M3/4/5P60 - GEOMETRIC COMPLEX ANALYSIS - HOME ASSIGNMENT 1

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Please submit your work by 16:00 on February 15 to the Undergraduate office on floor 6 in the Huxley building.

Each exercise is worth 5 point. Thus, the total score of this assessment is 25. The final mark will be out of 20, as usual (with 20/25 you get 20).

Exercise 1 - (Exercise 2.3 in Lecture notes)
Let $h\colon \mathbb{H} \to \mathbb{H}$ be a holomorphic map. Prove that for every $a \in \mathbb{H}$ we have

$$|h'(a)| \leq \frac{\mathrm{Im}\ h(a)}{\mathrm{Im}\ a}.$$

Exercise 2 - (Exercise 3.3 in Lecture notes)
Let $f\colon \Omega \to \mathbb{C}$ be a holomorphic map that has a zero of order $k \geq 1$ at some $z_0 \in \Omega$.

(1) Prove that there is $\delta > 0$ and a holomorphic function $\psi\colon B(z_0, \delta) \to \mathbb{C}$ such that $\psi(z_0) = 0$, $\psi'(z_0) \neq 0$, and $f(z) = (\psi(z))^k$ on $B(z_0, \delta)$.
(2) Conclude from part (1) that near 0 the map f is k-to-1, that is, every point near 0 has exactly k pre-images near z_0.

Exercise 3 - (Exercise 3.4 in Lecture notes) Let Ω be an open set in $\mathbb{C}$ and $f\colon \Omega \to \mathbb{C}$ be a holomorphic map.

(1) Using Exercise 2, prove that if f is not constant, it is an *open map*, that is, f maps every open set in Ω to an open set in $\mathbb{C}$.
(2) using part (1), prove the maximum principle (Theorem 1.6 in the Lecture notes)

Exercise 4 - (Exercise 4.2 in Lecture notes) Let $z_i, i \geq 1$ be an infinite sequence in $\mathbb{D}$, and ρ be the Poincaré metric on $\mathbb{D}$. Show that z_i converges to some point z in $\mathbb{D}$ with respect to d_ρ if and only if it converges to $z \in \mathbb{D}$ with respect to the Euclidean metric.

Exercise 5 - (Exercise 4.3 in Lecture notes) Show that the disk $\mathbb{D}$ equipped with the Poincaré metric ρ is a complete metric space. That is, every Cauchy sequence in $\mathbb{D}$ with respect to d_ρ converges to some point in $\mathbb{D}$ with respect to the distance d_ρ.