Imperial College London

Course:	M3P60/M4P60/M5P60
Setter:	Cheraghi
Checker:	Holzegel
Editor:	Helm
External:	Carbery
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BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2017

M3P60/M4P60/M5P60

Geometric Complex Analysis

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Imperial College London

Temporary front page

BSc, MSc and MSci EXAMINATIONS (MATHEMATICS)

May – June 2017

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Geometric Complex Analysis

Date: ??

Time: ??

Time Allowed: 2 Hours for M3 paper; 2.5 Hours for M4/5 paper

This paper has *4 Questions (M3 version); 5 Questions (M4/5 version)*.

Candidates should start their solutions to each question in a new main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw Mark	Up to 12	13	14	15	16	17	18	19	20
Extra Credit	0	$1/2$	1	$1\,1/2$	2	$2\,1/2$	3	$3\,1/2$	4

- Each question carries equal weight.
- Calculators may not be used.

In all questions below, $\mathbb{D}$ denotes the open unit disk $\{w \in \mathbb{C} \mid |w| < 1\}$.

1. Define the class of maps

$$\mathcal{S} = \{f : \mathbb{D} \to \mathbb{C} \mid f \text{ is holomorphic and one-to-one}, f(0) = 0, f'(0) = 1\}.$$

For $k \geq 2$, define

$$\Lambda_k = \{a_k \in \mathbb{C} \mid \text{ there is } f \in \mathcal{S} \text{ with } f(z) = z + a_2 z^2 + \cdots + a_k z^k + \dots\}.$$

Prove that for every $k \geq 2$, the following statements hold.

(a) For all $w \in \Lambda_k$ and all $\theta \in [0, 2\pi]$, $e^{i\theta} \cdot w \in \Lambda_k$;

(b) For all $w \in \Lambda_k$ and all $r \in (0, 1)$, $r \cdot w \in \Lambda_k$;

(c) There is $C > 0$, independent of k, such that for all $w \in \Lambda_k$ we have $|w| \leq Ck^2$.

2. (a) Let Ω be a connected open set in $\mathbb{C}$. Define the notion of a *conformal metric* on Ω.

(b) Let $f : \Omega \to \mathbb{D}$ be a biholomorphic map, and define the function

$$\rho_f(z) = |f'(z)| \cdot \frac{1}{1 - |f(z)|^2}, \quad \forall z \in \Omega.$$

Prove that ρ_f is independent of the choice of the Riemann map $f : \Omega \to \mathbb{D}$. That is, if $g : \Omega \to \mathbb{D}$ is another biholomorphic map, then for all $z \in \Omega$, $\rho_f(z) = \rho_g(z)$.

(c) Assume that $f_1 : \Omega_1 \to \mathbb{D}$ and $f_2 : \Omega_2 \to \mathbb{D}$ are two biholomorphic maps, where $\Omega_1 \subset \Omega_2 \subset \mathbb{C}$ but $\Omega_1 \neq \Omega_2$. Let ρ_{f_1} and ρ_{f_2} be the functions defined in Part (b). Prove that for all $z \in \Omega_1$ we have $\rho_{f_2}(z) < \rho_{f_1}(z)$.

3. Let $f : \mathbb{D} \to \mathbb{C}$ be a one-to-one holomorphic map, with $f(0) = 0$ and $f'(0) = 1$. Assume that there are $\alpha \in \mathbb{C} \setminus f(\mathbb{D})$ and $\beta \in \mathbb{C} \setminus f(\mathbb{D})$ with $\arg \alpha = \arg \beta + \pi$.

(a) State the Koebe $1/4$-Theorem.

(b) Prove that

$$\max\{|\alpha|, |\beta|\} \geq 1/2.$$

[hint: compose f with a Mobius transformation]

(c) By an example show that the inequality in Part (b) is sharp.

4. Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic map, and define the sequence of maps $f_n : \mathbb{C} \to \mathbb{C}$ as follows:

$$f_1 \equiv f, \quad f_{n+1} \equiv f \circ f_n, \forall n \geq 1.$$

(a) State what it means for the family of maps $\{f_n\}_{n=1}^{\infty}$ to be a normal family on some open set $U \subset \mathbb{C}$.

(b) Assume that there is $z_0 \in \mathbb{C}$ with $f(z_0) = z_0$ and $0 < |f'(z_0)| < 1$. Prove that there is $r_1 > 0$ such that the family $\{f_n\}_{n=1}^{\infty}$ is normal on the open ball $B(z_0, r_1)$.

(c) Define the sequence of maps $\phi_n : \mathbb{C} \to \mathbb{C}$ as

$$\phi_n(z) = \frac{f_n(z) - z_0}{f'(z_0)^n}.$$

Prove that there is $r_2 > 0$ such that each map ϕ_n is univalent on $B(z_0, r_2)$.

(d) Prove that the family $\{\phi_n\}_{n=1}^{\infty}$ is normal on $B(z_0, r_2)$.

5. (a) State (without proof) the *measurable Riemann mapping theorem* for the Beltrami equation with continuous coefficients.

(b) Define the function

$$\mu(z) = \begin{cases} \frac{1}{2} \cdot \frac{i - \sin(2\pi z)}{i + \sin(2\pi z)} & \text{if } \operatorname{Im} z \geq 0 \\ \frac{1}{2} \cdot \frac{i - \sin(2\pi \overline{z})}{i + \sin(2\pi \overline{z})} & \text{if } \operatorname{Im} z < 0 \end{cases},$$

where $\operatorname{Im} z$ denotes the imaginary part of z, and $\overline{z}$ denotes the complex conjugate of z. Why does the Beltrami equation with coefficient μ have a solution?

(c) Let $\phi : \mathbb{C} \to \mathbb{C}$ be the solution of the Beltrami equation with coefficient μ and normalization $\phi(0) = 0$ and $\phi(1) = 1$. Prove that the map $\Phi(z) = \phi(\phi^{-1}(z) + 1)$ is of the form $Az + B$ for some complex constants A and B. [hint: use $\mu(z + 1) = \mu(z)$]