

# Chapter 4

## Conformal geometry on the disk

### 4.1 Poincare metric

Let  $X$  be a set. Recall that a *metric* on  $X$  is a function  $d : X \times X \rightarrow \mathbb{R}$  such that for all  $x, y$ , and  $z$  in  $X$  we have

- (i)  $d(x, y) = d(y, x)$ ,
- (ii)  $d(x, y) \geq 0$ , and  $d(x, y) = 0$  if and only if  $x = y$ ,
- (iii)  $d(x, y) \leq d(x, z) + d(z, y)$ .

A metric on a space  $X$  allows us to talk about distances on  $X$ . The most familiar example is probably the *Euclidean distance* on  $\mathbb{R}$  given by the absolute value. That is,  $d(x, y) = |x - y|$ , for  $x$  and  $y$  in  $\mathbb{R}$ . This notion of distance respects the underlying operation of addition that is described by the relation  $d(x, y) = d(x + c, y + c)$ , for all  $c \in \mathbb{R}$ . That is,  $d$  is *invariant* under translations.

Another example of a metric on  $\mathbb{R}$  is given by the function

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y. \end{cases}$$

The above notion of metric is rather general for our purpose. There is a more restrictive notion of metric that is suitable in the world of complex analysis. Before we define that, recall that a set  $A \subset \mathbb{C}$  is called a discrete set, if for every  $z \in A$  there is an open set  $U$  in  $\mathbb{C}$  with  $A \cap U = \{z\}$ .

**Definition 4.1.** Let  $\Omega$  be a domain in  $\mathbb{C}$ . A *conformal metric* on  $\Omega$  is a continuously differentiable ( $C^1$ ) function

$$\rho : \Omega \rightarrow [0, \infty),$$

where  $\rho(z) \neq 0$  except on a discrete subset of  $\Omega$ . If  $z \in \Omega$  and  $\xi \in \mathbb{C}$  is a vector, we define the *length of  $\xi$  at  $z$  with respect to the metric  $\rho$*  as

$$\|\xi\|_{\rho, z} = \rho(z) \cdot |\xi|.$$

Here,  $|\xi|$  denotes the Euclidean norm of  $\xi$ , i.e.  $\sqrt{\xi\bar{\xi}}$ .

*Remark 4.2.* In contrast to what we learn in a calculus course that a vector has only direction and magnitude, in the above notion of the metric, a vector has direction, magnitude, and position. That is the length of a vector also depends on its position.

**Definition 4.3.** Assume that  $\gamma : [a, b] \rightarrow \Omega$  is a  $C^1$  curve. The *length of  $\gamma$  with respect to the metric  $\rho$*  is defined as

$$\ell_\rho(\gamma) = \int_a^b \left\| \frac{\partial\gamma(t)}{\partial t} \right\|_{\rho, \gamma(t)} dt = \int_a^b \rho(\gamma(t)) \cdot \left| \frac{\partial\gamma(t)}{\partial t} \right| dt.$$

The length of a piece-wise  $C^1$  curve is defined as the sum of the lengths of its  $C^1$  parts.

As in the definition of the integration along a curve in complex analysis, the above notion of length is independent of the parameterization of the curve.

It is convenient to think of the tangent vector to  $\gamma'(t)$  at  $\gamma(t)$  as a vector based at  $\gamma(t)$ . See Figure 4.1

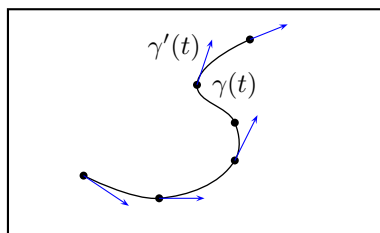


Figure 4.1: The tangent vectors to a  $C^1$  curve  $\gamma$  in the calculation of the length of  $\gamma$  with respect to a conformal metric.

For every  $c \in \mathbb{C}$  with  $|c| = 1$  we have

$$\|c \cdot \xi\|_{\rho, z} = \|\xi\|_{\rho, z}.$$

By the above relation, the length of a vector  $\xi$  at some  $z \in \Omega$  is independent of its direction. This feature makes conformal metrics natural in complex analysis, as we shall see in this section.

In the classical literature in analysis, sometimes you find the notation

$$\ell_\rho(\gamma) = \int_\gamma \rho(z) |dz|,$$

for the length of  $\gamma$  with respect to the metric  $\rho$ . This is consistent with the definition of integration along curves you learn in complex analysis.

**Definition 4.4.** A set  $A$  in  $\mathbb{C}$  is called *path connected* if for every two points  $z$  and  $w$  in  $A$  there is a continuous map  $\gamma : [0, 1] \rightarrow A$  with  $\gamma(0) = z$  and  $\gamma(1) = w$ .

It follows that any path connected subset of  $\mathbb{C}$  is connected, but there are connected subsets of  $\mathbb{C}$  that are not path connected.

**Definition 4.5.** Let  $\rho$  be a conformal metric defined on an open and path connected set  $\Omega \subseteq \mathbb{C}$ . Given  $z$  and  $w$  in  $\Omega$  let  $\Gamma_{z,w}$  denote the set of all piece-wise  $C^1$  curves  $\gamma : [0, 1] \rightarrow \Omega$  with  $\gamma(0) = z$  and  $\gamma(1) = w$ . Define

$$d_\rho(z, w) = \inf\{\ell_\rho(\gamma) : \gamma \in \Gamma_{z,w}\}.$$

It follows (Exercise 4.1) that  $d_\rho$  defines a metric on  $\Omega$ , that is,  $d_\rho : \Omega \times \Omega \rightarrow [0, \infty)$  satisfies the required properties listed at the beginning of this chapter.

*Remark 4.6.* One should not confuse the notion of  $d_\rho(z, w)$  with the notion of the length of the vector  $w - z$  at  $z$  with respect to  $\rho$ . In general, these are different values and not related.

**Example 4.7.** When  $\rho(z) \equiv 1$  on  $\Omega$ , the length of a piece-wise  $C^1$  curve  $\gamma$  with respect to  $\rho$ ,  $\ell_\rho(\gamma)$ , becomes the Euclidean length of  $\gamma$  (which we learn in calculus). When  $\Omega = \mathbb{C}$ ,  $d_\rho$  becomes the Euclidean distance. In general, when  $\Omega$  is a convex set, that is, the line segment connecting any two points in  $\Omega$  lies in  $\Omega$ , then  $d_\rho$  is the restriction of the Euclidean metric to  $\Omega$ . But in general, there may not be a curve of shortest length between two points in  $\Omega$ . See Figure 4.2.

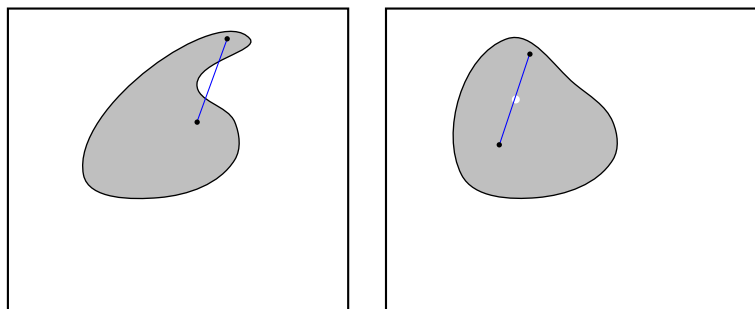


Figure 4.2: Examples of non-convex domains; one with a point omitted, and the other with a special shape.

**Definition 4.8.** The Poincaré metric on  $\mathbb{D}$  is defined as

$$\rho(z) = \frac{1}{1 - |z|^2}.$$

The Poincaré metric has been used to gain deep understanding into the complex analysis on the disk and beyond. We shall study this metric in details.

For any vector  $\xi \in \mathbb{C}$  we have

$$\begin{aligned}\|\xi\|_{\rho,0} &= \rho(0) \cdot |\xi| = |\xi|, \\ \|\xi\|_{\rho,(1/2+0i)} &= \rho(1/2 + 0i) \cdot |\xi| = \frac{4}{3} \cdot |\xi|, \\ \|\xi\|_{\rho,(0+i/2)} &= \rho(0 + i/2) \cdot |\xi| = \frac{4}{3} \cdot |\xi|, \\ \|\xi\|_{\rho,(0.99+0i)} &= \rho(0.99 + 0i) \cdot |\xi| = (50.251256\dots) \cdot |\xi|.\end{aligned}$$

The metric  $\rho$  has a rotational symmetry about 0, i.e.  $\rho(c \cdot z) = \rho(z)$  for all  $c \in \mathbb{C}$  with  $|c| = 1$ . Also,  $\rho(z) \rightarrow \infty$  as  $|z|$  tends to  $+1$  from below. There are many conformal metrics on  $\mathbb{D}$  with rotational symmetry and diverging to  $+\infty$  near the boundary, but the speed of divergence in the Poincaré metric is chosen to guarantee some significant properties.

**Example 4.9.** Let us calculate the length of the curve  $[0, 1 - \varepsilon]$  with respect to the Poincaré metric  $\rho$  on  $\mathbb{D}$ . Define  $\gamma(t) = t + 0i$ , for  $t \in [0, 1 - \varepsilon]$ . Then,

$$\ell_\rho(\gamma) = \int_0^{1-\varepsilon} \rho(\gamma(t)) \cdot |\gamma'(t)| dt = \int_0^{1-\varepsilon} \frac{1}{1-t^2} dt = \frac{1}{2} \log \left( \frac{1+t}{1-t} \right) \Big|_{t=0}^{t=1-\varepsilon} = \frac{1}{2} \log \left( \frac{2-\varepsilon}{\varepsilon} \right).$$

We note that the above quantity tends to  $+\infty$  as  $\varepsilon$  tends to 0. This means that the point  $+1$  is at distance  $\infty$  from the point 0 along the curve  $\gamma$ , with respect to the Poincaré metric on  $\mathbb{D}$ .

**Proposition 4.10.** *Let  $\rho$  be the Poincaré metric on  $\mathbb{D}$ . We have*

$$d_\rho(0, 1 - \varepsilon) = \frac{1}{2} \log \left( \frac{2 - \varepsilon}{\varepsilon} \right).$$

*Proof.* Let  $\eta : [a, b] \rightarrow \mathbb{D}$  be a  $C^1$  curve with  $\eta(a) = 0$  and  $\eta(b) = 1 - \varepsilon + 0i$ . In coordinates, let  $\eta(t) = \eta_1(t) + i\eta_2(t)$ , for  $t \in [a, b]$ . Both  $\eta_1$  and  $\eta_2$  are  $C^1$  and moreover, for all  $t \in [a, b]$ ,

$$|\eta'(t)| = |\eta_1'(t) + i\eta_2'(t)| \geq |\eta_1'(t)|.$$

Also, since  $|\eta(t)| \geq |\eta_1(t)|$ , for all  $t \in [a, b]$  we have

$$\rho(\eta(t)) \geq \rho(\eta_1(t)).$$

Note that  $\eta_1 : [a, b] \rightarrow (-1, 1)$  is a  $C^1$  curve with  $\eta_1(a) = 0$  and  $\eta_1(b) = 1 - \varepsilon$ . However,  $\eta_1$  may not be a monotone function of  $t \in [a, b]$ . Its image may cover some parts of  $[0, 1 - \varepsilon]$  several times. If necessary, we may throw away parts of this curve and keep a piece-wise

$C^1$  and monotone part of  $\eta_1$  that maps a subset of  $[a, b]$  to  $[0, 1 - \varepsilon]$ . Let  $A \subseteq [a, b]$  be that set. Using the above inequalities,

$$\begin{aligned} \ell_\rho(\eta) &= \int_a^b \rho(\eta(t)) \cdot |\eta'(t)| dt \\ &\geq \int_a^b \rho(\eta_1(t)) \cdot |\eta_1'(t)| dt \\ &\geq \int_A \rho(\eta(t)) \cdot |\eta'(t)| dt \\ &= \frac{1}{2} \log\left(\frac{2 - \varepsilon}{\varepsilon}\right). \end{aligned}$$

□

Figure 4.3 shows the graph of the function  $r \mapsto d_\rho(0, r)$ , on  $(0, 1)$ . Note how on a large interval  $(0, r)$  (with  $r$  close to 1) the distance of the points from 0 is less than 5.

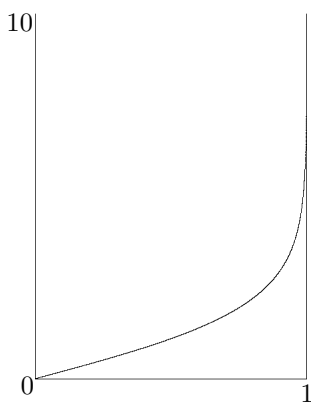


Figure 4.3: The graph of the function  $r \mapsto d_\rho(0, r)$ , for  $r \in (0, 1)$ .

## 4.2 Isometries

**Definition 4.11.** Assume that  $\Omega_1$  and  $\Omega_2$  are open sets in  $\mathbb{C}$  and

$$f : \Omega_1 \rightarrow \Omega_2$$

is a holomorphic map. Let  $\rho_2$  be a conformal metric on  $\Omega_2$ . The *pull-back* of  $\rho_2$  by  $f$  is defined as

$$(f^* \rho_2)(z) = \rho_2(f(z)) \cdot |f'(z)|.$$

It is clear that if  $\rho_2$  is  $C^1$  then  $f^*\rho_2$  is also  $C^1$ . On the other hand,  $f^*\rho_2(z) = 0$  if and only if either  $\rho_2(f(z)) = 0$  or  $f'(z) = 0$ . Since,  $f'$  is a holomorphic function on  $\Omega_1$ , the set of points where it becomes 0 is a discrete set. These imply that the pull-back of a conformal metric under a holomorphic map is a conformal metric. Indeed, this is the reason for the name conformal metric. These are metrics that behave well under holomorphic transformations.

By the above definition, if  $\xi$  is a vector in  $\mathbb{C}$  and  $z \in \Omega_1$ , then

$$\| \xi \|_{f^*\rho_2, z} = \rho_2(f(z)) \cdot |f'(z)| \cdot |\xi| = \rho_2(f(z)) \cdot |f'(z)| \cdot |\xi|.$$

Let us denote the metric  $f^*\rho_2$  on  $\Omega_1$  by  $\rho_1$ . If  $\gamma_1$  is a  $C^1$  curve in  $\Omega_1$ , and  $\gamma_2 = f \circ \gamma_1$ , then it follows that  $\ell_{\rho_1}(\gamma_1) = \ell_{\rho_2}(\gamma_2)$ .

$$\begin{aligned} \ell_{\rho_2}(\gamma_2) &= \int_a^b \rho_2(\gamma_2(t)) \cdot |\gamma_2'(t)| dt = \int_a^b \rho_2(f(\gamma_1(t))) \cdot |(f \circ \gamma_1)'(t)| dt = \\ &= \int_a^b \rho_2(f(\gamma_1(t))) \cdot |f'(\gamma_1(t))| \cdot |\gamma_1'(t)| dt = \int_a^b \rho_1(\gamma_1(t)) |\gamma_1'(t)| dt = \ell_{\rho_1}(\gamma_1) \end{aligned}$$

**Definition 4.12.** Let  $\Omega_1$  and  $\Omega_2$  be open sets in  $\mathbb{C}$  and  $f : \Omega_1 \rightarrow \Omega_2$  be an onto holomorphic map. Let  $\rho_i$  be a conformal metric on  $\Omega_i$ , for  $i = 1, 2$ . Then,  $f$  is called an *isometry* from  $(\Omega_1, \rho_1)$  to  $(\Omega_2, \rho_2)$  if for all  $z \in \Omega_1$  we have

$$f^*\rho_2(z) = \rho_1(z).$$

**Proposition 4.13.** Let  $\Omega_1$  and  $\Omega_2$  be open sets in  $\mathbb{C}$  with conformal metrics  $\rho_1$  and  $\rho_2$ , respectively. Assume that  $f$  is an isometry from  $(\Omega_1, \rho_1)$  to  $(\Omega_2, \rho_2)$ . Then for every  $C^1$  curve  $\gamma : [0, 1] \rightarrow \Omega_1$  we have

$$\ell_{\rho_1}(\gamma) = \ell_{\rho_2}(f \circ \gamma).$$

The curve  $f \circ \gamma$  is often called the *push-forward* of the curve  $\gamma$  by  $f$ , and is denoted by  $f_*\gamma$ . That is,

$$f_*\gamma(t) = f \circ \gamma(t), \text{ for } t \in [0, 1].$$

*Proof.* By Definition 4.3 we have

$$\ell_{\rho_1}(\gamma) = \int_0^1 \| \gamma'(t) \|_{\rho_1, \gamma(t)} dt, \quad \ell_{\rho_2}(f \circ \gamma) = \int_0^1 \| (f \circ \gamma)'(t) \|_{\rho_2, f \circ \gamma(t)} dt.$$

To prove that the above integrals give the same value, it is enough to show that the integrands are equal on  $[0, 1]$ . That is,

$$\begin{aligned}
\| (f \circ \gamma)'(t) \|_{\rho_2, f \circ \gamma(t)} &= \rho_2(f \circ \gamma(t)) \cdot |(f \circ \gamma)'(t)| \\
&= \rho_2(f \circ \gamma(t)) \cdot |f'(\gamma(t)) \cdot \gamma'(t)| \\
&= \rho_2(f \circ \gamma(t)) \cdot |f'(\gamma(t))| \cdot |\gamma'(t)| \\
&= (f^* \rho_2)(\gamma(t)) \cdot |\gamma'(t)| \\
&= \| \gamma'(t) \|_{\rho_1, \gamma(t)}.
\end{aligned}$$

□

Note that if  $f$  is an isometry from  $(\Omega_1, \rho_1)$  to  $(\Omega_2, \rho_2)$  we may not conclude that for every  $z$  and  $w$  in  $\Omega_1$  we have

$$d_{\rho_1}(z, w) = d_{\rho_2}(f(z), f(w)). \quad (4.1)$$

That is because not every curve in  $\Omega_2$  from  $f(z)$  to  $f(w)$  is obtained from push-forward of a curve in  $\Omega_1$  from  $z$  to  $w$ . We illustrate this by an example.

**Example 4.14.** Let

$$\Omega_1 = \{z \in \mathbb{C} : e^{-1} < |z| < e\}, \quad \Omega_2 = \{z \in \mathbb{C} : e^{-2} < |z| < e^2\}$$

and define

$$f : \Omega_1 \rightarrow \Omega_2, \quad f(z) = z^2.$$

Consider the conformal metrics

$$\rho_1(z) = \frac{\pi}{2} \cdot \frac{1}{|z| \cdot \cos\left(\frac{\pi \log |z|}{2}\right)},$$

$$\rho_2(z) = \frac{\pi}{4} \cdot \frac{1}{|z| \cdot \cos\left(\frac{\pi \log |z|}{4}\right)}.$$

We have

$$\begin{aligned}
f^* \rho_2(z) &= \rho_2(f(z)) \cdot |f'(z)| \\
&= \frac{\pi}{4} \cdot \frac{1}{|z|^2 \cdot \cos\left(\frac{\pi \log |z|^2}{4}\right)} \cdot 2|z| \\
&= \frac{\pi}{2} \cdot \frac{1}{|z| \cdot \cos\left(\frac{\pi \log |z|}{2}\right)} = \rho_1(z).
\end{aligned}$$

That means that  $f$  is an isometry from  $(\Omega_1, \rho_1)$  to  $(\Omega_2, \rho_2)$ .

Note that  $f$  is not one-to-one, for example,  $f(+1) = f(-1) = +1$ . If  $\gamma$  is a curve in  $\Omega_1$  connecting  $+1$  to  $-1$ ,  $f \circ \gamma$  is a curve in  $\Omega_2$  that connects  $+1$  to itself and wraps around in  $\Omega_2$  at least once. But, there is a constant curve with zero length from  $+1$  to  $+1$  in  $\Omega_2$ . The constant curve is not the image of any continuous curve from  $+1$  to  $-1$  in  $\Omega_1$ . This is the only issue that prevents us from having Equation (4.1). As you will show in Exercise 4.5, if  $f : \Omega_1 \rightarrow \Omega_2$  is one-to-one, then Equation (4.1) holds for all  $z$  and  $w$  in  $\Omega_1$ .

Let  $\Omega_1, \Omega_2$ , and  $\Omega_3$  be open sets in  $\mathbb{C}$  with conformal metrics  $\rho_1, \rho_2$ , and  $\rho_3$ , respectively. Assume that  $f$  is an isometry from  $(\Omega_1, \rho_1)$  to  $(\Omega_2, \rho_2)$ , and  $g$  is an isometry from  $(\Omega_2, \rho_2)$  to  $(\Omega_3, \rho_3)$ . You can show that  $g \circ f$  is an isometry from  $(\Omega_1, \rho_1)$  to  $(\Omega_3, \rho_3)$ .

**Theorem 4.15.** *Every automorphism of  $\mathbb{D}$  is an isometry from  $(\mathbb{D}, \rho)$  to  $(\mathbb{D}, \rho)$ , where  $\rho$  is the Poincaré metric on  $\mathbb{D}$ .*

*Proof.* By Theorem 2.5 every automorphism of  $\mathbb{D}$  is of the form

$$z \mapsto e^{i\theta} \cdot \frac{z - a}{1 - \bar{a}z},$$

for some  $\theta \in \mathbb{R}$  and  $a \in \mathbb{D}$ . We shall prove the theorem in two steps.

First assume that  $h(z) = e^{i\theta} \cdot z$ . We have

$$(h^* \rho)(z) = \rho(h(z)) \cdot |h'(z)| = \frac{1}{1 - |h(z)|^2} \cdot |e^{i\theta}| = \frac{1}{1 - |z|^2} = \rho(z).$$

Thus,  $h$  is an isometry of  $(\mathbb{D}, \rho)$ .

Now assume that  $h(z) = (z - a)/(1 - \bar{a}z)$ . we have

$$\begin{aligned} (h^* \rho)(z) &= \rho(h(z)) \cdot |h'(z)| \\ &= \frac{1}{1 - \left| \frac{z-a}{1-\bar{a}z} \right|^2} \cdot \frac{1 - |a|^2}{|1 - \bar{a}z|^2} \\ &= \frac{1 - |a|^2}{|1 - \bar{a}z|^2 - |z - a|^2} \\ &= \frac{1 - |a|^2}{1 - |z|^2 - |a|^2 + |a|^2|z|^2} \\ &= \rho(z). \end{aligned}$$

That is,  $h$  is an isometry of  $(\mathbb{D}, \rho)$ . Since the composition of isometries is an isometry, see Exercise 4.4, we conclude that any member of  $\text{Aut}(\mathbb{D})$  is an isometry of  $(\mathbb{D}, \rho)$ .  $\square$

As a corollary of the above theorem, and Proposition 4.10, we are able to calculate the Poincaré distance between any two points on  $\mathbb{D}$ .



**Proposition 4.16.** *Let  $p$  and  $q$  be two points in  $\mathbb{D}$  equipped with Poincaré metric  $\rho$ . We have*

$$d_\rho(p, q) = \frac{1}{2} \log \left( \frac{1 + \left| \frac{p-q}{1-\bar{p}q} \right|}{1 - \left| \frac{p-q}{1-\bar{p}q} \right|} \right).$$

*Proof.* When  $p = 0$  and  $q$  is a positive real number the formula in the proposition reduces to the one in Proposition 4.10. Now, define

$$\varphi_1(z) = \frac{z - p}{1 - \bar{p}z},$$

and

$$\varphi_2(z) = \frac{|\varphi_1(q)|}{\varphi_1(q)} \cdot z.$$

Note that both of  $\varphi_1$  and  $\varphi_2$  belong to  $\text{Aut}(\mathbb{D})$ . Then, by Theorem 4.15 and Exercise 4.5, we must have

$$d_\rho(p, q) = d_\rho(\varphi_1(p), \varphi_1(q)) = d_\rho(0, \varphi_1(q)) = d_\rho(\varphi_2(0), \varphi_2(\varphi_1(q))) = d_\rho(0, |\varphi_1(q)|).$$

Using Proposition 4.10 with  $1 - \varepsilon = |\varphi_1(q)|$  we obtain

$$d_\rho(0, |\varphi_1(q)|) = \frac{1}{2} \log \left( \frac{1 + |\varphi_1(q)|}{1 - |\varphi_1(q)|} \right).$$

This finishes the proof of the proposition. □

The proof of the above proposition also provides us with the shortest curve connecting the two points  $p$  and  $q$ . We state this in a separate proposition.

**Proposition 4.17.** *Let  $p$  and  $q$  be two distinct points in  $\mathbb{D}$ . The shortest curve with respect to  $\rho$  connecting  $p$  to  $q$  is given by the formula*

$$\gamma_{p,q}(t) = \frac{t \frac{q-p}{1-\bar{q}p} + p}{1 + t\bar{p} \frac{q-p}{1-\bar{q}p}}, \quad 0 \leq t \leq 1.$$

*Proof.* In Proposition 4.10 and its preceding example, the shortest curve connecting 0 to a point  $z$  on  $(0, 1) \subset \mathbb{D}$  is given by the interval  $[0, z]$ . As the rotation  $z \mapsto e^{i\theta} \cdot z$ , for each fixed  $\theta \in \mathbb{R}$ , is an isometry of  $(\mathbb{D}, \rho)$ , we conclude that the shortest curve connecting 0 to a given point  $z \in \mathbb{D}$  is the curve  $t \mapsto t \cdot z$ ,  $0 \leq t \leq 1$ .

Consider the automorphism

$$\varphi_1(z) = \frac{z - p}{1 - \bar{p}z}.$$

We have  $\varphi_1(p) = 0$  and  $\varphi_1(q) \in \mathbb{D}$ . The inverse of this map is given by the formula

$$\varphi_1^{-1}(z) = \frac{z + p}{1 + \bar{p}z}.$$

By the above paragraph, the shortest curve connecting 0 to  $\varphi_1(q)$  is given by the formula  $\theta(t) = t \cdot \varphi_1(q)$ . Since  $\varphi_1^{-1}$  is an isometry of the pair  $(\mathbb{D}, \rho)$ , the image of this curve under  $\varphi^{-1}$  is the shortest curve connecting  $p$  to  $q$ . The formula for this curve is

$$t \mapsto \varphi_1^{-1}(t \cdot \varphi_1(q)) = \frac{t \cdot \varphi_1^{-1}(q) + p}{1 + \bar{p}t \cdot \varphi^{-1}(q)}.$$

This finishes the proof of the proposition.  $\square$

**Definition 4.18.** Let  $\Omega$  be an open set in  $\mathbb{C}$  and  $\rho$  be a conformal metric on  $\Omega$ . A continuous curve  $\gamma : [a, b] \rightarrow \Omega$  is called *geodesic* if for every  $t \in [a, b]$  there is  $\varepsilon_t > 0$  such that for all  $x$  and  $y$  in  $[a, b] \cap [t - \varepsilon_t, t + \varepsilon_t]$  we have

$$d_\rho(\gamma(x), \gamma(y)) = \ell_\rho(\gamma : [x, y] \rightarrow \Omega).$$

In other words, the curve  $\gamma$  is locally the shortest curve connecting points together. For example, straight lines on  $\mathbb{C}$  are geodesics with respect to the conformal metric  $\rho \equiv 1$ . The curves  $\gamma_{p,q}$  in the above proposition provide examples of geodesics with respect to the Poincaré metric on  $\mathbb{D}$ .

There is an intuitive way to visualize the curve  $\gamma_{p,q}$ . To present this, we need to recall a basic property of holomorphic maps.

**Definition 4.19.** A holomorphic map  $f : \Omega \rightarrow \mathbb{C}$  is called *conformal* at  $z \in \Omega$  if  $f'(z) \neq 0$ . A holomorphic map  $f : \Omega \rightarrow \mathbb{C}$  is called *conformal*, if it is conformal at every point in  $\Omega$ .

If  $U \subseteq \mathbb{C}$  is open and  $f : U \rightarrow \mathbb{C}$  is one-to-one, it follows that  $f$  is conformal at every point in  $U$ ; see Exercise 3.3. In particular, biholomorphic maps are conformal at every point in their domain of definition. However, note that a map that is conformal at every point in its domain of definition is not necessarily one-to-one from its domain to its range. For example, the map  $z \mapsto z^2$  is conformal on the set

$$\{w \in \mathbb{C} \mid \arg(w) \in (-3\pi/4, 3\pi/4), |w| \in (1, 2)\}$$

but is not one-to-one on this set.

Recall from complex analysis that conformal maps preserve angles. We state this below for future reference.

**Proposition 4.20.** *Let  $U$  and  $V$  be two open subsets of  $\mathbb{C}$  and  $f : U \rightarrow V$  be a holomorphic map that is conformal at some  $z \in U$ . Then  $f$  preserves angles at  $z$ .*

The curve  $t \mapsto t \cdot \varphi_1(q)$  is part of a straight line segment in  $\mathbb{D}$ . By Exercise 3.5, the image of any line segment in  $\mathbb{D}$  under  $\varphi_1^{-1}$  is either a line segment or part of a circle. The image may be a line segment only if the three points  $p$ ,  $q$ , and  $0$  lie on a straight line segment, and otherwise the curve is part of a circle. Moreover, since the line segment passing through  $0$  is orthogonal to the boundary of  $\mathbb{D}$ , and conformal maps preserve angles, the circle passing through  $p$  and  $q$  is orthogonal to the circle  $|z| = 1$ . See Figure 4.4.

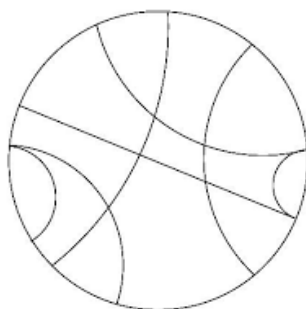


Figure 4.4: Some examples of geodesics with respect to the Poincaré metric  $\rho$  on  $\mathbb{D}$ .

### 4.3 Hyperbolic contractions

**Theorem 4.21** (Schwarz-Pick Lemma). *Let  $f : \mathbb{D} \rightarrow \mathbb{D}$  be a holomorphic map and  $\rho$  denote the Poincaré metric on  $\mathbb{D}$ . Then,  $f$  is distance decreasing with respect to  $\rho$ , that is, for every  $z \in \mathbb{D}$  we have*

$$f^* \rho(z) \leq \rho(z).$$

*In particular, if  $\gamma : [0, 1] \rightarrow \mathbb{D}$  is a  $C^1$  curve then*

$$\ell_\rho(f_* \gamma) \leq \ell_\rho(\gamma).$$

*Therefore, if  $z$  and  $w$  belong to  $\mathbb{D}$ , then*

$$d_\rho(f(z), f(w)) \leq d_\rho(z, w).$$

*Proof.* Recall that

$$f^* \rho(z) = \rho(f(z)) \cdot |f'(z)| = \frac{1}{1 - |f(z)|^2} \cdot |f'(z)|,$$

and

$$\rho(z) = \frac{1}{1 - |z|^2}.$$

Hence, the inequality in the theorem reduces to the Schwarz-Pick lemma we saw earlier in Exercise 2.2.

The latter parts of the theorem follow directly from the definitions.  $\square$

**Theorem 4.22** (Farkas-Ritt). *Let  $f : \mathbb{D} \rightarrow \mathbb{D}$  be a holomorphic map and assume that  $f(\mathbb{D})$  has a compact closure in  $\mathbb{D}$ , that is, every sequence in  $f(\mathbb{D})$  has a sub-sequence converging to some point in  $\mathbb{D}$ . Then,*

- i) there is a unique point  $p \in \mathbb{D}$  such that  $f(p) = p$ ;*
- ii) for every  $w_0$  in  $\mathbb{D}$  the sequence of points  $w_i$  defined as  $w_{i+1} = f(w_i)$ , for  $i \geq 0$ , converges to  $p$  in the Euclidean metric.*

*Proof.* Define

$$A = \{f(z) : z \in \mathbb{D}\}.$$

By the hypothesis, the closure of  $A$  is contained in  $\mathbb{D}$ . This implies that there is  $\delta > 0$  such that for every  $z \in \mathbb{C}$  with  $|z| \geq 1$  and every  $w \in A$  we have  $|w - z| > \delta$ .

Fix an arbitrary  $z_0 \in \mathbb{D}$ . Define the map

$$g(z) = f(z) + \frac{\delta}{2}(f(z) - f(z_0)), \quad \forall z \in \mathbb{D}.$$

The map  $g$  is holomorphic on  $\mathbb{D}$ , and maps  $\mathbb{D}$  into  $\mathbb{D}$  since

$$|g(z)| \leq |f(z)| + \frac{\delta}{2}|f(z) - f(z_0)| < (1 - \delta) + \frac{\delta}{2} \cdot 2 = 1.$$

We have  $g(z_0) = f(z_0)$  and  $g'(z_0) = (1 + \delta/2)f'(z_0)$ . By Theorem 4.21,  $g$  is non-expanding the Poincaré metric at  $z_0$ , that is,

$$g^* \rho(z_0) \leq \rho(z_0).$$

Writing the definition of  $g^*$ , this yields

$$(1 + \delta/2) \cdot |f'(z_0)| \cdot \rho(f(z_0)) \leq \rho(z_0).$$

As  $z_0 \in \mathbb{D}$  was arbitrary, we conclude that the above inequality holds for all  $z_0 \in \mathbb{D}$ . In particular, if  $\gamma$  is any  $C^1$  curve in  $\mathbb{D}$ , then

$$(1 + \delta/2) \cdot \ell_\rho(f \circ \gamma) \leq \ell_\rho(\gamma).$$

This implies that for arbitrary points  $z$  and  $w$  in  $\mathbb{D}$  we have

$$d_\rho(f(z), f(w)) \leq (1 + \delta/2)^{-1} \cdot d_\rho(z, w).$$

Fix an arbitrary  $w_0$  in  $\mathbb{D}$  and define the sequence of points  $w_{i+1} = f(w_i)$ , for  $i \geq 0$ . Inductively using the above inequality we conclude that for every  $i \geq 2$  we have

$$d_\rho(w_{i+1}, w_i) \leq (1 + \delta/2)^{-1} \cdot d_\rho(w_i, w_{i-1}) \leq \cdots \leq (1 + \delta/2)^{-i} \cdot d_\rho(w_1, w_0).$$

Since  $\sum_{i=0}^{\infty} (1 + \delta/2)^{-i}$  is finite, the sequence  $w_i$  is Cauchy with respect to  $d_\rho$ . The space  $\mathbb{D}$  with respect to  $d_\rho$  is a complete metric space, see Exercise 4.3. This means that any Cauchy sequence (w.r.t  $d_\rho$ ) in  $\mathbb{D}$  converges (w.r.t  $d_\rho$ ) to some point in  $\mathbb{D}$ . By Exercise 4.2, the sequence  $w_i$  converges with respect to the Euclidean metric on  $\mathbb{D}$ . Let  $p$  denote the limit of this sequence. Taking limit from the relation  $w_{i+1} = f(w_i)$  as  $i$  tends to  $+\infty$ , we conclude that  $f(p) = p$ .

If there is  $q$  in  $\mathbb{D}$  with  $f(q) = q$ , by the above inequalities,

$$d_\rho(p, q) = d_\rho(f(p), f(q)) \leq (1 + \delta/2)^{-1} d_\rho(p, q).$$

As  $\delta > 0$ , this is only possible if  $p = q$ . This shows the uniqueness of  $p$ . So far we have completed the proof of Part i).

By the above arguments,  $d_\rho(w_i, p) \leq (1 + \delta/2)^{-i} d_\rho(w_0, p)$ . Hence,  $w_i$  converges to  $p$  with respect to  $d_\rho$ . It follows that  $w_i$  converges to  $p$  with respect to the Euclidean metric, see Exercise 4.2.  $\square$

## 4.4 Exercises

**Exercise 4.1.** Show that  $d_\rho : \Omega \times \Omega \rightarrow \mathbb{R}$  defined in Definition 4.5 is a metric on  $\Omega$ .

**Exercise 4.2.** Let  $z_i$ ,  $i \geq 1$ , be an infinite sequence in  $\mathbb{D}$ , and  $\rho$  be the Poincaré metric on  $\mathbb{D}$ . Show that  $z_i$  converges to some point  $z$  in  $\mathbb{D}$  with respect to  $d_\rho$  iff it converges to  $z \in \mathbb{D}$  with respect to the Euclidean metric.

**Exercise 4.3.** Show that the disk  $\mathbb{D}$  equipped with the Poincaré metric  $\rho$  is a complete metric space. That is, every Cauchy sequence in  $\mathbb{D}$  with respect to  $d_\rho$  converges to some point in  $\mathbb{D}$  with respect to the distance  $d_\rho$ .

**Exercise 4.4.** Let  $\rho$  be the Poincaré metric on  $\mathbb{D}$ . For  $z \in \mathbb{D}$  and  $r > 0$ , the circle of radius  $r$  about  $z$  with respect to the metric  $d_\rho$  is defined as

$$\{w \in \mathbb{D} : d_\rho(z, w) = r\}.$$

Show that for every  $z \in \mathbb{D}$  and  $r > 0$ , the circle of radius  $r$  about  $z$  is an Euclidean circle. Find the center of this circle.

**Exercise 4.5.** Let  $\Omega_1$  and  $\Omega_2$  be open sets in  $\mathbb{C}$  with conformal metrics  $\rho_1$  and  $\rho_2$ , respectively. Assume that  $f : \Omega_1 \rightarrow \Omega_2$  is a one-to-one holomorphic map that is an isometry from  $(\Omega_1, \rho_1)$  to  $(\Omega_2, \rho_2)$ . Prove that for all  $z$  and  $w$  in  $\Omega_1$  we have

$$d_{\rho_1}(z, w) = d_{\rho_2}(f(z), f(w)).$$

**Exercise 4.6.** Recall the biholomorphic map  $F : \mathbb{H} \rightarrow \mathbb{D}$  given in Equation (2.1), and let  $\rho$  be the Poincaré metric on  $\rho$ . Show that for all  $w \in \mathbb{H}$  we have

$$(F^*\rho)(w) = \frac{1}{2|\operatorname{Im} w|}.$$