M3/4/5PA50 - INTRODUCTION TO RIEMANN SURFACES AND CONFORMAL DYNAMICS - HOME ASSIGNMENT 2

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Please submit your work by 16:00 on March 16 to the Undergraduate office on floor 6 in the Huxley building.

The total score of this assessment is 24. The final mark will be out of 20, as usual (with 20/24 you get 20).

Exercise 1 (4 points) Let M be a connected Riemann surface. Prove that M is path connected.

Exercise 2 (4 + 2 points) In this exercise we study the group of biholomorphism of an *annulus*

$$A_R = \{z \in \mathbb{C} \colon 1 < |z| < R\}.$$

We denote the boundary of A_R by $\partial\mathbb{D}_1 \cup \partial\mathbb{D}_R$.

(1) Prove that every biholomorphism ϕ of A_R extends to a homeomorphism of the boundary.
(2) Deduce that you have two possibilities: ϕ sends $\partial\mathbb{D}_1$ to $\partial\mathbb{D}_1$ and $\partial\mathbb{D}_R$ to $\partial\mathbb{D}_R$ or viceversa.

We consider in the following a biholomorphim ϕ sending $\partial\mathbb{D}_1$ to $\partial\mathbb{D}_1$ and $\partial\mathbb{D}_R$ to $\partial\mathbb{D}_R$.

(3) Consider the map $\psi(z) = \phi(z)/z$. Prove that $|\psi(z)| \to 1$ as z tends to the boundary of A_R.
(4) Deduce that $\psi \equiv 1$ on A_R (use the maximum principle!)
(5) Deduce that $|\phi|$ is a rotation of A_R.

Extra Let the group $\mathbb{Z}_2$ be the set $\{-1, +1\}$ with the product operation. Consider the set $(\mathbb{R}/\mathbb{Z}) \times \mathbb{Z}_2$ with an operation given by $(a, b) \cdot (c, d) = (a + b \cdot c, b \cdot d)$. Prove that this is indeed a group and that this is isomorphic to the group of biholomorphisms of A_R.

Exercise 3 (4 points) Let M be a Riemann surface and let g_1, g_2, g_3 be three holomorphic maps from M to the Riemann sphere $\widehat{\mathbb{C}}$. Let $\mathcal{F}$ be the family of all holomorphic maps f from M to $\widehat{\mathbb{C}}$ such that

$$f(\lambda) \neq g_i(\lambda) \qquad \forall \lambda \in M, \forall i \in \{1, 2, 3\}.$$

Prove that the family $\mathcal{F}$ is normal.

Exercise 4 (4 points) Let $f_c(z) := z^2 + c$. Assume that for some c_0 the map f_{c_0} has an attracting periodic cycle. Prove that f_c has an attracting periodic cycle for c sufficiently close to c_0.

Exercise 5 (6 points) Let $z_0 \in \mathbb{D}$. In this exercise we prove that there exists a constant $\sigma = \sigma(|z_0|) < 1$ such that, if $f \colon \mathbb{D} \to \mathbb{D}$ is a holomorphic function such that $f(0) = 0$ and $z_0 \notin f(\mathbb{D})$, then $|f'(0)| < \sigma$.

(1) Consider the maps (on $\mathbb{C}$) given by

$$g(z) = \frac{z - z_0}{1 - \bar{z}_0 z}, \; P(z) = z^2, \; h(z) = \frac{z - z_0^2}{1 - \bar{z}_0^2 z}.$$

Prove that the restriction of $B := h \circ P \circ g$ to $\mathbb{D}$ is a covering of degree 2 from $\mathbb{D}$ to $\mathbb{D}$, ramified at z_0.
(2) Prove that there exists a continuous path γ from z_0 to infinity avoiding $f(\mathbb{D})$.
(3) Prove that B admits an inverse on $\mathbb{C} \setminus \gamma$. Denote by A this inverse.
(4) Compute $|B'(0)|$ as a function of $|z_0|$. Prove that, as a function of $|z_0|$, $|B'(0)|$ is increasing and bounded from above by 1.
(5) Prove that $A \circ f$ maps $\mathbb{D}$ to $\mathbb{D}$. Deduce that $|(A \circ f)'| < 1$ (justify why not equal to 1!)
(6) Establish an upper bound for $|f'(0)|$ as a function of z_0.