Chapter 2

Schwarz lemma and automorphisms of the disk

2.1 Schwarz lemma

We denote the disk of radius 1 about 0 by the notation $\mathbb{D}$, that is,

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}.$$

Given $\theta \in \mathbb{R}$ the rotation of angle θ about 0, i.e. $z \mapsto e^{i\theta} \cdot z$, preserves $\mathbb{D}$. Due to the rotational symmetry of $\mathbb{D}$ most objects studied in complex analysis find special forms on $\mathbb{D}$ that have basic algebraic forms. We study some examples of these in this section, and will see more on this later on.

A main application of the maximum principle (Theorem 1.6) is the lemma of Schwarz. It has a simple proof, but has far reaching applications.

Lemma 2.1 (Schwarz lemma)**.** *Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic with $f(0) = 0$. Then,*

(i) for all $z \in \mathbb{D}$ we have $|f(z)| \leq |z|$;

(ii) $|f'(0)| \leq 1$;

(iii) if either $f(z) = z$ for some non-zero $z \in \mathbb{D}$, or $|f'(0)| = 1$, then f is a rotation about 0.

Proof. Since f is holomorphic on $\mathbb{D}$, we have a series expansion for f centered at 0,

$$f(z) = a_0 + a_1 z + a_2 z^2 + \dots,$$

that is convergent on $\mathbb{D}$. Since $f(0) = 0$, $a_0 = 0$, and we obtain

$$f(z) = a_1 z + a_2 z^2 + \dots = z(a_1 + a_2 z + a_3 z^2 + \dots).$$

In particular, the series in the above parenthesis is convergent on $\mathbb{D}$. In particular, the function $g(z) = f(z)/z = a_1 + a_2 z + a_3 z^2 + \dots$ is defined and holomorphic on $\mathbb{D}$. Note that $g(0) = a_1 = f'(0)$.

On each circle $|z| = r < 1$, as $|f(z)| < 1$, we have

$$|g(z)| < \frac{1}{r}.$$

Then by the maximum principle, we must have the above inequality on $|z| < r$. Taking limit as $r \to 1$ from the left, we conclude that on $\mathbb{D}$, $|g(z)| \le 1$. This implies part (i) and (ii) of the lemma.

To prove part (iii) of the lemma, note that if any of the two equality holds, then g attains its maximum in the interior of $\mathbb{D}$. Then, by the maximum principle, g must be a constant on $\mathbb{D}$, say $g(z) \equiv a$. Then, either of the relations $|f'(0)| = 1$ and $f(z) = z$ for some $z \neq 0$, implies that $|a| = 1$. This finishes the proof of part (iii). □

As an application of the Schwarz lemma we classify the one-to-one and onto holomorphic mappings of $\mathbb{D}$.

2.2 Automorphisms of the disk

Definition 2.2. Let U and V be open subsets of $\mathbb{C}$. A holomorphic map $f : U \to V$ that is one-to-one and onto is called a *biholomorphism* from U to V. A biholomorphism from U to U is called an *automorphism* of U. The set of all automorphisms of U is denoted by $\mathrm{Aut}(U)$.

Obviously, $\mathrm{Aut}(U)$ contains the identity map and hence is not empty. The composition of any two maps in $\mathrm{Aut}(U)$ is again an element of $\mathrm{Aut}(U)$. Indeed, $\mathrm{Aut}(U)$ forms a group with this operation.

Proposition 2.3. *For every non-empty and open set U in $\mathbb{C}$, $\mathrm{Aut}(U)$ forms a group with the operation being the composition of the maps.*

Proof. The composition of any pair of one-to-one and onto holomorphic maps from U to U is a one-to-one and onto holomorphic map from U to itself. Thus the operation is well defined on $\mathrm{Aut}(U)$. The identity map $z \mapsto z$ is the identity element of the group.

The associativity $(f \circ g) \circ h = f \circ (g \circ h)$ holds because the relation is valid for general maps. The inverse of every $f \in \mathrm{Aut}(U)$ is given by the inverse mapping f^{-1}. Note that the inverse of any one-to-one map is defined and is a holomorphic map. □

We have already seen that every rotation $z \mapsto e^{i\theta} \cdot z$, for a fixed $\theta \in \mathbb{R}$, is an automorphism of the disk. For $a \in \mathbb{D}$ define

$$\varphi_a(z) = \frac{a - z}{1 - \overline{a}z}.$$

Lemma 2.4. *For every $a \in \mathbb{D}$, φ_a is an automorphism of $\mathbb{D}$.*

Proof. First note that φ_a is defined and holomorphic at every $z \in \mathbb{C}$, except at $z = 1/\overline{a}$ where the denominator becomes 0. However, since $|a| < 1$, we have $|1/\overline{a}| = 1/|a| > 1$, and therefore, $1/\overline{a} \notin \mathbb{D}$. Hence, φ_a is holomorphic on $\mathbb{D}$.

To see that φ_a maps $\mathbb{D}$ into $\mathbb{D}$, fix an arbitrary $z \in \mathbb{C}$ with $|z| = 1$. Observe that

$$|\varphi_a(z)| = \left|\frac{a - z}{1 - \overline{a}z}\right| = \left|\frac{a - z}{1 - \overline{a}z}\right| \cdot \frac{1}{|\overline{z}|} = \left|\frac{a - z}{\overline{z} - \overline{a}}\right| = 1,$$

since $z\overline{z} = |z|^2 = 1$. By the maximum principle (Theorem 1.6), $|\varphi_a(z)| < 1$ on $\mathbb{D}$.

We observe that

$$\varphi_a(a) = 0, \text{ and } \varphi_a(0) = a.$$

Then,

$$\varphi_a \circ \varphi_a(0) = 0, \text{ and } \varphi_a \circ \varphi_a(a) = a.$$

By the Schwarz lemma, this implies that $\varphi_a \circ \varphi_a$ must be the identity map of $\mathbb{D}$. It follows that φ_a is both one-to-one and onto from $\mathbb{D}$ to $\mathbb{D}$. □

It turns out that the composition of rotations and the maps of the form φ_a are all the possible automorphisms of $\mathbb{D}$.

Theorem 2.5. *A map f is an automorphism of $\mathbb{D}$ iff there are $\theta \in \mathbb{R}$ and $a \in \mathbb{D}$ such that*

$$f(z) = e^{i\theta} \cdot \frac{a - z}{1 - \overline{a}z}.$$

Proof. Let f be an element of $\mathrm{Aut}(\mathbb{D})$. Since f is onto, there is $a \in \mathbb{D}$ with $f(a) = 0$. The map

$$g = f \circ \varphi_a$$

is an automorphism of $\mathbb{D}$ with $g(0) = 0$. By the Schwarz lemma, we must have $|g'(0)| \le 1$. Applying the Schwarz lemma to g^{-1} we also obtain $|(g^{-1})'(0)| \le 1$. By the two inequalities, we have $|g'(0)| = 1$. Thus, by the same lemma, $g(z) = e^{i\theta} \cdot z$, for some $\theta \in \mathbb{R}$. That is, $f \circ \varphi_a(z) = e^{i\theta} z$, for $z \in \mathbb{D}$. Since $\varphi_a \circ \varphi_a$ is the identity map, we conclude that $f(z) = e^{i\theta} \varphi_a(z)$.

On the other hand, for any $\theta \in \mathbb{R}$ and any $a \in \mathbb{D}$, f belongs to $\mathrm{Aut}(\mathbb{D})$. That is because, f is the composition of the automorphism φ_a (Lemma 2.4) and a rotation. □

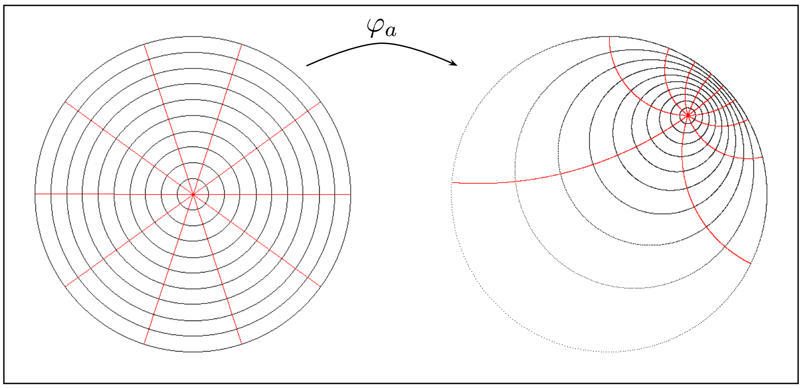

Figure 2.1: The images of the circles and rays under the map φ_a where $a = 0.5 + i0.5$.

2.3 Automorphisms of the half-plane

Define the upper half plane $\mathbb{H}$ as

$$\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}.$$

There are biholomorphic maps between $\mathbb{D}$ and $\mathbb{H}$ given by explicit formulae

$$F : \mathbb{H} \to \mathbb{D}, \quad F(z) = \frac{i - z}{i + z}, \quad F(i) = 0 \tag{2.1}$$

and

$$G : \mathbb{D} \to \mathbb{H}, \quad G(w) = i \cdot \frac{1 - w}{1 + w}, \quad G(0) = i.$$

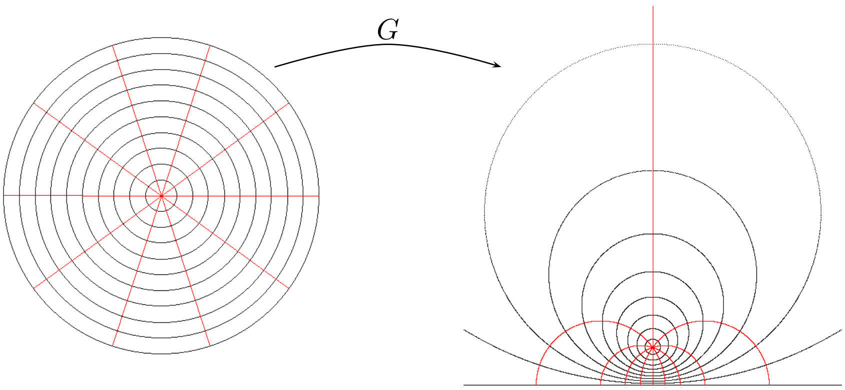

Figure 2.2: Similar line colors are mapped to one another by F.

Lemma 2.6. *The map $F : \mathbb{H} \to \mathbb{D}$ is a biholomorphic map with inverse $G : \mathbb{D} \to \mathbb{H}$.*

The proof of the above lemma is elementary and is left to the reader as an exercise.

The explicit biholomorphic map F allows us to identify $\mathrm{Aut}(\mathbb{H})$ in terms of $\mathrm{Aut}(\mathbb{D})$. That is, define

$$\Gamma : \mathrm{Aut}(\mathbb{D}) \to \mathrm{Aut}(\mathbb{H}) \text{ as } \Gamma(\varphi) = F^{-1} \circ \varphi \circ F.$$

It is clear that if $\varphi \in \mathrm{Aut}(\mathbb{D})$ then $\Gamma(\varphi) = F^{-1} \circ \varphi \circ F \in \mathrm{Aut}(\mathbb{H})$. The map Γ is one-to-one and onto with inverse given by $\Gamma^{-1}(\psi) = F \circ \psi \circ F^{-1}$. Indeed, Γ is more than just a bijection, it also preserves the operations on the groups $\mathrm{Aut}(\mathbb{D})$ and $\mathrm{Aut}(\mathbb{H})$. To see this, assume that φ_1 and φ_2 belong to $\mathrm{Aut}(\mathbb{D})$.

$$\Gamma(\varphi_1 \circ \varphi_2) = F^{-1} \circ (\varphi_1 \circ \varphi_2) \circ F = F^{-1} \circ \varphi_1 \circ F \circ F^{-1} \circ \varphi_2 \circ F = \Gamma(\varphi_1) \circ \Gamma(\varphi_2).$$

The isomorphism $\Gamma : \mathrm{Aut}(\mathbb{D}) \to \mathrm{Aut}(\mathbb{H})$ show that indeed the two groups are the same. However, we still would like to have explicit formulas for members of $\mathrm{Aut}(\mathbb{H})$. Using the explicit formulas for F and G, as well as the explicit formulas for elements of $\mathrm{Aut}(\mathbb{D})$ in Theorem 2.5, a long series of calculations shows that an element of $\mathrm{Aut}(\mathbb{H})$ is of the form

$$z \mapsto \frac{az+b}{cz+d},$$

where a, b, c, and d are real, and $ad - bc = 1$. We shall present an alternative proof of this, but before doing that we introduce some notations that simplify the presentation.

Define

$$\mathrm{SL}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \text{ and } ad - bc = 1 \right\}.$$

The set $\mathrm{SL}_2(\mathbb{R})$ forms a group with the operation of matrix-multiplication. This is called the *special linear group*. To each matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{SL}_2(\mathbb{R})$ we associate the map

$$f_M(z) = \frac{az+b}{cz+d}.$$

It is a straightforward calculation to see that for every M and M' in $\mathrm{SL}_2(\mathbb{R})$ we have

$$f_M \circ f_{M'} = f_{M \cdot M'}. \tag{2.2}$$

That is, the correspondence $M \mapsto f_M$ respects the group operations.

Theorem 2.7. *For every $M \in \mathrm{SL}_2(\mathbb{R})$ the map f_M is an automorphism of $\mathbb{H}$. Conversely, every automorphism of $\mathbb{H}$ is of the form f_M for some M in $\mathrm{SL}_2(\mathbb{R})$.*

Proof. We break the proof into several steps.

Step 1. Let $M \in \mathrm{SL}_2(\mathbb{R})$. The map f_M is holomorphic on $\mathbb{H}$. Moreover, for every $z \in \mathbb{H}$ we have

$$\operatorname{Im} f_M(z) = \operatorname{Im} \frac{az+b}{cz+d} = \operatorname{Im} \frac{(az+b)(\overline{cz+d})}{(cz+d)(\overline{cz+d})} = \frac{(ad-bc)\operatorname{Im} z}{|cz+d|^2} = \frac{\operatorname{Im} z}{|cz+d|^2} > 0.$$

Thus, f_M maps $\mathbb{H}$ into $\mathbb{H}$. As $\mathrm{SL}_2(\mathbb{R})$ forms a group, there is a matrix M^{-1} in $\mathrm{SL}_2(\mathbb{R})$ such that $M \cdot M^{-1} = M^{-1} \cdot M$ is the identity matrix. It follows that $f_M \circ f_{M^{-1}}$ is the identity map of $\mathbb{H}$. In particular, f_M is both one-to-one and onto. This proves the first part of the theorem.

Step 2. Let $h \in \operatorname{Aut}(\mathbb{H})$ with $h(i) = i$. Define $I = F \circ h \circ F^{-1}$, where F is the map in Equation (2.1). Then, $I \in \operatorname{Aut}(\mathbb{D})$ and $I(0) = 0$. Also, I^{-1} is a holomorphic map from $\mathbb{D}$ to $\mathbb{D}$ that sends 0 to 0. By the Schwarz lemma, $|I'(0)| \leq 1$, and $|(I^{-1})'(0)| \leq 1$. Thus, by the Schwarz lemma, I must be a rotation about 0, that is, there is $\theta \in \mathbb{R}$ such that

$$F \circ h \circ F^{-1}(z) = e^{i\theta} \cdot z. \tag{2.3}$$

Step 3. Let

$$Q = \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}.$$

The matrix Q belongs to $\mathrm{SL}_2(\mathbb{R})$, and one can verify that $f_Q(i) = i$ and $f'_Q(i) = e^{i\theta}$. Then the map $F \circ f_Q \circ F^{-1}$ is an automorphism of $\mathbb{D}$ that maps 0 to 0 and has derivative $e^{i\theta}$ at 0. By the Schwarz lemma, $F \circ f_Q \circ F^{-1}$ is the rotation $z \mapsto e^{i\theta} \cdot z$. That is, $F \circ f_Q \circ F^{-1} = F \circ h \circ F^{-1}$. Since F is one-to-one we conclude that $h = f_Q$, where h is the map in Step 1.

Step 4. We claim that for every $z_0 \in \mathbb{H}$ there is $N \in \mathrm{SL}_2(\mathbb{R})$ such that $f_N(i) = z_0$. First we choose a re-scaling about 0 that maps i to a point whose imaginary part is equal to $\operatorname{Im} z_0$. This map is given by the matrix

$$O = \begin{pmatrix} \sqrt{\operatorname{Im} z_0} & 0 \\ 0 & 1/\sqrt{\operatorname{Im} z_0} \end{pmatrix},$$

that is, $\operatorname{Im} f_O(i) = \operatorname{Im} z_0$. Then we use the translation $z + (z_0 - f_O(i))$ to map $f_O(i)$ to z_0. The latter map is obtained from the matrix

$$P = \begin{pmatrix} 1 & z_0 - f_M(i) \\ 0 & 1 \end{pmatrix}.$$

The map $f_P \circ f_O = f_{P \cdot O}$ maps i to z_0. Set $N = P \cdot O$.

Step 5. Let g be an automorphism of $\mathbb{H}$. There is z_0 in $\mathbb{H}$ with $g(z_0) = i$. By Step 4, there is $N \in \mathrm{SL}_2(\mathbb{R})$ such that $f_N(i) = z_0$. The composition $h = g \circ f_N$ belongs to $\mathrm{Aut}(\mathbb{H})$ and sends i to i. Thus, by Steps 2 and 3, $h = f_Q$. Now, using Equation (2.2), we have

$$g = h \circ (f_N)^{-1} = f_Q \circ f_{N^{-1}} = f_{Q \cdot N^{-1}}.$$

This shows that g has the desired form. $\square$

2.4 Exercises

Exercise 2.1. For $a \in \mathbb{C}$ and $r > 0$ let $B(a, r) = \{z \in \mathbb{C} : |z - a| < r\}$, that is, the open disk of radius r about a. Let a and b be arbitrary points in $\mathbb{C}$, and let r and s be positive real numbers. Prove that for every holomorphic map $f : B(a, r) \to B(b, s)$ with $f(a) = b$ we have $|f'(a)| \leq s/r$.

Exercise 2.2. Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic map.

(i) Prove that for every $a \in \mathbb{D}$ we have

$$\frac{|f'(a)|}{1 - |f(a)|^2} \leq \frac{1}{1 - |a|^2},$$

(ii) Prove that for every a and b in $\mathbb{D}$ we have

$$\left| \frac{f(a) - f(b)}{1 - f(a)\overline{f(b)}} \right| \leq \left| \frac{a - b}{1 - a\bar{b}} \right|.$$

The above inequalities are known as the Schwarz-Pick lemma.

Exercise 2.3. Let $h : \mathbb{H} \to \mathbb{H}$ be a holomorphic map. Prove that for every $a \in \mathbb{H}$ we have

$$|h'(a)| \leq \frac{\operatorname{Im} h(a)}{\operatorname{Im} a}.$$

Exercise 2.4. Prove that for every z and w in $\mathbb{D}$ there is $f \in \mathrm{Aut}(\mathbb{D})$ with $f(z) = w$.

[For an open set $U \subseteq \mathbb{C}$, we say that $\mathrm{Aut}(U)$ acts *transitively* on U, if for every z and w in U there is $f \in \mathrm{Aut}(U)$ such that $f(z) = w$. By the above statements, $\mathrm{Aut}(\mathbb{D})$ act transitively on $\mathbb{D}$.

Exercise 2.5. Prove Lemma 2.6