\}$$

Describe $A \cup B$, $A \cap B$, $A - B$, $A \times B$.

Solution:

$$\begin{aligned} A \cup B &= \{12,15,24,30,36,45,48,60,72,75,\ldots\}\\ A \cap B &= \{k \in \mathbb{N} \mid k \text{ is a multiple of } lcm(12,15) = 60\}\\ A - B &= \{12,24,36,48,72,84,\ldots\}\\ A \times B &= \{(a,b) \in \mathbb{N}^2 \mid a \text{ is a multiple of } 12 \text{ and } b \text{ is a multiple of } 15\} \end{aligned}$$

Exercise 3. *Given the sets*

$$A = \{(a,a^2) \mid a \in \mathbb{R}\} \text{ and } B = \{(b,b) \mid b \in \mathbb{R}\}$$

Describe $A \cup B$, $A \cap B$, $A - B$, $A \times B$.

Solution:

The set A is the parabola $y = x^2$, the set B the line $y = x$. The set $A \cup B$ is the union of the two curves. The set $A \cap B$ is just the two points $\{(0,0),(1,1)\}$. The set $A - B$ is the parabola minus the points $\{(0,0),(1,1)\}$, and

$$A \times B = \{(a,a^2,b,b) \in \mathbb{R}^4 \mid a,b \in \mathbb{R}\}$$

Exercise 4. *Draw on the $\mathbb{R}^2$ plane the sets*

$$[1,2]\times[1,2],\ [-1,2]\times[2,+\infty),\ \{(1,1),(2,3),(3,7)\}$$

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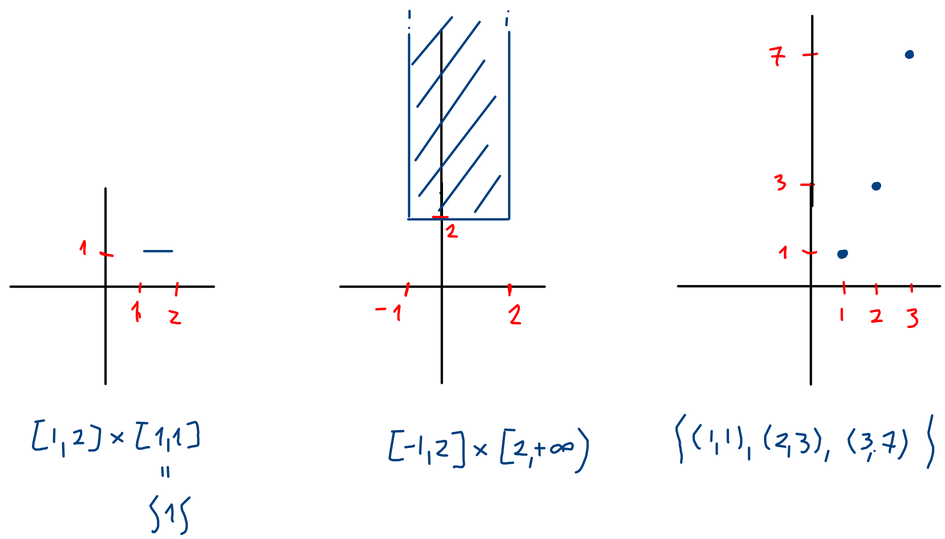


Figure 1:

Exercise 5. *Given the sets*

$$\begin{aligned} A &= \{a\in\mathbb{R} \mid x^3-4x^2+x+6=0\} \\ B &= \{a\in\mathbb{R} \mid x^4-5x^2+4=0\} \\ C &= \{a\in\mathbb{R} \mid x^4-2x^3+4x^2-6x+3=0\} \end{aligned}$$

Detail the equalities and inclusions between A, B, C

Solution:
We have that

$$\begin{aligned} x^3-4x^2+x+6 &= (x+1)(x-2)(x-3) \\ x^4-5x^2+4 &= (x+2)(x-2)(x+1)(x-1) \\ x^4-2x^3+4x^2-6x+3 &= (x^2+3)(x-1)^2 \end{aligned}$$

Hence

$$\begin{array}{rcl} A & = & \{-1, 2, 3\} \\ B & = & \{\pm 1, \pm 2\} \\ C & = & \{1\} \end{array}$$

Hence there are no equalities and the only inclusions are

$$C \subset A \text{ and } C \subset B$$

3 Functions

Definition 5. *For a function*

$$\begin{array}{rccc} F: & A & \longrightarrow & B \\ & a & \mapsto & F(a) \end{array},$$

we say that A is the domain *of F, B is the* codomain *of F and $F(a)$ is the* rule *or* formula *of F if for all $a \in A$, $F(a)$ is well-defined. Otherwise, we call F a mere* formula*.*

Example 2.

1. $\begin{array}{rccc} F: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & \frac{1}{x} \end{array}$, *is a mere formula, because $F(0) = \frac{1}{0}$ does not exist.*
2. $\begin{array}{rccc} F: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & x^2 - 1 \end{array}$, *is a function, because $F(x) = x^2 - 1$ always exists.*

Definition 6. *Given a formula $F : A \longrightarrow B$, the set*

$$\mathrm{EF}(F) = \{a \in A \mid F(a) \text{ exists}\}$$

is the field of definition *or* existence field *of F.*

Example 3.

1. *Given the formula* $\begin{array}{rccc} F: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & \frac{1}{x} \end{array}$ *we have* $\mathrm{ExF}(F) = \mathbb{R} - \{0\}$
2. *Given the formula* $\begin{array}{rccc} F: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & \frac{x}{x^2-4} \end{array}$ *we have* $\mathrm{ExF}(F) = \mathbb{R} - \{\pm 2\}$
3. *Given the formula* $\begin{array}{rccc} F: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & \sqrt{x} \end{array}$ *we have* $\mathrm{ExF}(F) = \mathbb{R}_0^+ = \{x \in \mathbb{R} \mid x \geq 0\}$

Exercise 6.

Given the formula

$$\begin{array}{rccc} F: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & x^3+2x^2-x+3 \end{array}$$

Find $\mathrm{ExF}(F)$.

Remark 1. *If* $F: A \longrightarrow B$ *is a formula,* $F: \mathrm{ExF}(F) \longrightarrow B$ *is a function.*

Definition 7. *Given the functions*

$$\begin{array}{rccc} F: & A & \longrightarrow & B \\ & a & \mapsto & F(a) \end{array}, \qquad \begin{array}{rccc} G: & C & \longrightarrow & D \\ & c & \mapsto & G(c) \end{array}$$

we have

$$F \equiv G$$

if and only if

$$A = C, \;\; B = D, \;\; \forall\, a \in A, \;\; F(a) = G(a)$$

We say that F and G are equal as functions.

Example 4.

1. *Given*

$$\begin{array}{rccc} F: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & a & \mapsto & \sin^2 a + \cos^2 a \end{array}, \qquad \begin{array}{rccc} G: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & c & \mapsto & 1 \end{array}$$

   we have $F \equiv G$ *because for every* $x \in \mathbb{R}$ *we have*

$$F(x) = \sin^2 x + \cos^2 x = 1 = G(x)$$

2. *Given*

$$\begin{array}{rccc} F: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & \sqrt{x^2} \end{array}, \qquad \begin{array}{rccc} G: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & x \end{array}$$

   we have $F \not\equiv G$ *because for* $x = -1$ *we have*

$$F(-1) = \sqrt{(-1)^2} = \sqrt{1} = 1 \neq -1 = G(-1)$$

Exercise 7. *We have the function*

$$\begin{array}{rccc} F: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & 3x^2-x+1 \end{array}$$

Compute

1. $F(1) = F(-1)$*? [* $F(-1) = 5 \neq 3 = F(1)$ *]*
2. $F(1), F(0), F(5)$*.* *[*$F(1) = 3, \;\; F(0) = 1, \;\; F(5) = 71$*]*

3. *Given $a \in \mathbb{R}$, compute $F(a-1), F(3a^2-2), F(\sqrt{a^2+1})$.*

$$\begin{aligned} F(a-1) &= 3(a-1)^2-(a-1)+1 = 3a^2-7a+5 \\ F(3a^2-2) &= 3(3a^2-2)^2-(3a^2-2)+1 = 27a^4-39a^2+15 \\ F(\sqrt{a^2+1}) &= 3(\sqrt{a^2+1})^2-(\sqrt{a^2+1})+1 = 3a^2-\sqrt{a^2+1}+4 \end{aligned}$$

4. *Given $\clubsuit \in \mathbb{R}$, with $\clubsuit > 0$, compute $F(\clubsuit-2)$, $F(2\clubsuit)$, $F(\sqrt{\clubsuit})$.*

$$F(\clubsuit-2) = 3\clubsuit^2-13\clubsuit+15,\ F(2\clubsuit) = 12\clubsuit^2-2\clubsuit+1,\ F(\sqrt{\clubsuit}) = 3\clubsuit-\sqrt{\clubsuit}+1$$

Exercise 8. *We have the function*

$$F: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & x^2+1 \end{array}$$

Compute

1. *$F(1) = F(-1)$? [YES]*
2. *For which $a \in \mathbb{R}$ do we have $F(a) = F(-a)$? [Solutions of $a^2+1 = (-a)^2+1$, so for all $a \in \mathbb{R}$]*
3. *For which $y \in \mathbb{R}$ do we have $F(y) = F(y+1)$? [Solutions of $y^2+1 = (y+1)^2+1$, so $y = -1/2$]*
4. *For which $b \in \mathbb{R}$ do we have $F(b+2) = F(2b+3)$? [Solutions of $(b+2)^2+1 = (2b+3)^2+1$, so $b = -5/3, -1$]*

Exercise 9. *We have the functions*

$$F: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & x^2 \end{array}, \qquad G: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & x \end{array}$$

1. *Is it true that $F \equiv G$? [NO, there is an $a \in \mathbb{R}$ such that $F(a) \neq G(a)$, for example $a = 2$, for which $F(2) = 4 \neq 2 = G(2)$]*
2. *Is there an $a \in \mathbb{R}$ such that $F(a) = G(a)$? [YES, for example $a = 1$, since $F(1) = 1 = G(1)$]*

Exercise 10. *We try to describe a function by*

$$F: \begin{array}{ccc} \mathbb{Q} & \longrightarrow & \mathbb{Q} \\ p/q & \mapsto & p^2/q^2 \end{array}$$

Is F a well-defined function?

Solution: YES.

Exercise 11. *We try to describe a function by*

$$F: \begin{array}{ccc} \mathbb{Q} & \longrightarrow & \mathbb{Q} \\ p/q & \mapsto & p+q \end{array}$$

Is F a well-defined function? If not, how can we modify the formula of F to have a well-defined function?

Solution: NO, because $\frac{6}{3} = \frac{4}{2}$ *but* $F\left(\frac{6}{3}\right) = 9 \neq 6 = F\left(\frac{4}{2}\right)$

4 Injective and Surjective Functions

Definition 8. *Given the sets A, B, the function*

$$F: \begin{array}{ccc} A & \longrightarrow & B \\ a & \mapsto & F(a) \end{array}$$

is injective *if and only if* $\forall b \in B$ *the equation $F(a) = b$ has at most 1 solution in A or, equivalently, if any element of B is reached by at most one element of A through F. More precisely, if*

$$\forall\ a, b \in A \quad F(a) = F(b) \Longleftrightarrow a = b$$

Definition 9. *Given the sets A, B, the function*

$$F: \begin{array}{ccc} A & \longrightarrow & B \\ a & \mapsto & F(a) \end{array}$$

is surjective *if and only if* $\forall b \in B$ *the equation $F(a) = b$ has at least 1 solution in A or, equivalently, if any element of B is reached by at least one element of A through F.*

Definition 10. *Given a function*

$$F: \begin{array}{ccc} A & \longrightarrow & B \\ a & \mapsto & F(a) \end{array}$$

the set

$$Im(F) = \{F(a) \ \mid \ a \in A\} = \{b \in B \ \mid \ \exists\ a \in A \text{ such that } b = F(a)\} \subseteq B$$

is the image *or* range *of F.*

Exercise 12. *Are the following functions injective?*

1.
$$F: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & F(x) = 3 - 2x \end{array}$$

Yes, because $\forall\ b \in \mathbb{R}$ the equation $F(x) = b \Longleftrightarrow 3 - 2x = b$ has exactly the solution $x = \frac{3-b}{2}$

2.

$$\begin{array}{rccc} F: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & F(x) = x^4 + 3x^2 + 3 \end{array}$$

No, because $F(1) = F(-1)$

3.

$$\begin{array}{rccc} F: & \mathbb{N} & \longrightarrow & \mathbb{N} \\ & n & \mapsto & F(n) = n^2 \end{array}$$

Yes, because $F(a) = F(b) \Longleftrightarrow a^2 = b^2 \Longleftrightarrow a = b$ *for natural numbers.*

4.

$$\begin{array}{rccc} F: & \mathbb{Z} & \longrightarrow & \mathbb{Z} \\ & n & \mapsto & F(n) = 3n^4 + n^2 \end{array}$$

No, because $F(-1) = F(1)$

5.

$$\begin{array}{rccc} F: & \{1,2,3,4,5,6\} & \longrightarrow & \{1,2,3,4,5,6\} \\ & 1 & \mapsto & 2 \\ & 2 & \mapsto & 4 \\ & 3 & \mapsto & 1 \\ & 4 & \mapsto & 6 \\ & 5 & \mapsto & 5 \\ & 6 & \mapsto & 3 \end{array}$$

Yes

6.

$$\begin{array}{rccc} F: & \{1,2,3,4,5,6\} & \longrightarrow & \{1,2,3,4,5,6\} \\ & 1 & \mapsto & 2 \\ & 2 & \mapsto & 4 \\ & 3 & \mapsto & 1 \\ & 4 & \mapsto & 2 \\ & 5 & \mapsto & 5 \\ & 6 & \mapsto & 3 \end{array}$$

No

Exercise 13. *Are the following functions surjective?*

1.

$$\begin{array}{rccc} F: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & F(x) = 3x + 1 \end{array}$$

Yes, because $\forall\ b \in \mathbb{R}$ *the equation* $F(x) = b \Longleftrightarrow 3x + 1 = b$ *has at least the solution* $x = \frac{b-1}{3}$

2.

$$\begin{array}{rccc} F: & \mathbb{N} & \longrightarrow & \mathbb{N} \\ & n & \mapsto & F(n) = n^2 \end{array}$$

No, because 3 *is not in the image. More precisely, because the equation* $F(n) = 3 \Longleftrightarrow n^2 = 3$ *has no solution in the natural numbers.*

3.

$$\begin{array}{rccc} F: & \{1,2,3,4,5,6\} & \longrightarrow & \{1,2,3,4,5,6\} \\ & 1 & \mapsto & 2 \\ & 2 & \mapsto & 4 \\ & 3 & \mapsto & 1 \\ & 4 & \mapsto & 6 \\ & 5 & \mapsto & 5 \\ & 6 & \mapsto & 3 \end{array}$$

Yes

4.

$$\begin{array}{rccc} F: & \{1,2,3,4,5,6\} & \longrightarrow & \{1,2,3,4\} \\ & 1 & \mapsto & 2 \\ & 2 & \mapsto & 1 \\ & 3 & \mapsto & 1 \\ & 4 & \mapsto & 2 \\ & 5 & \mapsto & 2 \\ & 6 & \mapsto & 3 \end{array}$$

No, because 4 *is not in the image.*

Exercise 14. *Let A be a set with* 10 *elements and B a set with* 9 *elements. Is it possible for a function $f : A \longrightarrow B$ to be injective ? [No]*

Exercise 15. *Let A be a set with* 10 *elements and B a set with* 11 *elements. Is it possible for a function $f : A \longrightarrow B$ to be surjective ? [No]*

Definition 11. *Given the function*

$$\begin{array}{rccc} F: & A & \longrightarrow & B \\ & a & \mapsto & F(a) \end{array}$$

and $C \subset A$, the function

$$\begin{array}{rccc} G: & C & \longrightarrow & B \\ & a & \mapsto & F(a) \end{array}$$

is called the restriction *of F to C and is written $F_{|C}$*

5 Function Composition. Inverse

Definition 12. *Given two functions*

$$\begin{array}{rccc} F: & A & \longrightarrow & B \\ & a & \mapsto & F(a) \end{array}, \quad \begin{array}{rccc} G: & D & \longrightarrow & C \\ & d & \mapsto & G(d) \end{array}$$

such that $D \subseteq Im(F)$ the function

$$\begin{array}{rccc} G \circ F: & A & \longrightarrow & C \\ & a & \mapsto & G(F(a)) \end{array}$$

is the composition *of G and F.*

Remark 2. *If we have the two functions*

$$F: \begin{array}{ccc} A & \longrightarrow & A \\ a & \mapsto & F(a) \end{array}, \qquad G: \begin{array}{ccc} A & \longrightarrow & A \\ a & \mapsto & G(a) \end{array}$$

it may happen that $F \circ G \not\equiv G \circ F$*. Consider for example*

$$F: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & F(x) = 5x - 2 \end{array}, \qquad G: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & G(x) = x^2 - 1 \end{array}$$

It is easy to verify that for most $x \in \mathbb{R}$

$$\begin{aligned} F \circ G(x) &= F(G(x)) \\ &= F(x^2 - 1) \\ &= 5(x^2 - 1) - 2 \\ &= 5x^2 - 7 \end{aligned}$$

and

$$\begin{aligned} G \circ F(x) &= G(F(x)) \\ &= G(5x - 2) \\ &= (5x - 2)^2 - 1 \\ &= 25x^2 - 20x + 3 \end{aligned}$$

and it is immediate to see that there exists at least one x *(for example* 0*) such that*

$$5x^2 - 7 \neq 25x^2 - 20x + 3$$

Hence $F \circ G \not\equiv G \circ F$

Definition 13. *The function*

$$\mathrm{id}_A: \begin{array}{ccc} A & \longrightarrow & A \\ a & \mapsto & \mathrm{id}_A(a) = a \end{array}$$

is called the identity *on* A*.*

Definition 14. *Given a function*

$$F: \begin{array}{ccc} A & \longrightarrow & B \\ a & \mapsto & F(a) \end{array}$$

if there is a function

$$G: \begin{array}{ccc} B & \longrightarrow & A \\ b & \mapsto & G(b) \end{array}$$

such that

$$F \circ G \equiv \mathrm{id}_B \text{ and } G \circ F \equiv \mathrm{id}_A$$

we say that F is invertible, G is the inverse of F *and we write* $G \equiv F^{-1}$

Proposition 1. *A function*

$$F: \begin{array}{ccc} A & \longrightarrow & B \\ a & \mapsto & F(a) \end{array}$$

is invertible if and only if

$$\forall\ b \in B\ \exists! a \in A \text{ such that } F(a) = b$$

or, rephrased, for every $b \in B$ the equation $F(a) = b$ has exactly one solution in A. In other words, a function is invertible if and only if it is both injective and surjective.

Remark 3. *Using another notation, perhaps more familiar, a function*

$$F: \begin{array}{ccc} A & \longrightarrow & B \\ a & \mapsto & F(a) \end{array}$$

is invertible if and only if F is a one-to-one correspondence between the sets A and B.

Remark 4. *If we have an invertible function $f : A \longrightarrow B$, it is immediate that $\left(f^{-1}\right)^{-1} = f$, since, by definition, if f^{-1} is the inverse of f, then f is the inverse of f^{-1}.*

Remark 5. *Let us consider the invertible functions $F : A \longrightarrow B$ and $G : B \longrightarrow C$ and the function*

$$G \circ F: \begin{array}{ccc} A & \longrightarrow & C \\ a & \mapsto & G \circ F(a) = G(F(a)) \end{array}$$

It is easy to see that $G \circ F$ is invertible and the function

$$H: \begin{array}{ccc} C & \longrightarrow & A \\ c & \mapsto & (F^{-1} \circ G^{-1})(c) = F^{-1}(G^{-1}(c)) \end{array}$$

is its inverse.

Since $F : A \longrightarrow B$ is invertible, the function $F^{-1} : B \longrightarrow A$ exists and $F^{-1} \circ F \equiv \mathrm{id}_A$, $F \circ F^{-1} \equiv \mathrm{id}_B$.

Since $G : B \longrightarrow C$ is invertible, the function $G^{-1} : C \longrightarrow B$ exists and $G \circ G^{-1} \equiv \mathrm{id}_C$, $G^{-1} \circ G \equiv \mathrm{id}_B$.

Hence, for all $a \in A$,

$$\begin{aligned} (F^{-1} \circ G^{-1}) \circ (G \circ F)(a) &= (F^{-1} \circ G^{-1} \circ G \circ F)(a) \\ &= (F^{-1} \circ \mathrm{id}_B \circ F)(a) = (F^{-1} \circ F)(a) \\ &= \mathrm{id}_A(a) = a \end{aligned}$$

so $(F^{-1} \circ G^{-1}) \circ (G \circ F) \equiv \mathrm{id}_A$.

For all $c \in C$,

$$\begin{aligned}(G \circ F) \circ (F^{-1} \circ G^{-1})(c) &= (G \circ F \circ F^{-1} \circ G^{-1})(c) \\ &= (G \circ \mathrm{id}_B \circ G^{-1})(c) \\ &= (G \circ G^{-1})(c) = c\end{aligned}$$

so $(G \circ F) \circ (F^{-1} \circ G^{-1}) \equiv \mathrm{id}_C$.

Definition 15. *Let* A, B *be sets. We say that* A has the same cardinality as B *if and only if there is a one-to-one correspondence between* A *and* B*. We write* $\mid A \mid = \mid B \mid$.

Example 5. *The sets*

$$\mathbb{N} = \{0, 1, 2, 3, 4, \ldots, n, \ldots\} \quad \textit{and} \quad \mathrm{EVEN} = \{0, 2, 4, 6, 8, \ldots, 2n, \ldots \mid n \in \mathbb{N}\}$$

have the same cardinality, because the function

$$\begin{array}{rccc} F: & \mathbb{N} & \longrightarrow & \mathrm{EVEN} \\ & n & \mapsto & 2n \end{array}$$

is invertible (and hence a one-to-one correspondence), since the function

$$\begin{array}{rccc} G: & \mathrm{EVEN} & \longrightarrow & \mathbb{N} \\ & n & \mapsto & n/2 \end{array}$$

is its inverse, as we can see from

$$\forall\ n \in \mathbb{N}\ G \circ F(n) = G(F(n)) = G(2n) = \frac{1}{2}2n = n \Longleftrightarrow G \circ F \equiv \mathrm{id}_{\mathbb{N}}$$

and

$$\forall\ n \in \mathrm{EVEN}\ F \circ G(n) = F(G(n)) = F(\frac{n}{2}) = 2\frac{n}{2} = n \Longleftrightarrow F \circ G \equiv \mathrm{id}_{\mathrm{EVEN}}$$

Exercise 16. *Are the following functions invertible? If the answer is yes, find the inverse if possible*

1. *if* ODD *is the set of the odd positive numbers* $\begin{array}{rccc} F: & \mathbb{N} & \longrightarrow & \mathrm{ODD} \\ & n & \mapsto & 2n+1 \end{array}$

2. $\begin{array}{rccc} F: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & -3x+4 \end{array}$

3. $\begin{array}{rccc} F: & [-1, +\infty) & \longrightarrow & \mathbb{R} \\ & x & \mapsto & \sqrt{x+1} \end{array}$

4. $$\begin{array}{rccc} F: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & \mid 2x-1 \mid \end{array}$$

Exercise 17. *Are the following functions invertible? If the answer is no, determine a restriction of the domain and/or codomain that yields an invertible function with the same formula*

1. $$\begin{array}{rccc} F: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & x^2-4 \end{array}$$

2. $$\begin{array}{rccc} F: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & x^2+x \end{array}$$

3. $$\begin{array}{rccc} F: & [-1,+\infty) & \longrightarrow & \mathbb{R} \\ & x & \mapsto & \sqrt{x+1} \end{array}$$

6 Infinity

Example 6 (The salary paradox)**.** *We have two employees, Joe and Jane. Each gets* 1000 *euros a month.*

- *Joe every month puts one euro on a pile on his desk and spends the other* 999*. He never withdraws any money from his pile.*
- *Jane every month puts* 1000 *euros on top of a pile of money on her desk and withdraws* 500 *euro* from the bottom of the pile.

I argue that "at infinity", Joe has an infinite amount of money, and Jane has nothing, because every single euro that she puts on the pile is, at some moment, spent.

Example 7. *Find a one-to-one correspondence between the sets*

$$A = \{n \in \mathbb{N} \mid n \text{ is a multiple of 3}\} \text{ and } B = \{n \in \mathbb{N} \mid n \text{ is a multiple of 4}\}$$

We consider the function

$$\begin{array}{rccc} F: & A & \longrightarrow & B \\ & n & \mapsto & \frac{4}{3}n \end{array}$$

that is well-defined because $n \in A$ and hence n is a multiple of 3*, so $n/3 \in \mathbb{N}$.*

The function

$$\begin{array}{rccc} G: & B & \longrightarrow & A \\ & n & \mapsto & \frac{3}{4}n \end{array}$$

is well-defined because $n \in B$ and hence n is a multiple of 4*, so $n/4 \in \mathbb{N}$.*

The function G is clearly the inverse of F, since

$$G \circ F(n) = G\left(\frac{4}{3}n\right) = \frac{3}{4}\left(\frac{4}{3}n\right) = n \text{ and } F \circ G(n) = F\left(\frac{3}{4}n\right) = \frac{4}{3}\left(\frac{3}{4}n\right) = n$$

and so F is invertible, a one-to-one correspondence, and $\mid A \mid = \mid B \mid$.

Example 8. *Find a one-to-one correspondence between the sets* $A = \{2n+2 \mid n \in \mathbb{N}\}$ *and* $B = \{n^2 \mid n \in \mathbb{N}\}$.

We have $\mid A \mid = \mid \mathbb{N} \mid$ *and* $\mid B \mid = \mid \mathbb{N} \mid$*, so our guess is that* $\mid A \mid = \mid B \mid$ *and thus there is a one-to-one correspondence between* A *and* B*, but we are not sure that the rules for equality apply to cardinality, so we need to find an explicit invertible function between* A *and* B*. This could be difficult. We know the one-to-one correspondences (invertible functions)*

$$\begin{array}{cccc} F: & \mathbb{N} & \longrightarrow & A \\ & n & \mapsto & 2n+2 \end{array} \quad \textit{and} \quad \begin{array}{cccc} G: & \mathbb{N} & \longrightarrow & B \\ & n & \mapsto & n^2 \end{array}$$

We find the inverses by solving the equations

$$2n+2 = a \textit{ and } n^2 = b$$

for $n \in \mathbb{N}$*,* $a \in A$ *and* $b \in B$*. We get*

$$n = \frac{a-2}{2} \textit{ and } n = \sqrt{b}$$

The inverses are

$$\begin{array}{cccc} F^{-1}: & A & \longrightarrow & \mathbb{N} \\ & n & \mapsto & \frac{n-2}{2} \end{array} \quad \textit{and} \quad \begin{array}{cccc} G^{-1}: & B & \longrightarrow & \mathbb{N} \\ & n & \mapsto & \sqrt{n} \end{array}$$

well-defined because in the first case, since $n \in A$ *we have* $\frac{n-2}{2} \in \mathbb{N}$ *and in the second case, since* $b \in B$*,* b *is a perfect square and* $\sqrt{b} \in \mathbb{N}$*.*

So we need an invertible function

$$A \xrightarrow{H} B$$

while we have

$$\mathbb{N} \underset{F^{-1}}{\overset{F}{\rightleftarrows}} A \quad \textit{and} \quad \mathbb{N} \underset{G^{-1}}{\overset{G}{\rightleftarrows}} B$$

The idea is to build the function $A \xrightarrow{H} B$ *using the functions we have*

$$A \xrightarrow{F^{-1}} \mathbb{N} \xrightarrow{G} B$$

so $H \equiv G \circ F^{-1}$ *and so for any* $n \in A$

$$G \circ F^{-1}(n) = G(F^{-1}(n)) = G\left(\frac{n-2}{2}\right) = \left(\frac{n-2}{2}\right)^2$$

and

$$\begin{array}{cccc} H: & A & \longrightarrow & B \\ & n & \mapsto & \left(\frac{n-2}{2}\right)^2 \end{array}$$

is invertible because it is a composition of invertible functions. If we want its explicit inverse,

$$\begin{array}{rccc} H^{-1}: & B & \longrightarrow & A \\ & b & \mapsto & F \circ G^{-1}(b) = F(\sqrt{b}) = 2\sqrt{b} + 2 \end{array}$$

since

$$H \equiv G \circ F^{-1} \Longrightarrow H^{-1} \equiv (G \circ F^{-1})^{-1} \equiv F \circ G^{-1}$$

Exercise 18. *Find a one-to-one correspondence between the sets* $A = \{n \in \mathbb{N} \mid n \text{ is a multiple of } 3\}$ *and* $B = \{n \in \mathbb{N} \mid n \text{ is a multiple of } 4\}$ *(this is a repeat of the example above).*

Exercise 19. *Find a one-to-one correspondence between the sets* $A = \{2n + 2 \mid n \in \mathbb{N}\}$ *and* $B = \{n^2 \mid n \in \mathbb{N}\}$ *(this is a repeat of the example above).*

Exercise 20. *Do the sets* $\mathbb{N}$ *and* $\mathbb{N}^2$ *have the same cardinality?*

Exercise 21. *Do the sets* $\mathbb{N}$ *and* $\mathbb{N}^3$ *have the same cardinality? [Difficult]*

Exercise 22 (Hilbert's hotel)**.** *We have a hotel with infinitely many rooms, all occupied. If a new customer comes, can we find a free room for him?*

Example 9. *Do the sets* $\mathbb{N}$ *and* $\mathbb{Z}$ *have the same cardinality?*

The question is, by definition, equivalent to: is there an invertible function (one-to-one correspondence) between $\mathbb{N}$ *and* $\mathbb{Z}$*? We build one such function.*

If we rearrange the elements of $\mathbb{Z}$ *listing them as* $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}$*, it is natural to define a function from* $\mathbb{N}$ *to* $\mathbb{Z}$ *that maps* 0 *to* 0*,* 1 *to* 1*,* 2 *to* -1*,* 3 *to* 2*,* 4 *to* -2*, and so on. If we write down the formula for this function we have*

$$\begin{array}{rccc} F: & \mathbb{N} & \longrightarrow & \mathbb{Z} \\ & n & \mapsto & \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ -\frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases} \end{array}$$

that is well-defined and invertible. It is well-defined because if n *is even,* $\frac{n}{2} \in \mathbb{Z}$ *and if* n *is odd,* $-\frac{n+1}{2} \in \mathbb{Z}$*, so in any case* n *is mapped to an integer. It is invertible because the function*

$$\begin{array}{rccc} G: & \mathbb{Z} & \longrightarrow & \mathbb{N} \\ & n & \mapsto & \begin{cases} 2n & \text{if } n \geq 0 \\ -2n - 1 & \text{if } n < 0 \end{cases} \end{array}$$

is its inverse, as we can check.

Definition 16. *Given* A, B, C *sets*

1. $\mid A \mid = \mid B \mid$ *if and only if there is an invertible function* $F : A \longrightarrow B$.

2. $\mid A \mid \neq \mid B \mid$ *if and only if there is no invertible function* $F : A \longrightarrow B$.

3. $\mid A \mid \leq \mid B \mid$ *if and only if there is an injective function* $F : A \longrightarrow B$.

4. $\mid A \mid < \mid B \mid$ *if and only if there is an injective function* $F : A \longrightarrow B$ *but there is no invertible function* $F : A \longrightarrow B$.

5. *If* $\mid A \mid = \mid B \mid$ *and* $\mid B \mid = \mid C \mid$ *then* $\mid A \mid = \mid C \mid$. *(Since the composition of invertible functions is an invertible function.)*

6. *If* A, B *are sets and* $A \subseteq B$ *then* $\mid A \mid \leq \mid B \mid$.

7. *(Cantor-Schröder-Bernstein theorem)*

$$\mid A \mid \leq \mid B \mid \text{ and } \mid B \mid \leq \mid A \mid \Longrightarrow \mid A \mid = \mid B \mid$$

We can define addition for infinite cardinalities, but it behaves strangely.

8. *If* A, B *are infinite sets and* $\mid A \mid = \mid B \mid$ *then* $\mid A \mid + \mid B \mid = \mid A \mid = \mid B \mid$.

9. *If* A, B *are infinite sets and* $\mid A \mid > \mid B \mid$ *then* $\mid A \mid + \mid B \mid = \mid A \mid$.

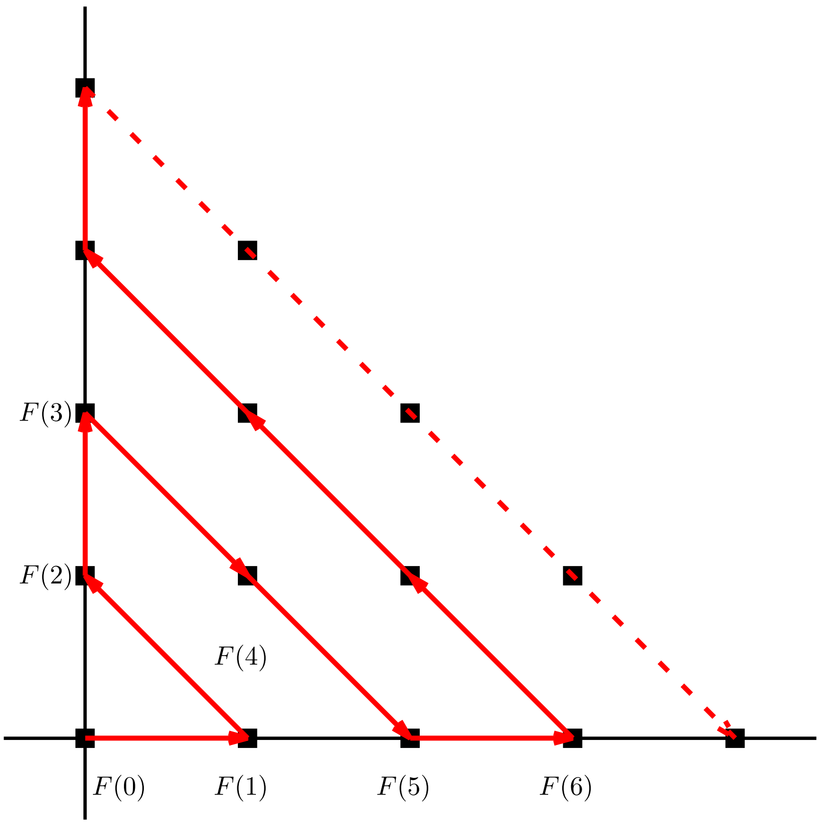
A consequence of the Cantor-Schröder-Bernstein theorem:

10. *If* A, B *are sets,* $\mid A \cup B \mid \leq \mid A \mid + \mid B \mid$ *and* $\mid A \cap B \mid \leq \mid A \mid, \mid B \mid$. *[This will be shown in class]*

Example 10. *We prove that* $\mid \mathbb{N} \mid = \mid \mathbb{N} \times \mathbb{N} \mid$ *by building a one-to-one correspondence* $F : \mathbb{N} \longrightarrow \mathbb{N} \times \mathbb{N}$.

We say that $F(0) = (0, 0)$, $F(1) = (1, 0)$, $F(2) = (0, 1)$, $F(3) = (0, 2)$, $F(4) = (1, 1)$, $F(5) = (2, 0)$ *and so on as in the image below.*

GNU1 - The one-to-one correspondence $F : \mathbb{N} \longrightarrow \mathbb{N} \times \mathbb{N}$



The function F is a one-to-one correspondence, and so

$$|\ \mathbb{N}\ |=|\ \mathbb{N} \times \mathbb{N}\ |$$

Example 11. *We prove that*

$$|\ \mathbb{N}\ |=|\ \mathbb{Q}\ |$$

by building a one-to-one correspondence $F : \mathbb{N} \longrightarrow \mathbb{Q}$. The idea is to mimic the one-to-one correspondence between $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$.

There is a natural function between $\mathbb{Q}$ and $\mathbb{Z} \times \mathbb{N}$,

$$\begin{array}{cccc} T: & \mathbb{Q} & \longrightarrow & \mathbb{Z} \times \mathbb{N} \\ & p/q & \mapsto & (p, q) \end{array}$$

with the usual provisos that $q > 0$ and p, q are coprime (no common factor, we don't consider rationals like $4/2$) to ensure that the representation p/q is unique.

If we restrict the codomain to the pairs (p,q) *such that* $q > 0$ *and* p,q *are coprime we have a one-to-one correspondence*

$$T' : \begin{array}{ccc} \mathbb{Q} & \longrightarrow & A \\ p/q & \mapsto & (p,q) \end{array} \quad \textit{where } A = \{(p,q) \in \mathbb{Z}\times\mathbb{N} \mid q > 0 \textit{ and } \gcd(p,q) = 1\}$$

So $\mid \mathbb{Q} \mid = \mid A \mid$.

We give a description of A*, detailing the elements of* $\mathbb{Z}\times\mathbb{N}$ *that we are taking out. We proceed diagonally*

$$(0,1) \qquad = 0/1 = 0 \in \mathbb{Q} \textit{ OK}$$

Listed elements of $\mathbb{Q}$*:* 0.

$$\begin{aligned} (-1,1) &\qquad = -1/1 = -1 \in \mathbb{Q} \textit{ OK} \\ (0,2) &\qquad = 0/1 = 0 \in \mathbb{Q} \textit{ we take it out, repeat} \\ (1,1) &\qquad = 1/1 = 1 \in \mathbb{Q} \textit{ OK} \end{aligned}$$

Listed elements of $\mathbb{Q}$*:* $0, \pm 1$.

$$\begin{aligned} (-2,1) &\qquad = -2/1 = -2 \in \mathbb{Q} \textit{ OK} \\ (-1,2) &\qquad = -1/2 \in \mathbb{Q} \textit{ OK} \\ (0,3) &\qquad = 0/1 = 0 \textit{ we take it out, repeat} \\ (1,2) &\qquad = 1/2 \in \mathbb{Q} \textit{ OK} \\ (2,1) &\qquad = 2/1 = 2 \in \mathbb{Q} \textit{ OK} \end{aligned}$$

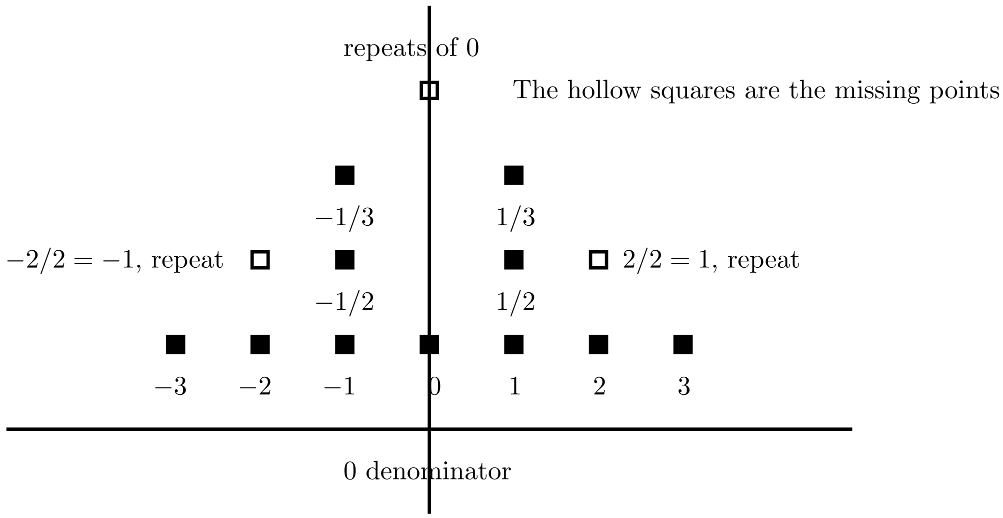
Listed elements of $\mathbb{Q}$*:* $0, \pm 1, \pm 2, \pm 1/2$.

$$\begin{aligned} (-3,1) &\qquad = -3/1 = -3 \in \mathbb{Q} \textit{ OK} \\ (-2,2) &\qquad = -2/2 = -1 \textit{ NO, repeat} \\ (-1,3) &\qquad = -1/3 \in \mathbb{Q} \textit{ OK} \\ (0,4) &\qquad = 0/1 = 0 \textit{ NO, repeat} \\ (1,3) &\qquad = 1/3 \in \mathbb{Q} \textit{ OK} \\ (2,2) &\qquad = 2/2 = 1 \textit{ NO, repeat} \\ (3,1) &\qquad = 3/1 = 3 \in \mathbb{Q} \textit{ OK} \end{aligned}$$

Listed elements of $\mathbb{Q}$*:* $0, \pm 1, \pm 2, \pm 1/2, \pm 3, \pm 1/3$.

graphically

GNU2 - The set $A \subseteq \mathbb{Z} \times \mathbb{N}$



From the description above and the drawing, it is clear that T' is a one-to-one correspondence.

If we prove that there is a one-to-one correspondence $G : \mathbb{N} \longrightarrow A$, then we are done, because then $\mid \mathbb{N} \mid = \mid A \mid$ and $\mid \mathbb{Q} \mid = \mid A \mid$ imply $\mid \mathbb{N} \mid = \mid \mathbb{Q} \mid$.

We can define G by following the diagonals and skipping the points not in A, similar to the previous example.

The function G is a one-to-one correspondence, and so

$$\mid \mathbb{N} \mid = \mid A \mid$$

Example 12. *Prove $\mid \mathbb{N} \mid \neq \mid \mathbb{R} \mid$.*

We have to prove that there is no one-to-one correspondence $F : \mathbb{N} \longrightarrow \mathbb{R}$. Since $\mid (0,1) \mid = \mid \mathbb{R} \mid$, we can prove that there is no one-to-one correspondence $F : \mathbb{N} \longrightarrow (0,1)$.

By contradiction, suppose that there is a one-to-one correspondence $F : \mathbb{N} \longrightarrow (0,1)$. That means that we can list ALL the real numbers in $(0,1)$.

That means that we can write them ALL in a list, where the a's, b's etc are digits (numbers $0, \ldots, 9$*)*

$$\begin{array}{rcl} F(0) & = & 0.a_{00}a_{01}a_{02}a_{03}\cdots \\ F(1) & = & 0.a_{10}a_{11}a_{12}a_{13}\cdots \\ F(2) & = & 0.a_{20}a_{21}a_{22}a_{23}\cdots \\ F(3) & = & 0.a_{30}a_{31}a_{32}a_{33}\cdots \\ \vdots & = & \vdots \end{array}$$

To find a contradiction, we build a real number y *belonging to* $(0,1)$ *that is NOT in the above list.*

$$y = 0.b_0b_1b_2b_3\cdots b_n\cdots$$

where the b_i*'s are chosen such that*

$$b_0 \neq a_{00},\ b_1 \neq a_{11},\ b_2 \neq a_{22},\ b_3 \neq a_{33}, \ldots b_n \neq a_{nn}, \ldots$$

(and also avoiding $b_i = 0$ *or* 9 *to prevent issues with non-unique decimal representations). Then* y *is different from* $F(0)$ *since their first digit differs,* y *is different from* $F(1)$ *since their second digit differs,* y *is different from* $F(2)$ *since their third digit differs, and so on.*

It is clear that $y \in (0,1)$ *but* y *cannot be in the list above, which supposedly contains all the elements of* $(0,1)$*, and here we have our contradiction. The hypothesis that the list above contains all the elements of* $(0,1)$ *is therefore false, and so it is impossible to find a one-to-one correspondence between* $\mathbb{N}$ *and* $(0,1)$*, and thus it is impossible to find a one-to-one correspondence between* $\mathbb{N}$ *and* $\mathbb{R}$*. Hence*

$$|\ \mathbb{N}\ | \neq |\ \mathbb{R}\ |$$

Exercise 23.

1. *Prove that* $|\ \mathbb{N}\ | = |\ \mathbb{Q}\ |$ *using the Cantor-Schröder-Bernstein theorem. [This is quite easy using the right idea].*
2. *Prove that* $|\ [0,1]\ | = |\ \mathbb{R}\ |$ *using the Cantor-Schröder-Bernstein theorem.*
3. *Prove that* $|\ [0,1]\ | = |\ \mathbb{R}\ |$ *by finding a one-to-one correspondence. [Difficult]*
4. *Prove that* $|\ \mathbb{N}\ | = |\ \mathbb{Q}\ |$ *directly by building a function* $F : \mathbb{N} \longrightarrow \mathbb{Q}$*.*
5. *Find an infinite set whose cardinality is greater than* $|\ \mathbb{R}\ |$*.*
6. *Prove that* $|\ \mathbb{N}\ | = |\ \mathbb{Z} \times \mathbb{Z}\ |$*.*
7. *Prove that* $|\ \mathbb{N}^3\ | = |\ \mathbb{N}\ |$*.*
8. *Prove that* $|\ \mathbb{Q}^2\ | = |\ \mathbb{N}\ |$*.*

9. *Prove that* $\forall\ n \in \mathbb{N}$, $\mid \mathbb{N}^n \mid = \mid \mathbb{N} \mid$.

10. *Prove that* $\forall\ n \in \mathbb{N}$, $\mid \mathbb{Q}^n \mid = \mid \mathbb{N} \mid$.

11. *Let* $\mathbb{Z}[x] = \{$*polynomials with coefficients in* $\mathbb{Z}\}$.
    Prove that $\mid \mathbb{Z}[x] \mid = \mid \mathbb{N} \mid$. *[Difficult]*

12. *Let* $\mathbb{Q}[x] = \{$*polynomials with coefficients in* $\mathbb{Q}\}$.
    Prove that $\mid \mathbb{Q}[x] \mid = \mid \mathbb{N} \mid$. *[Difficult]*

13. *Propose as many sets as possible whose cardinality is greater than the cardinality of* $\mathbb{R}$.

7 Graphs

Definition 17. *Given a function*

$$\begin{array}{rccc} F: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & F(x) \end{array}$$

the subset of $\mathbb{R}^2$

$$\text{Graph}(F) = \{(x, F(x)) \ \mid \ x \in \mathbb{R}\} \subset \mathbb{R}^2$$

is the graph of F

Definition 18. *Given* $D \subseteq \mathbb{R}^2$, D *is the graph of a function*

$$F_D : A \longrightarrow B$$

if and only if

$$\forall\ a \in A\ \exists!\ b \in B \ \ s.t.\ (a, b) \in D$$

And the associated *function is*

$$\begin{array}{rccc} F_D: & A & \longrightarrow & B \\ & a & \mapsto & F_D(a) = b \end{array}$$

Corollary 1. *For a subset* $D \subset \mathbb{R}^2$ *to be the graph of a function* $f : \mathbb{R} \longrightarrow \mathbb{R}$, *a vertical line must intersect* D *at most once.*

Corollary 2. *For a function*

$$\begin{array}{rccc} F: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & F(x) \end{array}$$

1. *it is* injective *if and only if every horizontal line* $y = c$ *intersects the graph of* F *at most once.*

2. *it is* surjective *if and only if every horizontal line $y = c$ intersects the graph of F at least once.*

3. *it is* a one-to-one correspondence *if and only if every horizontal line $y = c$ intersects the graph of F exactly once.*

Remark 6. *Consider the function*

$$F: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & F(x) \end{array}$$

whose graph is known, and constants $A, B, C, D \in \mathbb{R}$, and the function

$$F_1: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & AF(Bx + C) + D \end{array}$$

1. *The graph of F_1 is the graph of F shifted vertically by D. Up if $D > 0$, down if $D < 0$.*

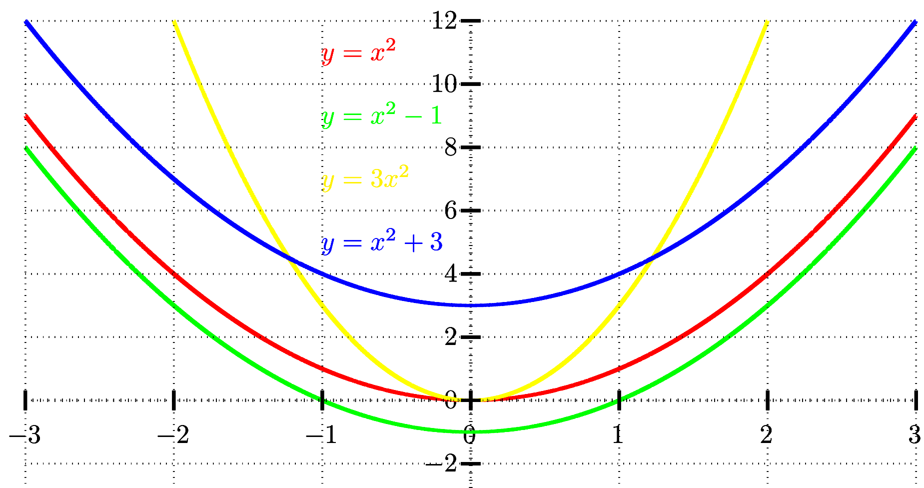
2. *The graph of F_1 is the graph of F shifted horizontally by $-C/B$ (provided $B \neq 0$). Left if $-C/B > 0$, right if $-C/B < 0$.*

3. *The graph of F_1 is the graph of F scaled vertically by A and horizontally by $1/B$ (provided $B \neq 0$).*

Example 13. *In the plane $\mathbb{R}^2$, draw the graphs of the functions*

$$F: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & x^2 \end{array}, \quad G: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & x^2 - 1 \end{array}, \quad H: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & 3x^2 \end{array}, \quad K: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & x^2 + 3 \end{array}$$

GNU11: Graphs



Example 14. *A full example of graph sketching: we want to sketch the graph of the formula*

$$\begin{array}{rccc} F: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & F(x) = \dfrac{x-1}{x^2-4} \end{array}$$

1. *Existence field: the existence field of F is $x \neq \pm 2$ or, if you prefer, $(-\infty,-2) \cup (-2,2) \cup (2,+\infty)$. The function associated to the formula is*

$$\begin{array}{rccc} F: & \mathbb{R} - \{\pm 2\} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & F(x) = \dfrac{x-1}{x^2-4} \end{array}$$

   with a slight abuse of notation, we call the formula and the function with the same name.

2. *We recall that rational functions are continuous (we can draw them without lifting the pen from the paper) on each interval of the domain, which are $(-\infty,-2)$, $(-2,2)$ and $(2,+\infty)$*

3. *Intersection with the y-axis ($x=0$): $F(0) = \frac{-1}{-4} = \frac{1}{4}$.*

4. *Zeros: solve*

$$F(x) = 0 \Longleftrightarrow \frac{x-1}{x^2-4} = 0 \Longleftrightarrow x = 1$$

   There is only one zero, $x = 1$.

5. *Sign: we have that*

$$x - 1 > 0 \Longleftrightarrow x > 1 \text{ and } x^2 - 4 > 0 \Longleftrightarrow x < -2 \text{ or } x > 2$$

   Hence,

$$F(x) > 0 \Longleftrightarrow \frac{x-1}{x^2-4} > 0 \Longleftrightarrow x \in (-2,1) \cup (2,+\infty)$$

   The function is positive on $(-2,1) \cup (2,+\infty)$, and negative on $(-\infty,-2) \cup (1,2)$.

6. *Vertical asymptotes: these can only occur at points not in the domain, hence $x = -2$ and $x = 2$. We examine the behavior near these points. When x approaches 2 from the right, $x^2 - 4$ is very small and positive, so $\frac{1}{x^2-4}$ is very large and positive. Using limits, we write*

$$\lim_{x \to 2^+} (x^2 - 4) = 0^+ \Longrightarrow \lim_{x \to 2^+} \frac{1}{x^2-4} = +\infty$$

   When x approaches 2 from the right, $x - 1$ approaches 1, so

$$\lim_{x \to 2^+} (x - 1) = 1$$

Hence

$$\lim_{x\to 2^+} \frac{x-1}{x^2-4} = +\infty$$

With the same reasoning, when x *approaches* 2 *from the left and* -2 *from the left and right, we have*

$$\lim_{x\to 2^-} \frac{x-1}{x^2-4} = -\infty \qquad \lim_{x\to -2^-} \frac{x-1}{x^2-4} = +\infty \qquad \lim_{x\to -2^+} \frac{x-1}{x^2-4} = -\infty$$

7. *Behavior at* $\pm\infty$*: We want to know what happens when* x *is very large and positive* ($x \to +\infty$) *or very large and negative* ($x \to -\infty$)*. We have that*

$$\frac{x-1}{x^2-4} = \frac{\frac{x-1}{x^2}}{\frac{x^2-4}{x^2}} = \frac{\frac{1}{x} - \frac{1}{x^2}}{1 - \frac{4}{x^2}}$$

and we have

$$\lim_{x\to\pm\infty} \frac{1}{x} = 0, \qquad \lim_{x\to\pm\infty} \frac{1}{x^2} = 0$$

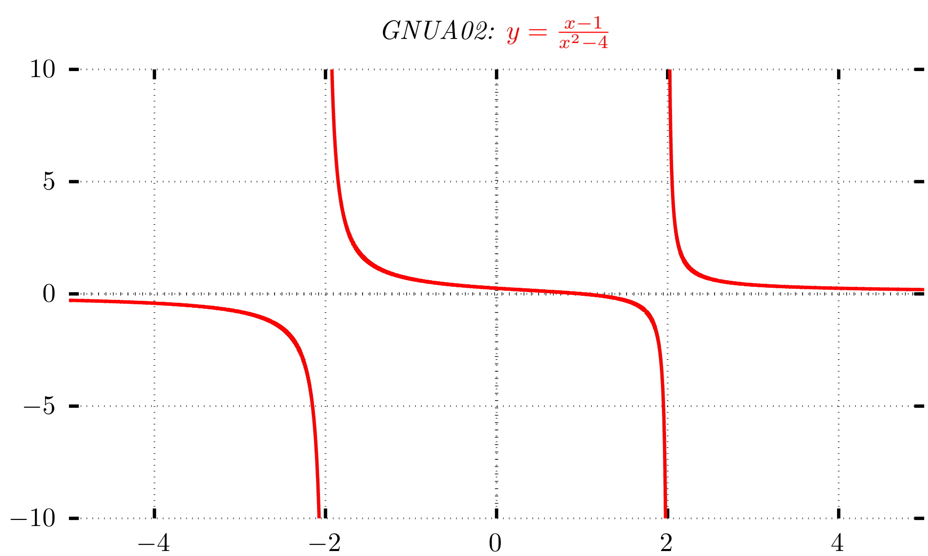
so

$$\lim_{x\to\pm\infty} \frac{x-1}{x^2-4} = 0$$

Thus the line $y = 0$ *is a horizontal asymptote. We want to check if the function intersects the line* $y = 0$ *when* x *is large, but we already know that the only zero is* $x = 1$*, so there are no intersections for large* $|x|$*.*

8. *We check that the sign and the limits at* $\pm 2, \pm\infty$ *are consistent.*

Then we can sketch the function



Proposition 2. *If $f(x) = ax^n + \cdots$ and $g(x) = bx^p + \cdots$ with $a, b \neq 0$ are two polynomials, then*

$$\lim_{x \to +\infty} \frac{f(x)}{g(x)} = \begin{cases} \infty & \textit{if } \deg(f(x)) > \deg(g(x)) \\ \\ \dfrac{a}{b} & \textit{if } \deg(f(x)) = \deg(g(x)) \\ \\ 0 & \textit{if } \deg(f(x)) < \deg(g(x)) \end{cases}$$

If the signs of a and b are negative, or if $x \to -\infty$, the result changes in the obvious ways.

Example 15. *If $f(x)$ and $g(x)$ are two polynomials, then*

$$\lim_{x \to +\infty} \frac{x^3 - x + 2}{10x^2 - 1} = +\infty$$

$$\lim_{x \to -\infty} \frac{x^3 - x + 2}{10x^2 - 1} = -\infty$$

$$\lim_{x \to +\infty} \frac{-x^3 - x + 2}{10x^2 - 1} = -\infty$$

$$\lim_{x \to +\infty} \frac{5x^3 + x + 2}{x^4 - 1} = 0$$

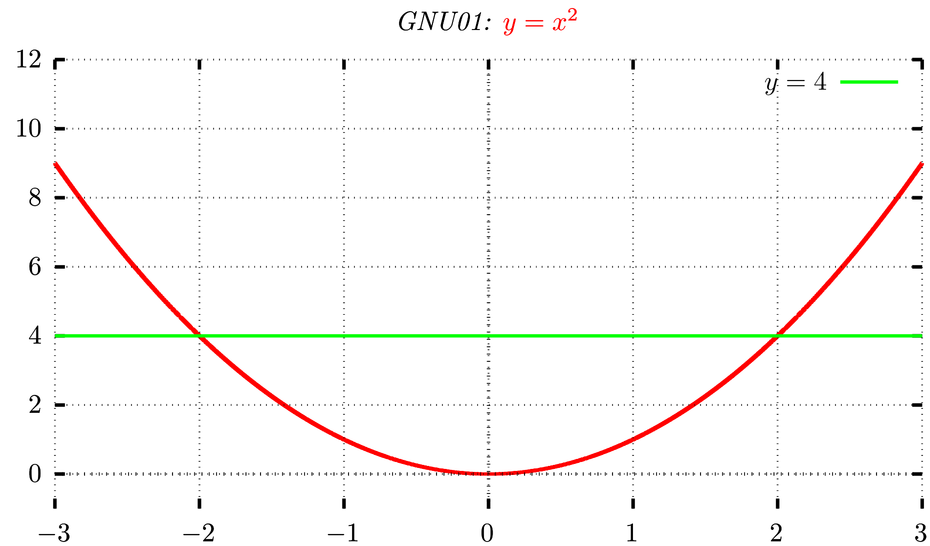
$$\lim_{x \to +\infty} \frac{5x^3 + x + 2}{3x^3 - 1} = \frac{5}{3}$$

$$\lim_{x \to +\infty} \frac{5x^3 + x + 2}{-3x^3 - 1} = -\frac{5}{3}$$

Example 16. *Determine if the following functions are injective, by looking at their graphs*

1.

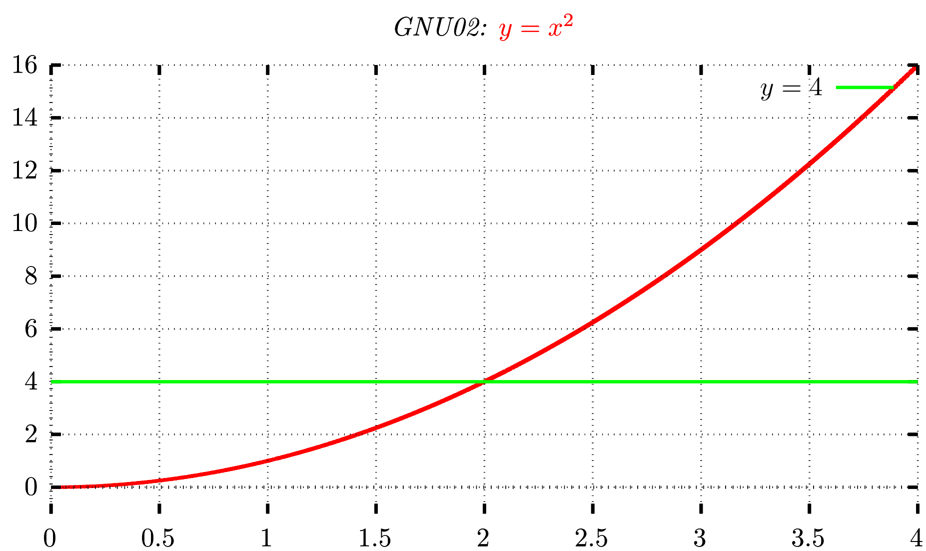
$$\begin{array}{rccc} F: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & F(x) = x^2 \end{array}$$



Not injective

2.

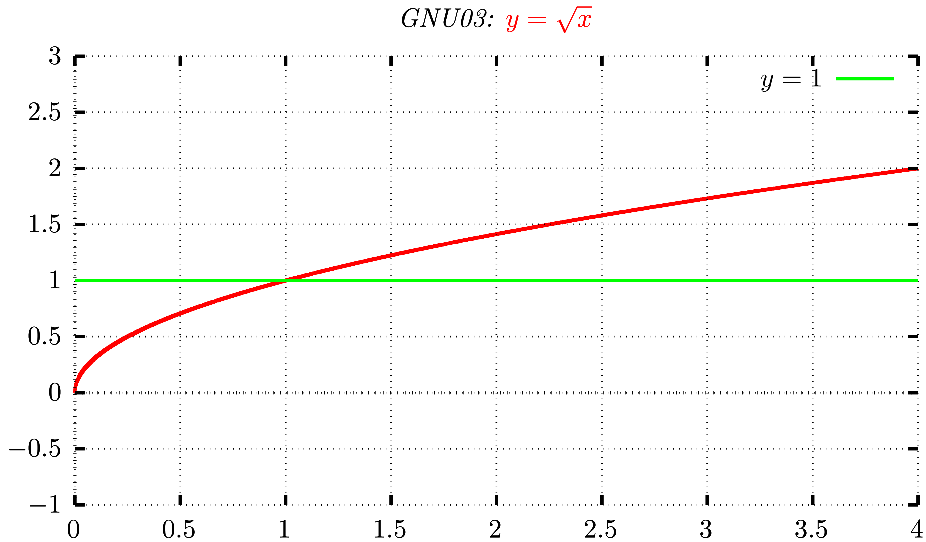
$$F : \begin{array}{ccc} \mathbb{R}_0^+ & \longrightarrow & \mathbb{R} \\ x & \mapsto & F(x) = x^2 \end{array}$$



Injective

3.

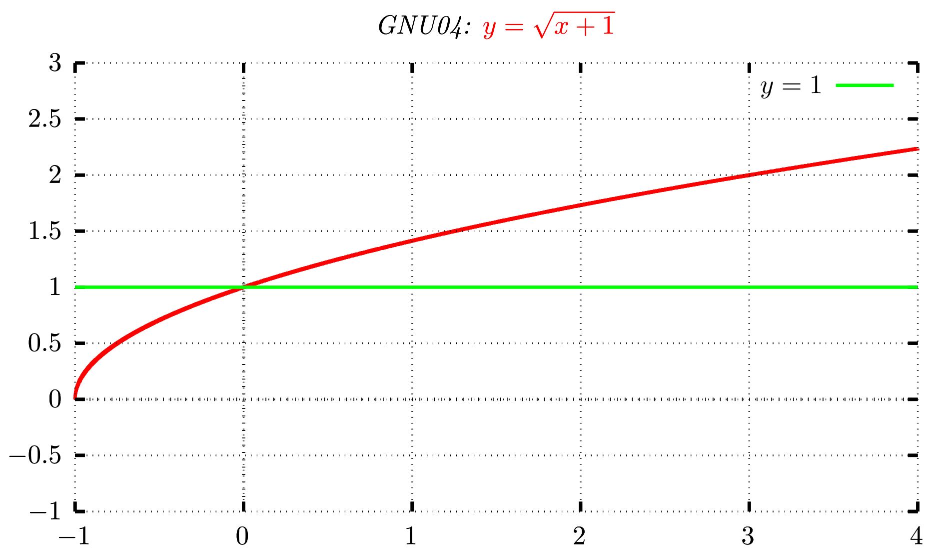
$$F : \begin{array}{ccc} \mathbb{R}_0^+ & \longrightarrow & \mathbb{R} \\ x & \mapsto & F(x) = \sqrt{x} \end{array}$$



Injective

4.

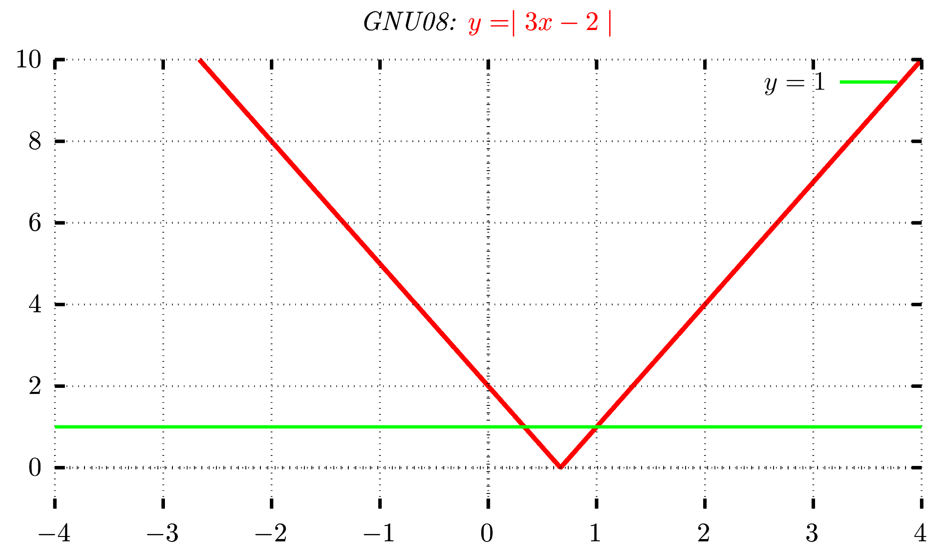
$$\begin{array}{rccc} F: & [-1, +\infty) & \longrightarrow & \mathbb{R} \\ & x & \mapsto & F(x) = \sqrt{x+1} \end{array}$$



Injective

5.

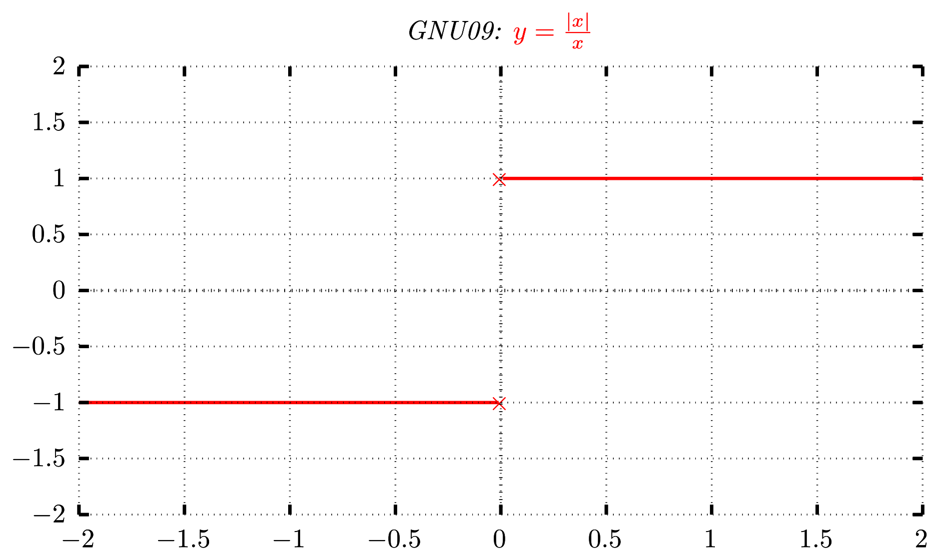
$$\begin{array}{rccc} F: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & F(x) = \mid 3x - 2 \mid \end{array}$$



Not injective

6.

$$\begin{array}{rccc} F: & \mathbb{R} \setminus \{0\} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & F(x) = \frac{|x|}{x} \end{array}$$



Not injective

Exercise 24 (A)**.** *Sketch the graph of the following formulas*

1. $\begin{array}{rccc} F_1: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & F(x) = x^2 + 2 \end{array}$

2. $\begin{array}{rccc} F_1: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & F(x) = (x-3)^2 \end{array}$

3. $\begin{array}{rccc} F_1: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & F(x) = (x-3)^2 - 1 \end{array}$

4. $\begin{array}{rccc} F_1: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & F(x) = \sqrt{x+2} \end{array}$

5. $\begin{array}{rccc} F_1: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & F(x) = \sqrt[3]{x+2} \end{array}$

6. $\begin{array}{rccc} F_1: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & F(x) = \sqrt{x+2} - 4 \end{array}$

Exercise 25 (B)**.** *Sketch the graph of the following formulas*

1. $\begin{array}{rccc} F_1: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & F(x) = \frac{x-3}{x+2} \end{array}$

2. $\begin{array}{rccc} F_1: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & F(x) = \frac{x^2-3}{x+2} \end{array}$

3. $\begin{array}{rccc} F_1: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & F(x) = \frac{x^2-8}{x^2-3} \end{array}$

4. $\begin{array}{rccc} F_1: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & F(x) = \frac{x^2-8}{4x^2+1} \end{array}$

5. $\begin{array}{rccc} F_1: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & F(x) = \frac{2x^2-1}{(x+1)^2} \end{array}$

6. $\begin{array}{rccc} F_1: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & F(x) = \frac{x^3-1}{x^2-4} \end{array}$

Exercise 26 (C)**.** *Sketch the graph of the following formulas*

1. $\begin{array}{rccc} F_1: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & F(x) = \frac{\sqrt{x-1}}{x} \end{array}$

2. $\begin{array}{rccc} F_1: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & F(x) = \frac{\sqrt{x^2-1}}{x} \end{array}$

3. $\begin{array}{rccc} F_1: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & F(x) = x\sqrt{x+2} \end{array}$

4. $\begin{array}{rccc} F_1: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & F(x) = x + \sqrt{x+2} \end{array}$

Exercise 27 (D)**.** *Sketch the graph of the following formulas and determine if they are injective and/or surjective*

1. $$\begin{array}{rccc} F_1: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & F(x) = \frac{3x+2}{x-4} \end{array}$$

2. $$\begin{array}{rccc} F_1: & \mathbb{R}^+ & \longrightarrow & \mathbb{R} \\ & x & \mapsto & F(x) = \frac{\sqrt{x^2}}{x} \end{array}$$

3. $$\begin{array}{rccc} F_1: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & F(x) = \mid x^2 - 1 \mid \end{array}$$

8 Third Lesson

Continuous functions

Definition 19. *Let A be a subset of $\mathbb{R}$ and $x_0 \in A$. A function $F : A \longrightarrow \mathbb{R}$ is* continuous *at x_0 if*

$$\lim_{x \to x_0} F(x) = F(x_0)$$

Proposition 3. *Let A be a subset of $\mathbb{R}$ and $f : A \longrightarrow \mathbb{R}$, $g : A \longrightarrow \mathbb{R}$ be functions. Then*

1. *$f + g$ is continuous on A.*
2. *$f \cdot g$ is continuous on A.*
3. *$\frac{f}{g}$ is continuous at all $x_0 \in A$ such that $g(x_0) \neq 0$.*
4. *If $h : Im\, f \longrightarrow \mathbb{R}$ is continuous, then $h \circ f$ is continuous on A.*

Derivatives

Definition 20. *Let $a \in \mathbb{Q}$ and $f : A \longrightarrow \mathbb{R}$, $f(x) = x^a$ be a function. Then the* derivative *of f is the function*

$$\begin{array}{rccc} f': & A' & \longrightarrow & \mathbb{R} \\ & x & \mapsto & ax^{a-1} \end{array}$$

where A' is the domain of existence of the derivative, which is a subset of A.

Proposition 4. *Let $f : A \longrightarrow \mathbb{R}$, $g : A \longrightarrow \mathbb{R}$ be differentiable functions. Then*

1. *$(f \pm g)' = f' \pm g'$.*
2. *$(fg)' = f'g + fg'$.*
3. *$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$, wherever $g \neq 0$.*
4. *$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$ (Chain Rule).*

Example 17.

1. $(x^2)' = 2x$

2. $(3x^2 + 7x)' = (3x^2)' + (7x)' = 6x + 7$

3. $(5)' = (5x^0)' = 0 \cdot 5x^{-1} = 0$

4. $(\sqrt{x})' = (x^{\frac{1}{2}})' = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2x^{\frac{1}{2}}} = \frac{1}{2\sqrt{x}}$

5. $$\begin{aligned} ((x^2+2x)(3x^5-1))' &= (x^2+2x)'(3x^5-1) + (x^2+2x)(3x^5-1)' \\ &= (2x+2)(3x^5-1) + (x^2+2x)(15x^4) \\ &= 6x^6 + 6x^5 - 2x - 2 + 15x^6 + 30x^5 \\ &= 21x^6 + 36x^5 - 2x - 2 \end{aligned}$$

6. $\left(\frac{1}{x}\right)' = (x^{-1})' = -x^{-2} = -\frac{1}{x^2}$

7. $\left(\frac{x^2+2x}{3x^5-1}\right)' = \frac{(x^2+2x)'(3x^5-1)-(x^2+2x)(3x^5-1)'}{(3x^5-1)^2} = \frac{(2x+2)(3x^5-1)-(x^2+2x)(15x^4)}{(3x^5-1)^2}$

8. $$\begin{aligned} \left((4x^3+x-3)^3\right)' &= 3(4x^3+x-3)^2 \cdot (4x^3+x-3)' \\ &= 3(4x^3+x-3)^2(12x^2+1) \end{aligned}$$

9. $$\begin{aligned} \left(\sqrt{3x^2-7x+2}\right)' &= \frac{1}{2\sqrt{3x^2-7x+2}} \cdot (3x^2-7x+2)' \\ &= \frac{6x-7}{2\sqrt{3x^2-7x+2}} \end{aligned}$$

Some useful propositions

Proposition 5. *Let $f : (a,b) \longrightarrow \mathbb{R}$ be a differentiable function. Then*

1. *f is increasing on an interval if $f'(x) > 0$ for all x in that interval.*

2. *f is decreasing on an interval if $f'(x) < 0$ for all x in that interval.*

Remark 7. *Let $f : (a,b) \longrightarrow \mathbb{R}$ be a differentiable function. Then f has a local maximum at $c \in (a,b)$ if f is increasing immediately before c and decreasing immediately after c (i.e., $f'(x) > 0$ for $x < c$ near c, and $f'(x) < 0$ for $x > c$ near c). Similarly for a local minimum.*

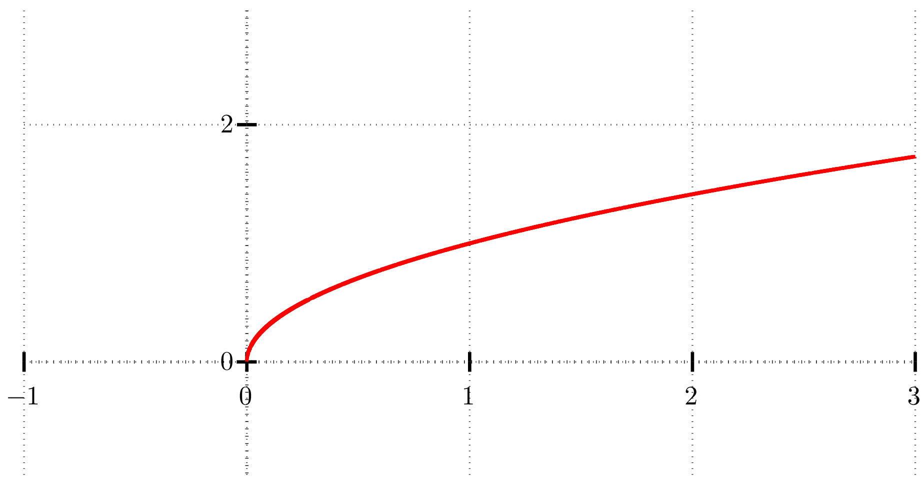
Proposition 6 (Intermediate Value Theorem)**.** *Let* $f : [a, b] \longrightarrow \mathbb{R}$ *be a continuous function and* $f(a) < 0$, $f(b) > 0$. *Then there exists* $c \in (a, b)$ *such that* $f(c) = 0$.

Proposition 7 (Extreme Value Theorem)**.** *Let* $f : [a, b] \longrightarrow \mathbb{R}$ *be a continuous function. Then* f *attains a maximum and a minimum value on* $[a, b]$.

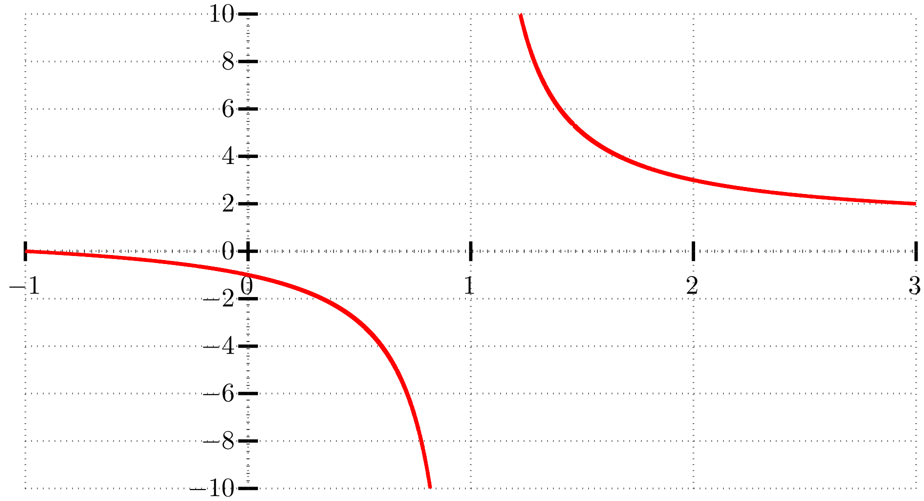
Proposition 8 (Rolle's Theorem)**.** *Let* $f : [a, b] \longrightarrow \mathbb{R}$ *be a continuous function on* $[a, b]$ *and differentiable on* (a, b), *with* $f(a) = f(b)$. *Then there exists* $c \in (a, b)$ *such that* $f'(c) = 0$.

Some examples

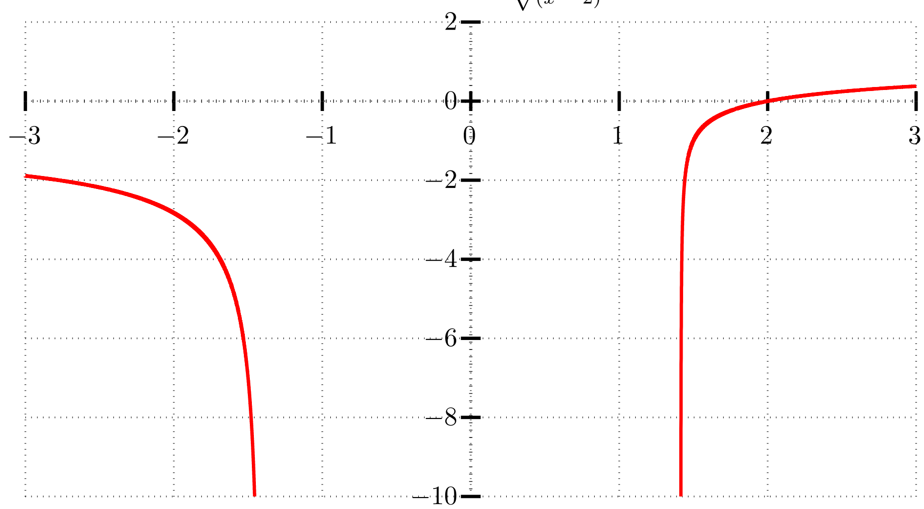
GNUA31: $y = \frac{x}{\sqrt{x}}$



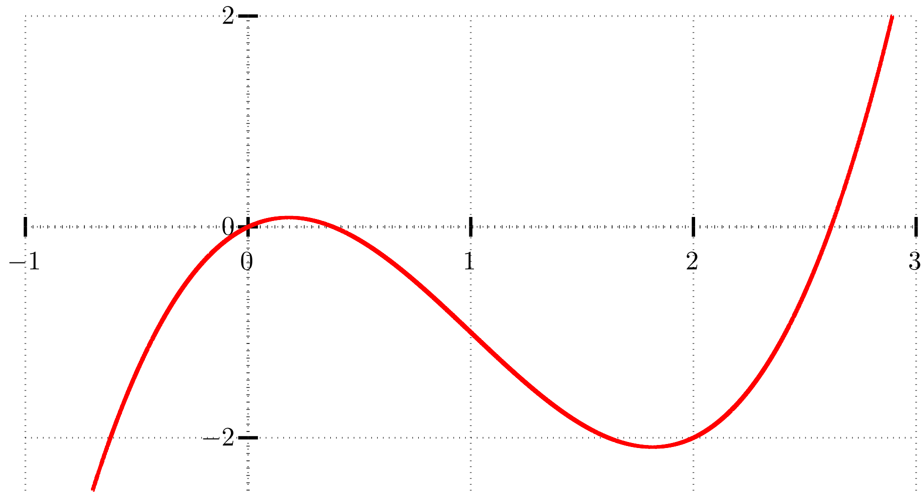
GNUA32: $y = \frac{(x^2-1)}{(x-1)**2}$



GNUA33: $y = \frac{(x-2)}{\sqrt{(x^2-2)}}$



GNUA34: $y = x^3 - 3x^2 + x$



Homework

Exercise 28. *Please, sketch the graph of the following:*

1. $\begin{array}{rccc} f: & (0,\infty) & \longrightarrow & \mathbb{R} \\ & x & \mapsto & \frac{x^2-\sqrt{x}}{\sqrt{x}} \end{array}$

2. $\begin{array}{rccc} g: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & \sqrt[3]{x^2} \end{array}$

3. $\begin{array}{rccc} h: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & \mid x^2-1 \mid \end{array}$

4. $\begin{array}{rccc} F: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & x^3-3x^2-3x+1 \end{array}$

5. $\begin{array}{rccc} G: & (0,\infty)\setminus\{2\} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & \frac{(x^2-1)\sqrt{x}}{(x-2)} \end{array}$