Holomorphic dynamics (Dinamica Olomorfa)
2023-2024, first semester
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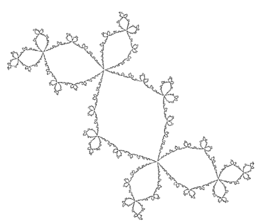
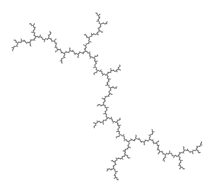
(Discrete) *Holomorphic Dynamics* is the study of the dynamical systems generated by the iteration of holomorphic self-maps on complex manifolds. It originated from the problem of the linearization of analytic germs in the XIX century, and from the works of Fatou and Julia at the beginning of the XX century. It is today a very active area of research, at the crossroad between complex analysis, ergodic theory, and fractal and dimension theory, and with strong and fruitful connections with complex and algebraic geometry, arithmetic and number theory, real dynamics, probability, differential geometry.

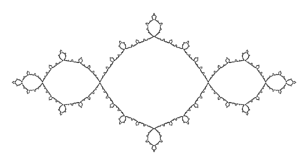
The goal of this course is to present an introduction to this domain, both from the local and the global points of view. We will mostly focus on the study of the iteration of polynomials on $\mathbb{C}$ and rational maps on the Riemann Sphere. In this case, one can decompose the phase space into two dynamically defined parts: the *Fatou set*, where the orbits are stable by a small perturbation of the starting point, and the *Julia set*, where a small perturbation of the point gives rise to a drastic change in the dynamics. The Julia set if a *fractal set*, i.e., a set with remarkable self-similarity.



$f(z) = z^2 - 0.1 + 0.75i$.



$f(z) = z^2 + i$

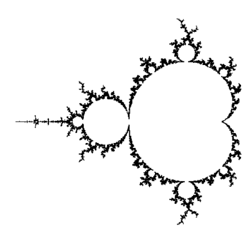


$f(z) = z^2 - 1$

In the first part of the course, we will cover the following topics:

- Poincaré metric; normal families, Montel Theorem;
- Fatou and Julia sets: definitions, examples;
- Structure of the Fatou set: linearization near attracting points, Fatou components, classification of periodic components;
- Structure of the Julia set: fractal geometry, connectivity, density of repelling periodic points.

In the same way as one can define the Fatou and Julia sets by considering the effect on orbits of the perturbation of the starting point, one can also study how, given a family of maps, the global dynamics depends on the specific map. In the case of the simplest family $f_\lambda(z) = z^2 + \lambda$, the *Mandelbrot set* is the set of parameters at which the global dynamics is very sensitive to a perturbation of the parameter. The Mandelbrot set is also a complicated fractal set: it has Hausdorff dimension 2, and small copies of the Mandelbrot set are dense in it! A complete understanding of the Mandelbrot set and its geometry are not achieved yet, and are among the main open questions in the field.



In this direction, we will cover the following topics:

- Holomorphic families of polynomials and rational maps;
- Stability and bifurcations: definitions and examples;
- Characterizations of stability; structural stability, hyperbolicity;
- Properties of bifurcation loci and of the Mandelbrot set.

If time allows, and depending on the interest of the students, the following are further topics that can be covered more in detail.

- Dynamics of Moebius transformations, Fuchsian groups;
- Quasiconformal maps, Sullivan's non-existence of wandering Fatou components;
- Ergodic theory of Julia sets: existence and uniqueness of the measure of maximal entropy, distribution of periodic points with respect to the measure of maximal entropy, statistical study of orbits and limit theorems;
- Hausdorff dimension and dimension theory of Julia sets and the Mandelbrot set;
- Elements of holomorphic dynamics in several complex variables.

Highlights. The course will give an introduction to the field of holomorphic dynamics, starting from the first definitions and arriving to a number of recent results and open problems in the field. In this theory many parts of modern mathematics come together – geometry, analysis, probability, combinatorics – and the techniques that have been introduced have had a number of applications also in other fields.

References.

- A. Beardon, *Iteration of Rational Functions*, Graduate Texts in Mathematics, 132, Springer-Verlag, New York, 1991.
- F. Berteloot F., V. Mayer, *Rudiments de dynamique holomorphe*, Paris, EDP Sciences, Les Ulis, 2001.
- L. Carleson, T. Gamelin, *Complex Dynamics*, Universitext: Tracts in Mathematics, Springer-Verlag, New York, 1993.
- T.-C. Dinh, N. Sibony, *Dynamics in several complex variables: endomorphisms of projective spaces and polynomial-like mappings*, Lecture Notes in Math., 1998, Springer, Berlin, 2010.
- J. Milnor, *Dynamics in One Complex Variable*, Princeton University Press, Princeton, 2006.

Prerequisites. Basic knowledge in complex analysis and topology. Some knowledge in dynamical systems is welcome, but not necessary.

Exam informations. Oral exam, including a short presentation of a topic of choice not covered in the lectures (from a list to be provided, or proposed by the student and approved by the lecturer) and questions on the content of the course.