M3/4/5PA50 - INTRODUCTION TO RIEMANN SURFACES AND CONFORMAL DYNAMICS - HOME ASSIGNMENT 1

FABRIZIO BIANCHI

Please submit your work by 16:00 on February 16 to the Undergraduate office on floor 6 in the Huxley building.

Recall that a *real Moebius transformation* is a self map of the upper half plane $\mathbb{H}$ of the form $z \mapsto \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$. The group of real Moebius transformations is denoted by $\mathrm{Mob}(\mathbb{H})$. Recall that a *Fuchsian group* is a discrete subgroup of $\mathrm{Mob}(\mathbb{H})$

Exercise 1 Prove that every real Moebius transformation is a finite composition of maps of the following type:

(1) translations $z \mapsto z + c$, with $c \in \mathbb{R}$;
(2) dilations $z \mapsto kz$, with $k \in \mathbb{R}^*$;
(3) the inversion $z \mapsto -1/z$.

Exercise 2 Let $A_1, A_2, A_3, B_1, B_2, B_3 \in \mathbb{H}$. Prove that there exists at maximum one real Moebius transformation γ such that $\gamma(A_i) = B_i$, for $i = 1, 2, 3$.

Exercise 3 Let F, G be two Moebius transformations with real coefficients and different from the identity. Recall that they induce a map on $\partial\mathbb{H}$. Assume that $F \circ G = G \circ F$. We denote by $\mathrm{Fix}(F)$ the set of the fixed points of F (in $\mathbb{H}$ as well as in $\partial\mathbb{H}$).

(1) Prove that G maps point in $\mathrm{Fix}(F)$ to point of $\mathrm{Fix}(F)$.
(2) Prove that G is injective on $\mathrm{Fix}(F)$.
(3) Prove that fixed points of F in $\mathbb{H}$ are sent by G to fixed points of F in $\mathbb{H}$, and fixed points of F in $\partial\mathbb{H}$ are sent by G to fixed points of F in $\partial\mathbb{H}$.
(4) Assume that F is elliptic. Prove that G is elliptic, too. Prove that the (unique) fixed point of F coincides with the unique fixed point for G.

Exercise 4 Let γ be a real Moebius transformation. Let Γ be the subgroup of the real Moebius transformations generated by γ.

(1) Assume that γ is parabolic. Prove that Γ is Fuchsian.
(2) Assume that γ is hyperbolic. Prove that Γ is Fuchsian.
(3) Assume that γ is elliptic. Prove that Γ is Fuchsian if and only if there exists $N \in \mathbb{N}$ such that $\gamma^{\circ N}$ (the composition of γ with itself N times) is equal to the identity (hint: it may be easier to work in the Poincaré disc model)