Examination solutions 2015-16
Course: M3P60, M4P60, M5P60

Solution to Question 1. [**(a) seen similar 5pts ; (b) seen similar 5pts ; (c) from exercises 10pts**]

Part a) There is $f \in S$ whose k-th coefficient is equal to w. For $\theta' \in \mathbb{R}$, consider the map $h(z) = e^{-i\theta'} \cdot f(e^{i\theta'} \cdot z)$, for $z \in \mathbb{D}$. The map h is holomorphic and one-to-one on $\mathbb{D}$, with $h(0) = 0$ and $h'(0) = 1$. [3pts] We have

$$h(z) = z + \cdots + e^{-i\theta'} \cdot w \cdot e^{ik\theta'} z^k + \dots.$$

Thus, for every $\theta' \in \mathbb{R}$, $e^{-i\theta'} e^{ik\theta'} w \in \Lambda_k$. With $\theta' = \theta/(k-1)$ we see that $e^{i\theta} w \in \Lambda_k$. [2pts]

Part b) There is $f \in S$ whose k-th coefficient is equal to w. For $s \in (0,1)$ consider $h(z) = s^{-1} \cdot f(s \cdot z)$, for $|z| < 1$. Then, h is holomorphic and one-to-one on $\mathbb{D}$, with $h(0) = 0$ and $h'(0) = 1$. [3pts]

We have

$$h(z) = z + \cdots + s^{-1} \cdot w \cdot z^k \cdot s^k + \dots.$$

Thus, for every $s \in (0,1)$, $s^{-1} s^k w \in \Lambda_k$. For $s = r^{1/(k-1)}$, we see that $rw \in \Lambda_k$. [2pts]

Part c) By the Cauchy integral formula for the derivatives, $\forall r \in (0,1)$, we have

$$a_k = \frac{f^{(k)}(0)}{k!} = \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z^{k+1}}\, dz.$$

[2pts]

By the growth theorem, we know that $|f(z)| \le |z|/(1-|z|)^2$, for all $z \in \mathbb{D}$. [2pts] Thus,

$$|a_k| \le \frac{1}{2\pi} \int_{|z|=r} \frac{|f(z)|}{|z|^{k+1}}\, |dz| \le \frac{1}{2\pi} \int_0^{2\pi} \frac{r}{(1-r)^2 r^{k+1}}\, r d\theta \le \frac{r}{(1-r)^2 r^k}.$$

[3pts]

The above bound holds for all $r \in (0,1)$. By differentiating, we may find that the minimum of $\frac{1}{(1-r)^2 r^{k-1}}$ occurs at $r = (k-1)/(k+1)$, and the minimum value is

$$\frac{(k+1)^2}{4} \cdot \Big(1 + \frac{2}{k-1}\Big)^{k-1} = O(k^2) \cdot e^2.$$

[3pts]

Solution to Question 2. [(a) seen 3pts; (b) unseen 12pts ; (c) unseen 5pts]

Part a) A conformal metric on Ω is a continuously differentiable (C^1) function

$$\rho : \Omega \to [0, \infty),$$

where $\rho(z) \neq 0$ except on a discrete subset of Ω. [3pts]

Part b) The function ρ_f is the pull-back of the Poincaré metric ρ on $\mathbb{D}$, that is, $f^*\rho$. [3pts]

Since f and g are biholomorphisms, $g \circ f^{-1} : \mathbb{D} \to \mathbb{D}$ is biholomorphic. By a theorem in the lectures, this must be an isometry of the Poincaré metric. That is, $(g \circ f^{-1})^*\rho = \rho$. [3pts]

This implies that for all $z \in \mathbb{D}$ we have

$$|(g \circ f^{-1})'(z)| \frac{1}{1 - |g \circ f^{-1}(z)|^2} = \frac{1}{1 - |z|^2}.$$

Let $w = f^{-1}(z)$, then $z = f(w)$, and the above equation reduces to

$$\frac{|g'(w)|}{|f'(w)|} \cdot \frac{1}{1 - |g(w)|^2} = \frac{1}{1 - |f(w)|^2}.$$

This implies that ρ_f and ρ_g are equal. [3pts]

Part c) Fix an arbitrary $z \in \Omega_1$. There are biholomorphic maps $f : \Omega_1 \to \mathbb{D}$ and $g : \Omega_2 \to \mathbb{D}$ with $f(z) = g(z) = 0$. [3pts]

Since, $\Omega_1 \subsetneq \Omega_2$, the holomorphic map $g \circ f^{-1} : \mathbb{D} \to \mathbb{D}$ is not onto. By the Schwarz lemma, this implies that $|(g \circ f^{-1})'(0)| < 1$. Or, $|g'(z)| < |f'(z)|$. [3pts]

Thus,

$$\rho_g(z) = \frac{|g'(0)|}{1 - |g(z)|^2} = |g'(0)| < |f'(0)| = \frac{|g'(0)|}{1 - |f(z)|^2} = \rho_f(z).$$

[2pts]

Solution to Question 3. [**(a) seen 5pts; (b) unseen and difficult 9pts ; (c) unseen 6pts**]

Part a) Theorem: Let $f : \mathbb{D} \to \mathbb{C}$ be a one-to-one holomorphic map with $f(0) = 0$ and $f'(0) = 1$. The set $f(\mathbb{D})$ contains the open ball $B(0, 1/4)$. [5pts]

Part b) We find a Möbius transformation $M(z) = (az + b)/(cz + d)$ satisfying $M(0) = 0$ and $M'(0) = 1$. Then, M must have the form $z/(cz + 1)$, for an arbitrary complex constant c. [3pts]

Without loss of generality we may assume that $|\beta| \geq |\alpha|$ (in the other case switch β to α). Choose $c = -1/\alpha$ so that $M(\alpha) = \infty$. We have $M(\beta) = \beta/(-\beta/\alpha + 1)$. The map $h = M \circ f : \mathbb{D} \to \mathbb{C}$ is defined, holomorphic, one-to-one, $h(0) = 0$, and $h'(0) = 1$. [3pts]

The map h omits $M(\beta)$ since f omits β. By the 1/4-theorem, we must have $|M(\beta)| \geq 1/4$. Hence

$$|\beta| \geq (-\beta/\alpha + 1)/4 \geq 2/4 = 1/2.$$

[3pts]

Part c) As an example of a biholomorphic map, in lectures we have seen that the map

$$z \mapsto \frac{z}{1 + z^2} : \mathbb{D} \to \mathbb{C} \setminus \big((-\infty, -1/2] \cup [1/2, +\infty)\big)$$

is biholomorphic. Here $\alpha = 1/2$ and $\beta = -1/2$ are omitted from the range of the map. [6pts]

Solution to Question 4. [(a) seen 4pts ; (b) seen similar 6pts ; (c) unseen 5pts ; (d) unseen 5pts]

Part a) We say that the family $\{f_n\}$ is normal on U if every sequence of maps g_n, for $n \geq 1$, in the set $\{f_n \mid n \in \mathbb{N}\}$ has a sub-sequence that converges uniformly on compact subsets of U. [4pts]

Part b) Let $\delta = |f'(z_0)|$ and choose $\delta' \in (\delta, 1)$. By the continuity of $z \mapsto f'(z)$ there is $r_1 > 0$ such that for all $z \in B(z_0, r_1)$ we have $|f'(z)| \leq \delta'$.

For $z \in B(z_0, r_1)$ we have

$$d(f(z), z_0) = d(f(z), f(z_0)) \leq \sup_{c \in B(z_0, r_1)} |f'(c)| \cdot d(z, z_0) \leq \delta' r_1 < r_1.$$

This implies that f maps $B(z_0, r_1)$ into $B(z_0, r_1)$. [3pts]

In particular, the maps f_n, for $n \geq 1$, are all defined on $B(z_0, r_1)$ and map into $B(z_0, r_1)$. That is, the family is uniformly bounded on compact sets. By MOntel's theorem, this implies that $\{f_n\}$ is a normal family. [3pts]

Part c) Since, $f'(z_0) \neq 0$, we may choose r_1 in Part (b) small enough so that f is one-to-one on the ball $B(z_0, r_1)$. As f maps $B(z_0, r_1)$ into itself, the composition of f with itself are defined on $B(z_0, r_1)$ and are one-to-one. [3pts]

Since translation by a constant and multiplication by a constant are one-one, each ϕ_n is one-to-one. [2pts]

Part d) We have $\phi_n(z_0) = 0$, and $\phi_n'(z_0) = 1$, for all $n \geq 1$.

Consider the maps

$$h_n(z) = r_2^{-1} \phi_n(z \cdot r_2), z \in \mathbb{D}.$$

Each h_n is univalent on $\mathbb{D}$ with $h_n(0) = 0$ and $h_n'(0) = 1$. [3pts]

By the growth theorem and the Montel's theorem, the family of maps h_n is normal. This implies that the family of maps ϕ_n is normal. [2pts]

Solution to Question 5. **[(a) seen 5pts ; (b) unseen 7pts ; (c) unseen and difficult 8pts]**

Part a) THM: Let $\mu : \mathbb{C} \to \mathbb{D}$ be a continuous map with $\sup_{z\in\mathbb{C}} |\mu(z)| < 1$. Then, there is a quasi-conformal map $f : \mathbb{C} \to \mathbb{C}$ such that $f_{\overline{z}}(z) = \mu(z) f_z(z)$ holds on $\mathbb{C}$. [3pts]

Moreover, the solution f is unique if we assume that $f(0) = 0$ and $f(1) = 1$. [2pts]

Part b) By MRMT, we need to verify that μ is continuous and $\sup_{\mathbb{C}} |\mu| < 1$.

We have seen in lectures that

$$z \mapsto \sin(2\pi z) : \{z \in \mathbb{C} \mid 0 < \operatorname{Re} z < 1, \operatorname{Im} z > 0\} \to \mathbb{H}$$

is a biholomorphism. Since, $\sin(2\pi z)$ is periodic of period $+1$, the above function maps the upper half plane to the upper half plane.

We have also seen in the lectures that $w \mapsto \frac{i-w}{i+w}$ is a biholomorphism from $\mathbb{H}$ to $\mathbb{D}$. These imply that $\mu : \mathbb{H} \to \mathbb{D}/2$ is holomorphic. In particular, it is continuous.

By the definition of μ and the above paragraph, μ maps the lower half plane into $\mathbb{D}/2$, and is continuous.

On $\operatorname{Im} z = 0$, we have $\overline{z} = z$, and the two formulas become the same. So μ is also continuous on $\mathbb{R}$.

By the above paragraphs, we have $\sup_{z\in\mathbb{C}} |\mu(z)| \le 1/2 < 1$. Hence all condition of the MRMT are satisfied.

[3pts to understand sine + 2 pts to understand the fraction + 1 pt for continuity + 2pts for norm.]

Part c) The map ϕ is a diffeomorphism from $\mathbb{C}$ to $\mathbb{C}$. It maps the field of ellipses defined by μ to the field of circles. Since $\mu(z+1) = \mu(z)$, the field of ellipses defined by μ is periodic of period $+1$. This implies that Φ preserves the field of circles on all of $\mathbb{C}$, that is $\Phi_{\overline{z}} \equiv 0$. Hence, by Weyl's lemma, $\Phi : \mathbb{C} \to \mathbb{C}$ is holomorphic. [6pts]

Since $\phi : \mathbb{C} \to \mathbb{C}$ is one-to-one and onto, then, Φ is biholomorphic. By a theorem in the lectures, Φ must be of the form $Az + B$, for some complex constants A and B. [2pts]