M3/4/5P60 - GEOMETRIC COMPLEX ANALYSIS - HOME ASSIGNMENT 2

FABRIZIO BIANCHI

Please submit your work by 16:00 on March 15 to the Undergraduate office on floor 6 in the Huxley building.

Recall that the class of maps $\mathcal{S}$ is defined by

$$\mathcal{S} := \{f\colon \mathbb{D} \to \mathbb{C}\colon f \text{ is univalent on } \mathbb{D}, f(0) = 0, f'(0) = 1\}.$$

The following result may be particularly helpful in the solutions. You can use it without proving it (as everything else covered during the lectures).

Theorem A - (Theorem 6.9 in the Lecture Notes) *For each $f \in \mathcal{S}$ and every $z \in \mathbb{D}$ such that $|z| = r$,*

$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2}.$$

Exercise 1 - (Exercise 5.8 in Lecture notes)
Let Ω be an open subset of $\mathbb{C}$ and $\mathrm{Hol}(\Omega)$ denote the space of holomorphic maps from Ω to $\mathbb{C}$. Define the map $D\colon \mathrm{Hol}(\Omega) \to \mathrm{Hol}(\Omega)$ as $D(f) = f'$, that is, $D(f)(z) = f'(z)$. Prove that D is continuous from $\mathrm{Hol}(\Omega)$ to $\mathrm{Hol}(\Omega)$ with respect to the metric d'' ((5.4) in the lecture notes).

Exercise 2 - (Exercise 5.9 in Lecture notes)
Let $\mathcal{F}$ denote the space of all analytic functions $f\colon \mathbb{D} \to \mathbb{C}$ of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots,$$

such that for all $n \geq 2$ we have $|a_n| \leq n$. Prove that the family $\mathcal{F}$ is normal.

Exercise 3 - (Exercise 6.1 in Lecture notes)
Prove that $\mathcal{S}$ is a normal family.

Exercise 4[1] - (Exercise 6.3 in Lecture notes)
Let $k \geq 2$ be an integer and define

$$\Lambda_k := \left\{ f^{(k)}(0)\colon f \in \mathcal{S} \right\}.$$

Prove that

(1) for every $k \geq 2$, there is $r_k > 0$ such that $\Lambda_k = \{w \in \mathbb{C}\colon |w| \leq r_k\}$;
(2) there is a constant $C > 0$ such that for all $n \geq 1$ we have $r_n \leq C n^2 n!$.

Exercise 5 - (Exercise 6.5 in Lecture notes)
Let Ω be a non-empty, connected, simply connected subset of $\mathbb{C}$ that is not equal to $\mathbb{C}$. For $z \in \Omega$, the *conformal radius* of Ω at z is defined as

$$\mathrm{rad}_{\mathrm{conf}}(\Omega, z) = |\phi'(0)|,$$

where $\phi\colon \mathbb{D} \to \Omega$ is the Riemann mapping with $\phi(0) = z$.

(1) Prove that the quantity $\mathrm{rad}_{\mathrm{conf}}$ is independent of the choice of the Riemann map ϕ.

[1] This is the most difficult exercise of this homework.

(2) Define

$$r_z = \sup\{r > 0 \colon B(z,r) \subset \Omega\}.$$

Prove that

$$r_z \le \mathrm{rad}_{\mathrm{conf}}(\Omega, z) \le 4 r_z.$$

(3) Let $\Omega' \subsetneq \Omega$ be a connected, simply connected open set that contains z. Prove that

$$\mathrm{rad}_{\mathrm{conf}}(\Omega', z) < \mathrm{rad}_{\mathrm{conf}}(\Omega, z).$$