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BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2017

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 draft cover

1. Let Ω be a non-empty open set in $\mathbb{C}$.

    (a) Define the notion of *conformal metric* on Ω. What is the *Poincaré conformal metric* on the open unit disk $\mathbb{D}$.

    (b) Define the notion of *automorphism* of Ω. What is the *automorphism group* of the unit disk $\mathbb{D}$.

    (c) Prove that every automorphism of $\mathbb{D}$ is an *isometry* with respect to the Poincaré metric.

2. (a) Define the notion of *normal family*.

    (b) State (without proof) *Montel's normal family theorem*.

    (c) Let $\mathcal{F}$ be the set of holomorphic maps $f : \mathbb{D} \to \mathbb{C}$ where $f(z) = \sum_{n=1}^{\infty} a_n z^n$ with $|a_n| \leq n$, for all $n \geq 0$. Prove that $\mathcal{F}$ forms a *normal family*.

    (d) Let $\mathcal{F}$ be the set of holomorphic maps defined on $\mathbb{D}$ with values in $\mathbb{C} \setminus (-\infty, -1/4]$. Prove that $\mathcal{F}$ forms a *normal family*.

3. Let $\Omega \subsetneq \mathbb{C}$ be a non-empty and simply connected open set.

    (a) State (without proof) the *Schwarz lemma*.

    (b) Let $f : \Omega \to \Omega$ be a holomorphic map. Assume that for some $z \in \Omega$, $f(z) = z$. Prove that $|f'(z)| \leq 1$.

    (c) Let $f : \Omega \to \mathbb{C}$ be a one-to-one holomorphic map. Assume that $g : \Omega \to \mathbb{C}$ is an arbitrary holomorphic map such that $g(\Omega) \subseteq f(\Omega)$, and $f(z) = g(z)$ for some $z \in \Omega$. Prove that $|g'(z)| \leq |f'(z)|$.

4. Let $f : \mathbb{D} \to \mathbb{C}$ be a one-to-one holomorphic map, with $f(0) = 0$ and $f'(0) = 1$. Assume that there are $\alpha \in \mathbb{C} \setminus f(\mathbb{D})$ and $\beta \in \mathbb{C} \setminus f(\mathbb{D})$ with $\arg \alpha = \arg \beta + \pi$.

    (a) State (without proof) the *Koebe* $1/4$-*Theorem*.

    (b) Prove that

    $$|\alpha| + |\beta| \geq 1.$$

    [hint: compose f with a Mobius transformation]

5. (a) State (without proof) the *measurable Riemann mapping theorem* for the Beltrami equation with continuous coefficients.

(b) Define the function

$$\mu(z) = \begin{cases} \frac{1}{4} \cdot (e^{iz} + e^{-i\overline{z}}) & \text{if } \operatorname{Im} z \geq 0 \\ \frac{1}{4} \cdot \left(e^{-iz} + e^{i\overline{z}}\right) & \text{if } \operatorname{Im} z < 0 \end{cases}.$$

Does the Beltrami equation with coefficient μ have a solution?

(c) Let $\phi : \mathbb{C} \to \mathbb{C}$ be the solution of the Beltrami equation with coefficient μ and normalization $\phi(0) = 0$ and $\phi(1) = 1$. Prove that $\phi(\overline{z}) = \overline{\phi(z)}$, for all $z \in \mathbb{C}$. [hint: use $\mu(\overline{z}) = \overline{\mu(z)}$]