# Solutions to mock exam, 2016-17
## Course: M3P16, M4P16, M5P16

**Solution to Question 1.** [**(a) seen 6pts; (b) seen 4pts ; (c) seen 10pts**]

**Part a)** A conformal metric on $\Omega$ is a continuously differentiable ($C^1$) function

$$\rho : \Omega \to [0, \infty),$$

where $\rho(z) \neq 0$ except on a discrete subset of $\Omega$. [4pts]

The Poincaré metric on $\mathbb{D}$ is defined as

$$\rho(z) = \frac{1}{1 - |z|^2}. \text{ [2pt]}$$

**Part b)** A one-to-one and onto holomorphic map $f : \Omega \to \Omega$ is called an *automorphism* of $\Omega$. The set of all automorphisms of $U$ is called the automorphism group of $\Omega$. [2pt]

By a theorem in the lectures, every automorphism of $\mathbb{D}$ is of the form

$$z \mapsto e^{i\theta} \cdot \frac{z - a}{1 - \overline{a}z},$$

for some $\theta \in \mathbb{R}$ and $a \in \mathbb{D}$. [2pt]

**Part c)** Let $h : \mathbb{D} \to \mathbb{D}$ be an automorphism of $\mathbb{D}$. First assume that $h(z) = e^{i\theta} \cdot z$. We have

$$(h^*\rho)(z) = \rho(h(z)) \cdot |h'(z)| = \frac{1}{1 - |h(z)|^2} \cdot |e^{i\theta}| = \frac{1}{1 - |z|^2} = \rho(z).$$

Thus, $h$ is an isometry of $(\mathbb{D}, \rho)$. [3pt]

Now assume that $h(z) = (z - a)/(1 - \overline{a}z)$. we have

$$\begin{aligned}
(h^*\rho)(z) &= \rho(h(z)) \cdot |h'(z)| \\
&= \frac{1}{1 - \left|\frac{z-a}{1-\overline{a}z}\right|^2} \cdot \frac{1 - |a|^2}{|1 - \overline{a}z|^2} \\
&= \frac{1 - |a|^2}{|1 - \overline{a}z|^2 - |z - a|^2} \\
&= \frac{1 - |a|^2}{1 - |z|^2 - |a|^2 + |a|^2|z|^2} \\
&= \rho(z).
\end{aligned}$$

That is, $h$ is an isometry of $(\mathbb{D}, \rho)$ [5pt]. Since the composition of isometries is an isometry, we conclude that any member of $\mathrm{Aut}(\mathbb{D})$ is an isometry of $(\mathbb{D}, \rho)$ [2pt].

**Solution to Question 2.** [**(a) seen 4pts ; (b) seen 4pts ; (c) unseen 6pts ; (d) unseen 6pts**]

**Part a)** Let $\Omega$ be an open set in $\mathbb{C}$ and $\mathcal{F}$ be a family (set) of maps that are defined on $\Omega$. We say that the family $\mathcal{F}$ is *normal* if every sequence of maps $f_n$, $n \geq 1$, in $\mathcal{F}$ has a sub-sequence that converges uniformly on every compact subset of $\Omega$. [4pt]

**Part b)** Theorem: Let $\mathcal{F}$ be a family of holomorphic maps defined on an open set $\Omega \subseteq \mathbb{C}$. If $\mathcal{F}$ is uniformly bounded on every compact subset of $\Omega$, then

(i) $\mathcal{F}$ is equicontinuous on every compact subset of $\Omega$;

(ii) $\mathcal{F}$ is a normal family.

**Part c)** By Montel's theorem, it is enough to prove that the family is uniformly bounded on every compact subset of $\mathbb{D}$. Let $K \subset \mathbb{D}$ be an arbitrary compact set. There is $r < 1$ such that for all $z \in K$, $|z| < r$. For every $f \in \mathcal{F}$, and every $z \in K$ we have

$$|f(z)| = \big|\sum_{n=1}^{\infty} a_n z^n\big| \leq \sum_{n=1}^{\infty} |a_n z^n| \leq \sum_{n=1}^{\infty} a_n z^n \leq \sum_{n=1}^{\infty} n r^n = \frac{r}{(1-r)^2} < +\infty.$$

This shows that the family is uniformly bounded on $K$. [6pt]

It is not necessary to find the precise value of the above series, but any finite upper bound on the series suffices. However, the above sum have been calculated as follows. Set $S_n = r + 2r^2 + 3r^3 + \cdots + nr^n$, for $n \geq 5$. Then,

$$S_n(1-r) = S_n - rS_n = r + r^2 + r^3 + r^n = r(1-r^n)/(1-r).$$

As $n \to +\infty$, $r^n \to 0$, and hence $S_n \to r/(1-r)^2$.

**Part d)** Let $k : \mathbb{D} \to \mathbb{C} \setminus (-\infty, -1/4]$ be the Koebe function. That is, $k$ is a one-to-one and onto holomorphic map. Define the new class of maps

$$\mathcal{G} = \{k^{-1} \circ f \mid f \in \mathcal{F}\}.$$

Every element of $\mathcal{G}$ is a holomorphic map defined on $\mathbb{D}$ with vales in $\mathbb{D}$. In particular, the family $\mathcal{G}$ is uniformly bounded on compact sets. By Montel's theorem, $\mathcal{G}$ forms a normal family.

Let $f_n$, $n \geq 1$, be an arbitrary sequence in $\mathcal{F}$. By the above paragraph, the sequence of maps $k^{-1} \circ f_n$, for $n \geq 1$, has a convergence sub-sequence, say, $k^{-1} \circ f_{n_i}$, for $i \geq 1$, which converges uniformly on compact subsets of $\mathbb{D}$. Since $k$ is a continuous function, it follows that the sequence $f_{n_i} = k \circ (k^{-1} \circ f_{n_i})$ converges uniformly on compact subsets of $\mathbb{D}$. [6pt]

**Solution to Question 3.** [**(a) seen 4pts ; (b) seen similar 6pts ; (c) unseen 10pts**]

**Part a)** Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic with $f(0) = 0$. Then,

(i) for all $z \in \mathbb{D}$ we have $|f(z)| \leq |z|$;

(ii) $|f'(0)| \leq 1$;

(iii) if either $f(z) = z$ for some non-zero $z \in \mathbb{D}$, or $|f'(0)| = 1$, then $f$ is a rotation about 0.

[4pt]

**Part b)** By the Riemann mapping theorem, there is a one-to-one and onto holomorphic map $\varphi : \Omega \to \mathbb{D}$. Moreover, by composition the Riemann map with a Möbius transformation, we may assume that $\varphi(z) = 0$. [3pt]

The map $g = \varphi \circ f \circ \varphi^{-1}$ is holomorphic and sends $\mathbb{D}$ into $\mathbb{D}$, with $g(0) = 0$. By the Schwarz lemma, we must have $|g'(0)| \leq 1$. Thus,

$$|f'(z)| = |\varphi'(z) \cdot f'(z) \cdot \frac{1}{\varphi'(z)}| = |g'(0)| \leq 1.$$

[3pt]

**Part c)** Since, $f$ is one-to-one, its inverse is define and holomorphic on $f(\Omega)$. Since $g(\Omega) \subset f(\Omega)$, the map $h = f^{-1} \circ g$ is defined an holomorphic on $\Omega$ with values in $\Omega$. Moreover, since $f(z) = g(z)$, we have $h(z) = z$.

By part (b) of the problem, we must have $|h'(z)| \leq 1$. This implies that $|g'(z)| \leq |f'(z)|$.

[10pt for this. There may be other ways to reduce the problem to the Schwarz lemma]

*Remark: Part (c) introduces a method known as "subordination": it allows one to give an upper bound on the derivative of an arbitrary holomorphic function by finding a conformal map which covers the image of that function.*

**Solution to Question 4. [(a) seen 6pts; (b) unseen and difficult 14pts]**

**Part a)** Theorem: Let $f : \mathbb{D} \to \mathbb{C}$ be a one-to-one holomorphic map with $f(0) = 0$ and $f'(0) = 1$. The set $f(\mathbb{D})$ contains the open ball $B(0, 1/4)$. [6pts]

**Part b)** The plan is to compose the map $f$ with a suitable Möbius transformation $M$, and apply the 1/4-theorem to the map $M \circ f$. But, for the 1/4-theorem, we need that $M \circ f(0) = 0$ and $(M \circ f)'(0) = 1$. This is guaranteed by requiring $M(0) = 0$ and $M'(0) = 1$. [3pt, for the idea to post-compose with a suitable Möbius transformation]

The set of Möbius transformations $M(z) = (az+b)/(cz+d)$ satisfying $M(0) = 0$ and $M'(0) = 1$ are of the form $z/(cz+1)$, for an arbitrary complex constant $c$. [2pt]

Let us first assume that $|\beta| \geq |\alpha|$. We choose $c = -1/\alpha$ so that $M(\alpha) = \infty$. We have $M(\beta) = \beta/(-\beta/\alpha + 1)$. The map $h = M \circ f : \mathbb{D} \to \mathbb{C}$ is defined, holomorphic, one-to-one, $h(0) = 0$, and $h'(0) = 1$. The map $h$ omits $M(\beta)$ since $f$ omits $\beta$. By the 1/4-theorem, we must have $|M(\beta)| \geq 1/4$. Hence

$$|\beta| \geq (-\beta/\alpha + 1)/4 \geq 2/4 = 1/2. \text{ [5pt]}$$

By a similar argument (replace $\alpha$ with $\beta$), we note that if $|\alpha| \geq |\beta|$, then $|\alpha| \geq 1/2$.

We choose the parameter $c = (-\beta - \alpha)/2\alpha\beta$ so that $M(\alpha) = -M(\beta)$. Apply the Koebe 1/4-theorem to the map $h = M \circ f$, which omits two values $M(\alpha)$ and $M(\beta)$ of the same size. So, we must have $|M(\alpha)| \geq 1/2$, which gives us

$$|\beta - \alpha| \geq 1/4.$$

Finally since for all $x$ and $y$, $xy \leq (x+y)^2/4$, we conclude that $|\beta - \alpha| \geq 1$. [4pt]

*Remark: Note that the biholomorphic map* $f : \mathbb{D} \to \mathbb{C} \setminus ((-\infty, -1/2] \cup [1/2, +\infty))$,

$$f(z) = \frac{z}{1+z^2}$$

*we studied in the lectures shows that the above inequality is sharp.*

**Solution to Question 5. [(a) seen 5pts ; (b) unseen 7pts ; (c) unseen and difficult 8pts]**

**Part a)** Theorem: Let $\mu : \mathbb{C} \to \mathbb{D}$ be a continuous map with $\sup_{z\in\mathbb{C}} |\mu(z)| < 1$. Then, there is a quasi-conformal map $f : \mathbb{C} \to \mathbb{C}$ such that $f_{\overline{z}}(z) = \mu(z) f_z(z)$ holds on $\mathbb{C}$. [3pts]

Moreover, the solution $f$ is unique if we assume that $f(0) = 0$ and $f(1) = 1$. [2pts]

**Part b)** By MRMT, we need to verify that $\mu$ is continuous and $\sup_{\mathbb{C}} |\mu| < 1$. [2pt]

Continuity: On $\operatorname{Im} z > 0$ and $\operatorname{Im} z < 0$ $\mu$ is continuous since it is given by a linear combination of the exponential maps. We need to show that the two formulas match on $\operatorname{Im} z = 0$. The first formula becomes

$$\mu(z) = \frac{1}{4}(e^{iz} + e^{-i\overline{z}}) = \frac{1}{4}(e^{iz} + e^{-iz}).$$

The second one becomes

$$\mu(z) = \frac{1}{4}(e^{-iz} + e^{i\overline{z}}) = \frac{1}{4}(e^{-iz} + e^{iz}).$$

This shows that $\mu$ is continuous on $\mathbb{C}$. [2pt]

Size of $\mu$: Let $z = x + iy$. When $\operatorname{Im} z = y \geq 0$, we have

$$|\mu(z)| \leq \frac{1}{4}\big(|e^{iz}| + |e^{-i\overline{z}}|\big) = \frac{1}{4}\big(|e^{-y}| + |e^{-y}|\big) \leq \frac{1}{4}(1+1) = \frac{1}{2}.$$

When, $\operatorname{Im} z = y \leq 0$,

$$|\mu(z)| \leq \frac{1}{4}\big(|e^{-iz}| + |e^{i\overline{z}}|\big) = \frac{1}{4}\big(|e^{y}| + |e^{y}|\big) \leq \frac{1}{4}(1+1) = \frac{1}{2}.$$

Hence, $\sup_{\mathbb{C}} |\mu| \leq 1/2 < 1$. [3pt]

**Part c)** First we show that $\mu(\overline{z}) = \overline{\mu(z)}$. When $\operatorname{Im} z \geq 0$,

$$\overline{\mu(z)} = \frac{1}{4}\overline{(e^{iz} + e^{-i\overline{z}})} = \frac{1}{4}(e^{-i\overline{z}} + e^{iz}),$$

and (as $\operatorname{Im} \overline{z} \leq 0$)

$$\mu(\overline{z}) = \frac{1}{4}(e^{-i\overline{z}} + e^{iz}).$$

When $\operatorname{Im} z \leq 0$,

$$\overline{\mu(z)} = \frac{1}{4}\overline{(e^{-iz} + e^{i\overline{z}})} = \frac{1}{4}(e^{-i\overline{z}} + e^{-iz}),$$

and (as $\operatorname{Im} \overline{z} \geq 0$)

$$\mu(\overline{z}) = \frac{1}{4}(e^{-i\overline{z}} + e^{-iz}).$$

These show that $\mu(\overline{z}) = \overline{\mu(z)}$ on $\mathbb{C}$. [2pt]

For the remaining part we present two solutions.

*Solution 1:* Define $G(z) = \overline{\varphi(\overline{z})}$. We claim that $G$ is also the solution of the Beltrami equation with coefficient $\mu$. Since, $G(0) = 0$ and $G(1) = 1$, by the uniqueness in the MRMT, we conclude that $G(z) \equiv \varphi(z)$, which gives the desired functional equation.

By MRMT, $\varphi$ is a diffeomorphism from $\mathbb{C}$ to $\mathbb{C}$. Thus, $G : \mathbb{C} \to \mathbb{C}$ is a diffeomorphism. So, to prove that $G$ is the solution of the Beltrami equation, it is enough to show that $G$ sends the field of ellipses generated by $\mu(z)$ to the field of circles on $\mathbb{C}$.

The map $z \to \overline{z}$ sends the infinitesimal ellipse given by $\mu(z)$ at $z$ to the ellipse $\overline{\mu(z)}$ at $\overline{z}$. This implies that $z \to \overline{z}$ sends the infinitesimal ellipse given by $\overline{\mu(\overline{z})}$ at $\overline{z}$ to the ellipse $\mu(z)$ at $z$. Then, $\varphi$ sends the infinitesimal ellipse given by $\mu(z)$ at $z$ to the infinitesimal circle at $\varphi(z)$. Combining these together, we conclude that the map $z \mapsto \varphi(\overline{z})$, sends the ellipse given by $\overline{\mu(\overline{z})}$ at $\overline{z}$ to the infinitesimal circle at $\varphi(\overline{z})$. Finally, since $z \mapsto \overline{z}$ preserves the field of circles (this is the second conjugation in $G$), the map $G$ sends the field of ellipses generated by $\overline{\mu(\overline{z})}$ to the field of circles. However, we already showed that $\overline{\mu(\overline{z})} = \mu(z)$. [6pt]

Solution 2: By MRMT, $\varphi$ is a diffeomorphism from $\mathbb{C}$ to $\mathbb{C}$. This implies that the map $H(z) = \varphi\big(\overline{\varphi^{-1}(\overline{z})}\big)$ is a well-defined diffeomorphism of $\mathbb{C}$. We aim to show that $H$ is biholomorphic from $\mathbb{C}$ to $\mathbb{C}$. Evidently, being the composition of the diffeomorphisms $\varphi$, $\varphi^{-1}$, and the complex conjugation, $H$ is one-to-one and onto. It remains to show that $H$ is holomorphic.

To prove that $H$ is holomorphic, by the Weyl's lemma, it is enough to show that $H_{\overline{z}} \equiv 0$. But, since $H$ is a diffeomorphism, this is equivalent to showing that $H$ maps the field of circles to the field of circles.

By MRMT, $\varphi$ sends the field of ellipses generated by $\mu(z)$ to the field of circles on $\mathbb{C}$. Thus, $\varphi^{-1}$ sends the field of circles to the field of ellipses generated by $\mu(z)$. The complex conjugation sends the field of circles to the field of circles. Hence, $\varphi^{-1}(\overline{z})$ sends the field of circles to the field of ellipses generated by $\mu(z)$. This implies that $\overline{\varphi^{-1}(\overline{z})}$ sends the field of circles to the field of ellipses generated by $\overline{\mu(\overline{z})} = \mu(z)$. As $\varphi$ sends those ellipses to the field of circles, $H$ sends the field of circles to the field of circles.

By a theorem in the lectures, the biholomorphic map $H : \mathbb{C} \to \mathbb{C}$ must be of the form $Az + B$, for some complex constants $A$ and $B$. On the other hand, as $\varphi(0) = 0$ and $\varphi(1) = 1$, we have $H(0) = 0$ and $H(1) = 1$. These imply that $B = 0$ and $A = 1$. That is, $H(z) = \varphi(\overline{\varphi^{-1}(\overline{z})}) \equiv z$. This implies that $\varphi(\overline{w}) = \overline{\varphi(w)}$, for all $w \in \mathbb{C}$. [6pt]