Complex Analysis and Geometry (Geometria e Analisi Complessa)
2024-2025, first semester
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This course is an introduction to the theory of complex analysis in several variables, mainly from a geometric and analytic viewpoint. It will extend the foundational principles of single-variable complex analysis to $\mathbb{C}^n$, present the main differences of the theory, and new tools and techniques which are characteristic of the higher-dimensional context. Applications of complex analysis in several variables are far-reaching, influencing areas such as differential geometry, algebraic geometry, and complex dynamics, as well as mathematical physics.

Central to the topic and the course is the study of holomorphic functions in several variables, characterized by satisfying the multi-dimensional Cauchy-Riemann equations. We will investigate the generalizations of Cauchy's Theorem and other integral formulas to higher dimensions, and study the solvability of the so-called $\bar\partial$-*equation* $\bar\partial f = g$ (for g a function, and more generally a form, satisfying some necessary conditions).

A preliminary program for the first part of the course is as follows:
(1) Holomorphic functions and $\bar\partial$ equation (*);
(2) Cauchy formula and applications (*);
(3) Complex manifolds and analytic subsets;
(4) Bochner-Martinelli and Leray Formulas;
(5) De Rham currents.

Thanks to the tools developed so far, we will already encounter a new phenomenon with respect to the one-dimensional theory: the existence of domains which are not *domains of holomorphy*, meaning that every holomorphic function can be extend to a larger domain. The characterization of the domains of holomorphy (the so-called *Levi problem*) will be the main goal of the second part of the course. This will require to study some properties of (pluri)subharmonic functions, and to introduce the so-called L^2-*method*, giving a strong control on the solutions of the $\bar\partial$-equation. A preliminary program for this second part will be as follows:
(1) Subharmonic functions on $\mathbb{R}^n$ (*);
(2) Plurisubharmonic functions and L^2-method;
(3) Characterization and properties of Stein manifolds.

Depending on the time available and the interest of the students, the course will also present an introduction to the theory of positive closed currents, as well as some applications.

By the end of this course, students will have a deeper understanding of both the theoretical side and practical applications of complex analysis in several variables, equipping them with the analytical tools necessary for advanced research in the domain.

Main references.

- J.-P. Demailly, *Complex Analytic and Differential Geometry* , available at `https://www-fourier.ujf-grenoble.fr/~Demailly/manuscripts/agbook.pdf`
- L. Hormander, *An introduction to complex analysis in several variables*, Elsevier, 1973.
- S. G. Krantz, *Function Theory of Several Complex Variables*, AMS Chelsea Publishing, 1992

- L. Kaup, B. Kaup, *Holomorphic functions of several variables: an introduction to the fundamental theory.* Walter de Gruyter, 2011.

Prerequisites. Basic knowledge in complex analysis and topology. It is advised, but not necessary, to have followed "Elementi di Analisi Complessa". The arguments marked with (*) are introduced also in that course. They will be recalled here and studied in more detail, as they are the foundations for the following topics.

Exam informations. Oral exam, including a short presentation of a topic of choice not covered in the lectures (from a list to be provided, or proposed by the student and approved by the lecturer) and questions on the content of the course.