Chapter 3

Riemann sphere and rational maps

3.1 Riemann sphere

It is sometimes convenient, and fruitful, to work with holomorphic (or in general continuous) functions on a compact space. However, we wish to still "keep" all of $\mathbb{C}$ in the space we work on, but see it as a subset of a compact space. There are sequences in $\mathbb{C}$ that have no sub-sequence converging to a point in $\mathbb{C}$. The least one needs to do is to add the limiting values of convergent sub-sequences to $\mathbb{C}$. It turns out that one may achieve this by adding a single point to $\mathbb{C}$ in a suitable fashion. We denote this point with the notation ∞. Below we discuss the construction in more details.

Let us introduce the notation $\hat{\mathbb{C}}$ for the set $\mathbb{C} \cup \{\infty\}$, where ∞ is an element not in $\mathbb{C}$. The arithmetic on $\mathbb{C}$ may be extended, to some extent, by assuming that

- for all finite $a \in \mathbb{C}$, $\infty + a = a + \infty = \infty$.
- for all non-zero $b \in \mathbb{C} \cup \{\infty\}$, $b \cdot \infty = \infty \cdot b = \infty$.

Remark 3.1. It is not possible to define $\infty + \infty$ and $0 \cdot \infty$ without violating the laws of arithmetic. But, by convention, for $a \in \mathbb{C} \setminus \{0\}$ we write $a/0 = \infty$, and for $b \in \mathbb{C}$ we write $b/\infty = 0$.

We "attach" the point ∞ to $\mathbb{C}$ by requiring that every sequence $z_i \in \mathbb{C}$, for $i \geq 1$, with $|z_i|$ diverging to infinity converges to ∞. This is rather like adding the point $+1$ to the set $(0, 1)$. With this definition, it is easy to see that every sequence in $\hat{\mathbb{C}}$ has a convergent sub-sequence. We have also kept a copy of $\mathbb{C}$ in $\hat{\mathbb{C}}$.

There is a familiar model for the set $\mathbb{C} \cup \{\infty\}$ obtain from a process known as "stereographic projection". To see that, let

$$S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}.$$

Let $N = (0, 0, 1) \in S$. We define a homeomorphism $\pi : S \to \hat{\mathbb{C}}$ as follows. Let $\pi(N) = \infty$, and for every point $(x_1, x_2, x_3) \neq N$ in S define

$$\pi(x_1, x_2, x_3) = \frac{x_1 + ix_2}{1 - x_3}. \tag{3.1}$$

By the above formula,

$$|\pi(X)|^2 = \frac{x_1^2 + x_2^2}{(1-x_3)^2} = \frac{1+x_3}{1-x_3},$$

which implies that

$$x_3 = \frac{|\pi(X)|^2 - 1}{|\pi(X)|^2 + 1}, \quad x_1 = \frac{\pi(X) + \overline{\pi(X)}}{1 + |\pi(X)|^2}, \quad x_2 = \frac{\pi(X) - \overline{\pi(X)}}{i(1 + |\pi(X)|^2)}.$$

The above relations imply that π is one-to-one and onto.

The continuity of π at every point on $S \setminus \{N\}$ is evident from the formula. To see that π is continuous at N, we observe that if X tends to N on S, then x_3 tends to $+1$ from below. This implies that $|\pi(X)|$ tends to $+\infty$, that is, $\pi(X)$ tends to ∞ in $\hat{\mathbb{C}}$.

If we regard the plane $(x_1, x_2, 0)$ in $\mathbb{R}^3$ as the complex plane $x_1 + ix_2$, there is a nice geometric description of the map π, called stereographic projection. That is the points N, X, and $\pi(X)$ lie on a straight line in $\mathbb{R}^3$. See Figure 3.1.

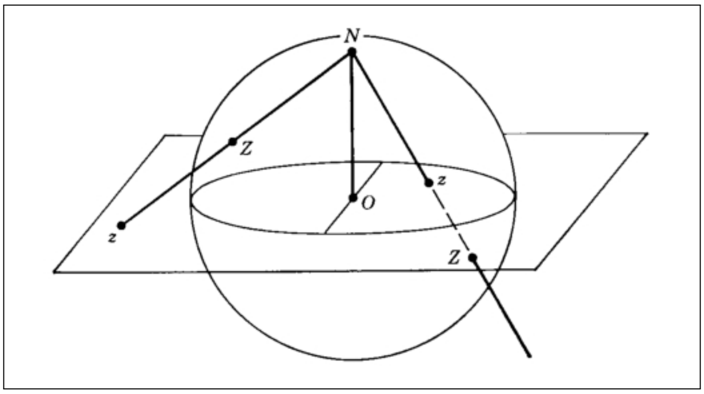

Figure 3.1: Presentation of the map π.

The set $\hat{\mathbb{C}}$, with the convergence of sequences described above, is known as the *Riemann sphere*. In view of the above construction, as we know S as a symmetric space, $\hat{\mathbb{C}}$ should be also viewed as a symmetric space. To discuss this further, we need to give some basic definitions.

Let Ω be an open set in $\mathbb{C}$. Recall that $f : \Omega \to \mathbb{C}$ is called continuous at a point $z \in \Omega$, if for every $\varepsilon > 0$ there is $\delta > 0$ such that for all $z' \in \Omega$ with $|z - z'| < \delta$ we have $|f(z) - f(z')| < \varepsilon$. This is equivalent to saying that f is continuous at z if and only if for every sequence z_n, $n \geq 1$, in Ω that converges to z, the sequence $f(z_n)$ converges to $f(z)$.

We use the above idea to define the notion of continuity for maps $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. That is, $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is called continuous at $z \in \hat{\mathbb{C}}$, if every sequence that converges to z is mapped by f to a sequence that converges to $f(z)$.

When f maps ∞ to ∞, the continuity of f at ∞ is equivalent to the continuity of the map $z \mapsto 1/f(1/z)$ at 0. Similarly, when $f(\infty) = a \neq \infty$, the continuity of f at ∞ is equivalent to the continuity of the map $z \mapsto f(1/z)$ at 0. When $f(a) = \infty$ for some $a \in \mathbb{C}$, the continuity of f at a is equivalent to the continuity of the map $z \mapsto 1/f(z)$ at a.

As usual, $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is called continuous, if it is continuous at every point in $\hat{\mathbb{C}}$.

Definition 3.2. Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a continuous map and $a \in \hat{\mathbb{C}}$. Then,

(i) When $a = \infty$ and $f(a) = \infty$, we say that f is holomorphic at a if the map $z \mapsto 1/f(1/z)$ is holomorphic at 0.

(ii) If $a = \infty$ and $f(a) \neq \infty$, then f is called holomorphic at a if the map $z \mapsto f(1/z)$ is holomorphic at 0.

(iii) If $a \neq \infty$ but $f(a) = \infty$, then f is called holomorphic at a if the map $z \mapsto 1/f(z)$ is holomorphic at a.

Continuous and Holomorphic maps from $\mathbb{C}$ to $\hat{\mathbb{C}}$, from $\mathbb{D}$ to $\hat{\mathbb{C}}$, and vice versa, are defined accordingly.

Example 3.3. You have already seen that every polynomial $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ is holomorphic from $\mathbb{C}$ to $\mathbb{C}$. As z tends to ∞ in $\mathbb{C}$, $P(z)$ tends to ∞ in $\mathbb{C}$. Hence, we may extend P to a continuous map from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$ by defining $P(\infty) = \infty$. To see whether P is holomorphic at ∞ we look at

$$\frac{1}{P(1/z)} = \frac{z^n}{a_n + a_{n-1}z + \cdots + a_0 z^n},$$

which is well-defined and holomorphic near 0. When $n > 1$, the complex derivative of the above map at 0 is equal to 0. When $n = 1$, its derivative becomes $1/a_1$. Thus, P is a holomorphic map of $\hat{\mathbb{C}}$.

Proposition 3.4. *If $f : \hat{\mathbb{C}} \to \mathbb{C}$ is a holomorphic map, then f is a constant map.*

Proof. We break the proof into several steps.

Step1. There is $z_0 \in \hat{\mathbb{C}}$ such that for all $z \in \hat{\mathbb{C}}$ we have $|f(z)| \leq |f(z_0)|$. That is, $|f|$ attains its maximum value at some point.

To prove the above statement, first we note that there is $M > 0$ such that for all $z \in \hat{\mathbb{C}}$, we have $|f(z)| \leq M$. If this is not the case, there are $z_n \in \hat{\mathbb{C}}$, for $n \geq 1$, with $|f(z_n)| \geq n$. As $\hat{\mathbb{C}}$ is a compact set, the sequence z_n has a sub-sequence, say z_{n_k} that converges to some point $w \in \hat{\mathbb{C}}$. By the continuity of f we must have $f(w) = \lim_{k\to+\infty} f(z_{n_k}) = \infty$. This contradicts with $f : \hat{\mathbb{C}} \to \mathbb{C}$.

The set $V = \{|f(z)| : z \in \hat{\mathbb{C}}\}$ is a subset of $\mathbb{R}$, and by the above paragraph, it is bounded from above. In particular, V has a supremum, say s. For any $n \geq 1$, since s is the least upper bound, there is $z_n \in \hat{\mathbb{C}}$ such that $|f(z_n)| \geq s - 1/n$. The sequence z_n is contained in the compact set $\hat{\mathbb{C}}$. Thus, there is a sub-sequence z_{n_l}, for $l \geq 1$, that converges to some point z_0 in $\hat{\mathbb{C}}$. It follows from the continuity of $|f(z)|$ that $|f(z_0)| = s$. Therefore, for all $z \in \hat{\mathbb{C}}$, $|f(z)| \leq |f(z_0)|$.

Step 2. If $z_0 \in \mathbb{C}$, then the map $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and $|f|$ attains its maximum value at a point inside $\mathbb{C}$. By the maximum principle, f must be constant on $\mathbb{C}$. Then, by the continuity of $f : \hat{\mathbb{C}} \to \mathbb{C}$, we conclude that f is constant on $\hat{\mathbb{C}}$.

Step 3. If $z_0 = \infty$, then we look at the map $h(z) = f(1/z)$. Buy definition, $h : \mathbb{C} \to \mathbb{C}$ is holomorphic and $|h|$ attains its maximum value at 0. Again, by the maximum principle, h must be constant on $\mathbb{C}$. Equivalently, f is constant on $\hat{\mathbb{C}} \setminus \{0\}$. As in the above paragraph, the continuity of $f : \hat{\mathbb{C}} \to \mathbb{C}$, implies that indeed f is constant on $\hat{\mathbb{C}}$. □

Example 3.5. The exponential map $z \mapsto e^z$ is holomorphic from $\mathbb{C}$ to $\mathbb{C}$. As z tends to infinity along the positive real axis, e^z tends to ∞ along the positive real axis. But as z tends to ∞ along the negative real axis, e^z tends to 0. Hence there is no continuous extension of the exponential map from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$.

Definition 3.6. Let Ω be an open set in $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ be a holomorphic map. We say that f has a *zero of order* $k \in \mathbb{N}$ at $z_0 \in \Omega$, if $f^{(i)}(z_0) = 0$ for $0 \leq i \leq k-1$, and $f^{(k)}(z_0) \neq 0$. Similarly, we can say that f attains value w_0 at z_0 of order k, if z_0 is a zero of order k for the function $z \mapsto f(z) - w_0$. Here, the series expansion of f at z_0 has the form $f(z) = w_0 + a_k(z-z_0)^k + a_{k+1}(z-z_0)^{k+1} \dots$, with $a_k \neq 0$.

Definition 3.7. Definition 3.6 may be extended to holomorphic maps $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. That is, we say that f attains ∞ at $z_0 \in \mathbb{C}$ of order k, if z_0 is a zero of order k for the map $z \mapsto 1/f(z)$. Then, near z_0 we have

$$1/f(z) = a_k(z-z_0)^k + a_{k+1}(z-z_0)^{k+1} + a_{k+2}(z-z_0)^{k+2} + \dots.$$

This implies that

$$
\begin{aligned}
f(z) &= \frac{1}{a_k(z-z_0)^k + a_{k+1}(z-z_0)^{k+1} + a_{k+2}(z-z_0)^{k+2} + \dots} \\
&= \frac{1}{z^k(a_k + a_{k+1}(z-z_0) + a_{k+2}(z-z_0)^2 + \dots)} \\
&= \frac{1}{(z-z_0)^k}(b_0 + b_1(z-z_0) + b_2(z-z_0)^2 + \dots) \\
&= \frac{b_0}{(z-z_0)^k} + \frac{b_1}{(z-z_0)^{k-1}} + \frac{b_2}{(z-z_0)^{k-2}} + \dots.
\end{aligned}
$$

Recall that z_0 is also called *a pole of order* k for f.

Similarly, if $f(\infty) = \infty$, we say that f attains ∞ at ∞ of order k, if the map $z \mapsto 1/f(1/z)$ has a zero of order k at 0

Proposition 3.8. *Let $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a holomorphic map such that for every $z \in \mathbb{C}$, $g(z) \in \mathbb{C}$. Then, g is a polynomial.*

Proof. The map g has a convergent power series on all of $\mathbb{C}$ as

$$g(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots.$$

We consider two possibilities.

If $g(\infty) \neq \infty$, then $g : \hat{\mathbb{C}} \to \mathbb{C}$ is holomorphic and by Proposition 3.4, g must be constant on $\hat{\mathbb{C}}$. Therefore, $g(z) \equiv a_0$ is a polynomial.

The other possibility is that $g(\infty) = \infty$. To understand the behavior of g near ∞, we consider the map $h(w) = 1/g(1/w)$ near 0. We have $h(0) = 0$. Let $n \geq 1$ be the order of 0 at 0 for the map h, that is, $h(w) = a_n w^n + a_{n+1} w^{n+1} + \dots$ near 0. This implies that there is $\delta > 0$ such that for $|w| \leq \delta$ we have

$$|h(w)| \geq \frac{|a_n w^n|}{2}.$$

In terms of g, this means that for $|z| \geq 1/\delta$ we have $|g(z)| \leq 2|z^n|/|a_n|$. Then, by the Cauchy integral formula for the derivatives, for every $j \geq n+1$ and $R > 0$ we have

$$g^{(j)}(0) = \frac{j!}{2\pi i} \int_{\partial B(0,R)} \frac{g(z)}{z^{j+1}}\, dz.$$

Then, for $R > 1/\delta$,

$$|g^{(j)}(0)| \leq \frac{2 \cdot j!}{2\pi |a_n|} \int_{\partial B(0,R)} \frac{|z|^n}{|z^{j+1}|}\, dz \leq \frac{2 \cdot j!}{2\pi |a_n|} \cdot 2\pi R \cdot \frac{1}{R^{j+1-n}}.$$

Now we let $R \to +\infty$, and conclude that for all $j \geq n+1$, $g^{(j)}(0) = 0$. Therefore, for all $j \geq n+1$, $a_j = g^{(j)}(0)/j! = 0$, and thus, g is a polynomial of degree n. □

3.2 Rational functions

Example 3.9. If $Q(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ is a polynomial, then Q attains ∞ of order n at ∞. The map $1/Q(z) : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is well-defined and holomorphic. At every point z_0 where $Q(z_0) \neq 0$, $1/Q(z)$ is well defined near z_0. If z_0 is a zero of order k for $Q(z)$, then $1/Q(z_0) = \infty$ and z_0 is a pole of order k.

Definition 3.10. If P and Q are polynomials, the map $z \mapsto P(z)/Q(z)$ is a well-defined holomorphic map from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$. Any such map is called a *rational function*.

Theorem 3.11. *Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a holomorphic map. Then, there are polynomials $P(z)$ and $Q(z)$ such that*

$$f(z) = \frac{P(z)}{Q(z)}.$$

Before we present a proof of the above theorem, we recall a basic result from complex analysis.

Proposition 3.12. *Let Ω be a connected and open set in $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ be a holomorphic map. Assume that there is a sequence of distinct points z_j in Ω converging to some $z \in \Omega$ such that f takes the same value on the sequence z_j. Then, f is constant on Ω.*

In the above proposition, the connectivity of Ω is necessary and is imposed to avoid trivial counter examples. For example, one may set $\Omega = \mathbb{D} \cup (\mathbb{D} + 5)$ and defined f as $+1$ on $\mathbb{D}$ and as -1 on $\mathbb{D}+5$. It is also necessary to assume that the limiting point z belongs to Ω. For instance, the map $\sin(1/z)$ is defined and holomorphic on $\Omega = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ and has a sequence of zeros at points $1/(2\pi n)$, but it is not identically equal to 0.

Proof. Without loss of generality we may assume that the value of f on the sequence z_j is 0 (otherwise consider $f - c$). Since f is holomorphic at z, it has a convergent power series for ζ in a neighborhood of z as

$$f(\zeta) = a_1(\zeta - z) + a_2(\zeta - z)^2 + a_3(\zeta - z)^3 + \dots.$$

If f is not identically equal to 0, there the smallest integer $n \geq 1$ with $a_n \neq 0$. Then, $f(\zeta) = (\zeta - z)^{n-1} \cdot h(\zeta)$, for some holomorphic function h defined on a neighborhood U of z with $h(z) \neq 0$. But for large enough j, z_j belongs to U and we have $f(z_j) = 0$. This is a contradiction that shows for all $n \geq 1$, $a_n = 0$. In particular, f is identically 0 on U.

Let us define the set $E \subseteq \Omega$ as the set of points w in Ω such that for all $n \geq 1$ we have $f^{(n)}(w) = 0$. By the above paragraph, E contains z and hence it is not empty. Also, the argument shows that E is an open subset of Ω (see Theorem 1.4).

If $E = \Omega$ then we are done and f is identically equal to 0. Otherwise, there must be an integer $n \geq 1$ and $w \in \Omega$ such that $f^{(n)}(w) \neq 0$. Let us define the sets

$$F_n = \{w \in \Omega : f^{(n)}(w) \neq 0\}, \text{for } n \geq 1.$$

By the continuity of the map $z \mapsto f^{(n)}(z)$ on Ω, each F_n is an open set. In particular, the union $F = \cup_{n \geq 1} F_n$ is an open set. Now, $\Omega = E \cup F$, where E and F are non-empty and open sets. This contradicts the connectivity of Ω. □

By the above proposition, if holomorphic functions f and g defined on Ω are equal on a sequence converging to some point in Ω, they must be equal. This follows from considering the function $f - g$ in the above proposition. In other words, a holomorphic function is determined by its values on a sequence whose limit is in the domain of the function. However, this does not mean that we know how to identify the values of the function all over the domain.

Proof of Theorem 3.11. If the map f is identically equal to a constant $c \neq \infty$ we choose $P \equiv c$ and $Q \equiv 1$. If the map f is identically equal to ∞ we choose, $P \equiv 1$ and $Q \equiv 0$. Below we assume that f is not constant on $\hat{\mathbb{C}}$.

If f does not attain ∞ at any point on $\hat{\mathbb{C}}$, then $f : \hat{\mathbb{C}} \to \mathbb{C}$ is holomorphic, and by Proposition 3.4 it must be constant on $\hat{\mathbb{C}}$. So, if f is not constant, it must attain ∞ at some points in $\hat{\mathbb{C}}$.

There are at most a finite number of points in $\mathbb{C}$, denoted by $a_1, a_2, \ldots, a_n$, where $f(a_i) = \infty$. That is because, if f attains ∞ at an infinite number of distinct points in $\mathbb{C}$, since $\hat{\mathbb{C}}$ is a compact set, there will be a sub-sequence of those points converging to some z_0 in $\hat{\mathbb{C}}$. Then, we apply proposition 3.12 to the map $1/f(z)$ or $1/f(1/z)$ (depending on the value of z_0), and conclude that f is identically equal to ∞.

Each pole a_i of Q has some finite order $k_i \geq 1$. Define

$$Q(z) = (z - a_1)^{k_1}(z - a_2)^{k_2} \ldots (z - a_n)^{k_n}.$$

Consider the map $g(z) = f(z)Q(z)$. Since f and Q are holomorphic functions from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$, g is holomorphic from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$. The map g is holomorphic from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$. Moreover, since the order of zero of Q at a_i is equal to the order of the pole of f at a_i, g is finite at any point in $\mathbb{C}$. Thus, by Proposition 3.8, g is a polynomial in variable z. This finishes the proof of the theorem. □

The degree of a rational map $f = P/Q$, where P and Q have no common factors, is defined as the maximum of the degrees of P and Q. There is an intuitive meaning of the degree of a rational map as in the case of polynomials. Recall that by the fundamental theorem of algebra, for $c \in \mathbb{C}$, the equation $P(z) = c$ has $\deg(P)$ solutions, counted with the multiplicities given by the orders of the solutions. As the equation $f(z) = c$ reduces to $cQ(z) - P(z) = 0$, the number of solutions of $f(z) = c$ counted with multiplicities is given by $\deg(f)$.

Theorem 3.13. *A holomorphic map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is an automorphism of $\hat{\mathbb{C}}$, iff there are constants a, b, c, and d in $\mathbb{C}$ with $ad - bc = 1$ and*

$$f(z) = \frac{az + b}{cz + d}. \tag{3.2}$$

Proof. By Example 3.9, every map f of this form is holomorphic from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$. Moreover, one can verify that the map $g(z) = (dz - b)/(-cz + a)$ satisfies $f \circ g(z) = g \circ f(z) = z$, for all $z \in \hat{\mathbb{C}}$. Hence, f is both on-to-one and onto from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$. This proves one side of the theorem.

On the other hand, if f is an automorphism of $\hat{\mathbb{C}}$, by Theorem 3.11, there are polynomials P and Q such that $f = P/Q$. Let us assume that P and Q have no common factors. Since f is one-to-one, every point has a single pre-image. Thus, by the paragraph preceding the theorem, we must have $\max\{\deg(P), \deg(Q)\} = 1$. Then, there are complex constants a, b, c, d such that $P(z) = az + b$ and $Q(z) = cz + d$, where at least one of a and c is non-zero.

Since P and Q have no common factors, we must have $ad - bc \neq 0$. We may multiply both P and Q by some constant to make $ad - bc = 1$. □

Definition 3.14. Every map of the form in Equation (3.2), where a, b, c, and d are constants in $\mathbb{C}$ with $ad - bc = 1$ is called a *Möbius* transformation. By Theorems 2.5 and 2.7, every automorphism of $\mathbb{D}$ and $\mathbb{C}$ is a Möbius transformation.

Theorem 3.15. *Every automorphism of $\mathbb{C}$ is of the form $az + b$ for some constants a and b in $\mathbb{C}$ with $a \neq 0$.*

Proof. Let $f : \mathbb{C} \to \mathbb{C}$ be an automorphism. We claim that when $|z| \to +\infty$, $|f(z)| \to +\infty$. If this is not the case, there is an infinite sequence of distinct points z_i with $|z_i| \to +\infty$ but $|f(z_i)|$ are uniformly bounded. As f is one-to-one, the values $f(z_i)$ are distinct for distinct values of i. There is a sub-sequence of $f(z_i)$ that converges to some point in $\mathbb{C}$, say w'. Since $f : \mathbb{C} \to \mathbb{C}$ is onto, there is $z' \in \mathbb{C}$ with $f(z') = w'$.

There is a holomorphic map $g : \mathbb{C} \to \mathbb{C}$ with $f \circ g(z) = g \circ f(z) = z$ on $\mathbb{C}$. We have $g(w') = z'$, and by the continuity of g, the points $z_i = g(f(z_i))$ must be close to z'. This contradicts $|z_i| \to +\infty$.

We extend f to the map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ by defining $f(\infty) = \infty$. By the above paragraph, f is continuous at ∞, and indeed holomorphic (see Exercise 3.2) from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$. It follows that $f \in \mathrm{Aut}(\hat{\mathbb{C}})$, and by Theorem 3.13, it must be of the form $(az+b)/(cz+d)$. However, since f maps $\mathbb{C}$ to $\mathbb{C}$, we must have $c = 0$. This finishes the proof of the theorem. □

Definition 3.16. The automorphisms $z \mapsto z + c$, for c constant, are called *translations*, and the automorphisms $z \mapsto c \cdot z$, for c constant, are called *dilations*. When c is real, these are also automorphisms of $\mathbb{H}$. When $|c| = 1$, the map $z \mapsto c \cdot z$ is called a *rotation* of $\mathbb{C}$.

3.3 Exercises

Exercise 3.1. Prove that

(i) for every z_1, z_2, w_1, and w_2 in $\mathbb{C}$ with $z_1 \neq z_2$ and $w_1 \neq w_2$, there is $f \in \mathrm{Aut}(\mathbb{C})$ with $f(z_1) = w_1$ and $f(z_2) = w_2$;

(ii) for all distinct points z_1, z_2, and z_3 in $\hat{\mathbb{C}}$ and all distinct points w_1, w_2, and w_3 in $\hat{\mathbb{C}}$, there is $f \in \mathrm{Aut}(\hat{\mathbb{C}})$ with $f(z_i) = w_i$, $i = 1, 2, 3$.

Exercise 3.2. Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a map whose restriction to $\mathbb{C}$ is holomorphic, and has a continuous extension to ∞. Show that $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is holomorphic.

Exercise 3.3. Let $f : \Omega \to \mathbb{C}$ be a holomorphic map that has a zero of order $k \geq 1$ at some $z_0 \in \Omega$.

(i) Prove that there is $\delta > 0$ and a holomorphic function $\psi : B(z_0, \delta) \to \mathbb{C}$ such that $\psi(z_0) = 0$, $\psi'(z_0) \neq 0$, and $f(z) = (\psi(z))^k$ on $B(z_0, \delta)$.

(ii) Conclude from part (i) that near 0 the map f is k-to-one, that is, every point near 0 has exactly k pre-images near z_0.

Exercise 3.4. Let Ω be an open set in $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ be a holomorphic map.

(i) Using Exercise 3.3, prove that if f is not constant, it is an *open map*, that is, f maps every open set in Ω to an open set in $\mathbb{C}$.

(ii) Using part (i), prove the maximum principle, Theorem 1.6.

Exercise 3.5. Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a Möbius transformation. Prove that the image of every straight line in $\mathbb{C}$ is either a straight line or a circle in $\mathbb{C}$. Also, the image of every circle in $\mathbb{C}$ is either a straight line or a circle in $\mathbb{C}$.

Exercise 3.6. Let $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a holomorphic map which maps $\mathbb{D}$ into $\mathbb{D}$ and maps $\partial\mathbb{D}$ to $\partial\mathbb{D}$. Prove that there are points a_1, a_2, $\ldots$, a_d (not necessarily distinct) in $\mathbb{D}$ and $\theta \in [0, 2\pi]$ such that

$$g(z) = e^{2\pi i\theta} \prod_{j=1}^{d} \frac{z - a_j}{1 - \overline{a_j} z}.$$

The maps of the above form are called *Blaschke products of degree d.*