

6

Global dynamics of holomorphic maps

In this chapter we study the dynamics of a holomorphic self map of a Riemann surface.

6.1 Dynamics on hyperbolic Riemann surfaces

We begin our study with the hyperbolic surfaces, i.e., surfaces covered by the unit disc. Let us start assuming that the surface is the disc and the map f a biholomorphism. Then f is a Moebius transformation. The dynamics of f follows immediately by the description of the previous chapters.

Exercise 6.1.1. *Study the dynamics of a Moebius transformation of the unit disc.*

We now move to self maps of any degree (and any hyperbolic surface). We will just state the results, without proofs. For all the details, one can consult Milnor's book. The following Lemma is a consequence of Montel Theorem.

Lemma 6.1.2. *The Julia set of any self map of a hyperbolic Riemann surface is empty.*

Theorem 6.1.3 (Fatou). *For every holomorphic self map f of a hyperbolic Riemann surface, exactly one of the following occurs:*

1. *f has an attracting fixed point, and all orbits converge to this point and the convergence is locally uniform on compact subsets;*
2. *some orbit has no accumulation point, and so no orbit has accumulation points;*
3. *f has two distinct periodic points, and some iterate of f is the identity map;*
4. *in all other case, S is biholomorphic to either the disc, the punctured disc or an annulus, and the dynamics is conjugated to an irrational rotation.*

Theorem 6.1.4 (Denjoy-Wolff). *Let f be a holomorphic self map of the unit disc. Then:*

1. f is conjugated to a rotation; or
2. the iterates of f converge locally uniformly on compact subsets to a constant map $h(z) = c_0$, where c_0 can belong to the disc or to its boundary.

6.2 Dynamics on Euclidean Riemann surfaces

We consider now self maps of surfaces whose universal covering is the complex plane. We have the following cases: the complex tori, the punctured plane (or cylinder) and the complex plane itself.

We start with the complex tori. In this case, the surface is compact and we have $\mathbb{T} = \mathbb{C}/\Lambda$. It turns out that the dynamics is quite easy to understand.

Theorem 6.2.1. *Every holomorphic map $f : \mathbb{T} \rightarrow \mathbb{T}$ is an affine map, $f(z) = \alpha z + c \pmod{\Lambda}$, with degree $d = |\alpha|^2$, and we have the following dichotomy:*

1. if $d \leq 1$, then the Julia set is empty;
2. if $d > 1$, then the Julia set equals \mathbb{T} .

Proof. Let us assume that the torus is given by $\mathbb{T}/\mathbb{C}/\Lambda$, where $\Lambda = \langle 1, \tau \rangle$, with $\tau \notin \mathbb{R}$. The map f lifts to a self map F of the covering space \mathbb{C} .

There exists a $\alpha \in \Lambda$ such that $F(z+1) = F(z) + \alpha$. Indeed, we must have $F(z+1) \equiv F(z) \pmod{\Lambda}$ and the difference function $F(z+1) - F(z)$ must be constant since takes value in a discrete set. In the same way, there exists $\beta \in \mathbb{C}$ such that $F(z+\tau) = F(z) + \beta$. Set $g(z) = F(z) - \alpha z$, so that $g(z) = g(z+1)$. Then

$$g(z+\tau) = F(z+\tau) - \alpha(z+\tau) = F(z) + \beta + \alpha z + \alpha\tau = g(z) + \beta - \alpha\tau.$$

It suffices to prove that g is constant. Notice that g gives rise to a (holomorphic) map from \mathbb{T} to $\mathbb{C}/(\beta - \alpha\tau)\mathbb{Z}$. The target surface can be \mathbb{C} or a cylinder. In either case, it is non compact, which implies that g is constant.

So, $F(z) = \alpha z + c$ as required. The area is multiplied by $|\alpha|^2$, so the degree is $|\alpha|^2$. The final assertions are proved by considering the derivative of the sequence $\{f^n\}$ at any point of \mathbb{T} . \square

In the non compact case the dynamical study is much more difficult. Remark that we may have two possibilities: the iteration of a polynomial map, or of a trascendental map. The study of the iteration of polynomial maps can be seen as a particular case of the study of rational maps on the Riemann sphere, that we will do in the next section. The study of trascendental maps is much more difficult and we do not do it here.

6.3 Conformal dynamics on the Riemann sphere

We study in this section the dynamics of a rational map on the Riemann sphere. We start with some general properties of the Julia set. Recall the following version of Montel Theorem.

Theorem 6.3.1 (Montel's Theorem). *Suppose that $V \subset \mathbb{C}$ is an open domain, and let \mathcal{F} be a family of meromorphic maps $f : V \rightarrow \widehat{\mathbb{C}}$ that whose range omits three distinct values in $\widehat{\mathbb{C}}$; that is, there exist three distinct points $\{a, b, c\}$ such that for any $f \in \mathcal{F}$, $f(\widehat{\mathbb{C}}) \cap \{a, b, c\} = \emptyset$. Then \mathcal{F} is a normal family.*

Lemma 6.3.2. *The Julia set of f is non-empty.*

Proof. We argue by contradiction. Suppose that $J(f)$ is empty. Then $\{f^n\}$ forms a normal family on $\widehat{\mathbb{C}}$. Then there exists a subsequence f^{n_j} of the iterates of g that converges uniformly to a meromorphic function g on $\widehat{\mathbb{C}}$. But now, all but a finite number of points have $d = \deg(g)$ preimages under g . Since f^{n_j} converges uniformly to g , for j large enough, f^{n_j} has the same property. But $\deg(f^{n_j})$ tends to infinity as $j \rightarrow \infty$, so that is impossible. Hence $J(f)$ is non-empty. \square

The *grand orbit* of a point z under $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ is the set of all points whose orbit eventually intersects the orbit of z . A point z' is in the grand orbit of z if there exist $m, n \in \mathbb{N}$ such that $f^m(z) = f^n(z')$. A point z is called *grand orbit finite* (or *exceptional*) if its grand orbit is a finite set.

Lemma 6.3.3 (Finite Grand Orbits). *Let $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ be a rational map with degree at least 2. The set of points with finite grand orbits can have at most two elements and these points, if they exist, must be super attracting fixed points and hence must belong to the Fatou set.*

Proof. Suppose that z is a point whose grand orbit is finite. A rational mapping $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ must map the grand orbit of any point onto itself. Since the grand orbit of z is finite, f must map the grand orbit of z bijectively onto itself. Hence the grand orbit of z must consist of a single periodic orbit

$$z = z_0 \mapsto z_1 \mapsto z_2 \mapsto \dots \mapsto z_m = z_0.$$

Notice that any point $\hat{z} \in \widehat{\mathbb{C}}$ has $d = \deg(f)$ preimages, counted with multiplicity. Thus every point z_i , $0 \leq i < m$, must be a critical point of f . Thus the number of grand orbit finite points is finite (bounded by the number of critical points of f) and is part of a super attracting cycle. Hence they are in the Fatou set.

Suppose that there are three distinct grand orbit finite points, then for any set U that does not contain these points, $\{f^n|U\}$ is a normal family (the range of every function in the family omits the three grand orbit finite points, so Montel's Theorem applies.) But now, every point in $\widehat{\mathbb{C}}$ is contained in the Fatou set of f , which contradicts the fact that the Julia set is non-empty. \square

Theorem 6.3.4 (Transitivity). *Let $z_0 \in J(f)$ and let N be an arbitrary neighbourhood of z_0 . Let $U = \cup_{n=0}^{\infty} f^n(N)$, be the union of forward images of N . Then U contains $J(f)$ and contains all but at most two points of $\widehat{\mathbb{C}}$. If N is chosen sufficiently small, then $\widehat{\mathbb{C}} \setminus U$ is the set of grand orbit finite points.*

Proof. Montel's Theorem implies that the complement of U can contain at most 2 points. Otherwise since $f(U) \subset U$ it follows from Montel's Theorem that $\{f^n\}$ is a normal family on N , which contradicts $z_0 \in J(f)$. Let $z \in \widehat{\mathbb{C}} \setminus U$. No preimage of z can be contained in U , so z is either fixed point or a periodic point of period 2, and it grand orbit finite. Since the points with finite grand orbit are contained in the Fatou set of f , it follows that U contains $J(f)$. If N is chosen small enough so that N does not contain any grand orbit finite point of f , then $\widehat{\mathbb{C}} \setminus U$ consists of grand orbit finite points. \square

Corollary 6.3.5. *If $J(f)$ has interior, then $J(f) = \widehat{\mathbb{C}}$.*

Proof. Suppose that N is an open set contained in $J(f)$, then $U = \cup_{n=0}^{\infty} f^n(N)$ is a union of open sets, and so it is open. By Theorem 6.3.4, U omits at most two points in $\widehat{\mathbb{C}}$, and by invariance, $U \subset J(f)$. Hence $J(f)$ omits at most two points in $\widehat{\mathbb{C}}$, but $J(f)$ is closed (so that it must contain its boundary points), thus $J(f) = \widehat{\mathbb{C}}$. \square

Corollary 6.3.6 (Iterated preimages are dense). *Suppose that $z_0 \in J(f)$, then $\cup_{n=0}^{\infty} f^{-n}(z_0)$ is dense in $J(f)$; that is,*

$$\{z : f^n(z) = z_0 \text{ for some } n \geq 0\}$$

is dense in $J(f)$.

Proof. Notice that z_0 is not grand orbit finite and use Theorem 6.3.4. The details are left as an exercise. \square

Corollary 6.3.7. *$J(f)$ has no isolated points.*

Proof. If $J(f)$ was a finite set, then it would consist of grand orbit finite points, but these are in the Fatou set of f . Hence $J(f)$ is infinite. Since $J(f)$ is compact, it must contain at least one limit point z_0 . The iterated preimages of z_0 forms a dense set of nonisolated points. So no point in $J(f)$ can be isolated. \square

We say that a property is *generic* if it holds on a countable intersection of dense open sets.

Corollary 6.3.8 (Topological transitivity). *For a generic $z \in J(f)$,*

$$\{f^n(z) : n \in \mathbb{N}\}$$

is dense in $J(f)$; that is, the forward orbit of z is dense in $J(f)$.

Proof. For each integer j , we can cover $J(f)$ by finitely many open sets, $N_{j,k}$, with diameter $< 1/j$ (in the spherical metric). For each $N_{j,k}$ let $U_{j,k}$ be the union of preimages of $N_{j,k}$: $U = \cup_{n=0}^{\infty} f^{-n}(N_{j,k})$. By the Iterated Preimages are Dense Lemma, the closure $\overline{U_{j,k} \cap J(f)} = J(f)$. Hence $U_{j,k} \cap J(f)$ is a dense open subset of $J(f)$. If a point z belongs to the intersection of $\cap_{j,k} U_{j,k}$, then the orbit of z intersects every $N_{j,k}$, and hence is dense in $J(f)$. \square

The next theorem gives a nice characterization of the Julia set.

Theorem 6.3.9 (Repelling cycles are dense). *The Julia set is the closure of the set of repelling periodic points.*

Notice the stark difference between Proposition 6.3.8 and Theorem 6.3.9: the forward orbit of a repelling periodic point can never be dense in the Julia set - it's finite, and repelling periodic points are dense in $J(f)$. But the set of repelling periodic points is small - it's countable - and generically, forward orbits of points are dense in the Julia set. We will prove this theorem later.

6.4 Most periodic orbits repel

Theorem 6.4.1. *Suppose that $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ is rational map with degree d at least 2. The number of attracting or indifferent cycles of F is bounded from above by $6d - 6$.*

For a rational map of degree d , f' has degree $2d - 2$, and so a rational map can have at most $2d - 2$ critical points. Thus we have already explained why the number of attracting or parabolic points is at most $2d - 2$ - to each attracting or parabolic point, there is an associated critical point in the immediate basin of attraction (in the attracting case) or in an attracting petal (in the parabolic case). Hence the theorem follow from

Lemma 6.4.2. *Let f be a rational map with degree d at least 2. Then f has at most $4d - 4$ indifferent cycles.*

Proof. Let k denote the number of indifferent cycles of f . The idea of the proof is to perturb $1/2$ of the indifferent cycles of f , so that they become attracting. Then $k/2 \leq 2d - 2$, which gives us that $k \leq 4d - 4$, which is what we want.

If f is the function $z \mapsto z^d$, then f has no indifferent cycles, and the estimate holds. So from now on, we assume that f is not identical to $z \mapsto z^d$, and that $k \neq 0$.

To start, we include f in a smooth family of rational maps: Suppose that

$$f(z) = \frac{p(z)}{q(z)}$$

is a quotient of polynomials without common factors. Set

$$f_t = \frac{p(z) - tz^d}{q(z) - t}, \quad f_0(z) = f(z), \quad f_\infty(z) = z^d.$$

Remember that f_t is a family of rational functions that depends on a complex parameter t .

For almost all values of t the family f_t varies smoothly with respect to t . The only time it doesn't is when a root of the numerator coincides with a root of the denominator at a point \hat{t} , so that as $t \rightarrow \hat{t}$, the degree of f_t is d for t close to \hat{t} , but $t \neq \hat{t}$, and the degree of $f_{\hat{t}}$ is $< d$. When this occurs we have that $p(z) - tz^d = 0 = q(z) - t$, and we see that

$$f(z) = \frac{p(z)}{q(z)} = \frac{tz^d}{t} = z^d.$$

Notice that the equation $f(z) = z^d$ does not depend on t , and that since f is a rational function that is not identical to z^d , it has at most finitely many solutions. Thus there are at most finitely many values of t such that $p(z) - tz^d = 0 = q(z) - t$. Thus, since $f = f_0$, there is a neighbourhood U of 0 such that $t \mapsto f_t$ is a smooth family of rational maps for $t \in U$.

For each indifferent cycle $\mathcal{O}_j(0)$, $j = 1, \dots, k$, of $f_0 = f$ choose a periodic point $z_j(0) \in \mathcal{O}_j(0)$. Let l_j be the period $z_j(0)$ and let $\lambda_j(0) = Df^{l_j}(z_j(0))$ be its multiplier.

Observe that $f_0^{l_j}(z_j(0)) = z_j(0)$, and that $t \mapsto f_t$ is a smooth family of rational maps, so that by the Implicit Function Theorem, there is a neighbourhood $U' \subset U$ of 0 such that for all $t \in U'$, there is a point $f_t^{l_j}(z_j(t)) = z_j(t)$. To each of these cycles we associate the multiplier mapping

$$t \mapsto \lambda_j(t) = Df_t^{l_j}(x_j(t)).$$

The points $z_j(t)$ and the multipliers $\lambda_j(t)$ all depend holomorphically on t .

Choose an angle $\theta \in [0, 2\pi)$ such that the ray

$$r \mapsto re^{i\theta} = R(r), \quad r \in [0, \infty]$$

avoids the finitely many points \hat{t} where f_t does not depend smoothly on t .

Claim: None of the functions $t \mapsto \lambda_j(R(r))$ is constant on a neighbourhood of 0 in $[0, \infty]$.

Proof of claim. We argue by contradiction. Suppose that for some j , $t \mapsto \lambda_j(t)$ is constant on a neighbourhood of 0. Let $A \subset [0, \infty]$ denote the set of all r such that $\lambda_j(R(r)) = \lambda_j(R(0))$. By assumption A is non-empty. We will show that A is open and closed. Let $r_i \rightarrow \hat{r}$ be any sequence of points in A . Since the periodic points $z_j(R(r_i))$ of period l_j under $f_{R(r_i)}$ depend continuously on r for all $r \in [0, \infty]$, $z_j(R(r_i)) \rightarrow z_j(R(\hat{r}))$ as $r_i \rightarrow \hat{r}$, and the multiplier, $\lambda_j(R(r_i)) \rightarrow \lambda_j(R(\hat{r}))$, since $\lambda_j(R(r_i)) = \lambda_j(0)$ for all i , we have that $\lambda_j(\hat{r}) = \lambda_j(0)$. So $\hat{r} \in A$ and A is closed. By the Implicit Function Theorem we have that any periodic orbit: $f_{R(r)}^{l_j}(z_j(R(r))) = z_j(R(r))$ varies smoothly with r , and since the multiplier map is holomorphic, and constant in A , that nearby maps have the same multiplier. Hence A is open.

This implies that $A = [0, \infty]$, but now $f_\infty(z) = z^d$ has an indifferent cycle, but this is not the case. \square

Let us return to the proof of the theorem. For $1 \leq j \leq k$, express

$$\frac{\lambda_j(t)}{\lambda_j(0)} = 1 + a_j t^{n_j} + \dots \quad a_j \neq 0, n_j \geq 1.$$

Then

$$|\lambda_j(t)| = 1 + \operatorname{Re}(a_j t^{n_j}) + o(t^{n_j}).$$

We divide the t -plane into n_j sectors where $\operatorname{Re}(a_j t^{n_j}) > 0$ and n_j sectors where $\operatorname{Re}(a_j t^{n_j}) < 0$.

Let

$$\sigma_j(\theta) = \operatorname{sign}(\operatorname{Re}(a_j e^{i\theta n_j})).$$

Then $\sigma_j(\theta) = 1$ iff $|\lambda_j(re^{i\theta})| > 1$ for small $r > 0$ and $\sigma_j(\theta) = -1$ iff $|\lambda_j(re^{i\theta})| < 1$ for small $r > 0$. For each j

$$\sigma_j(\theta) \rightarrow \{1, -1, 0\}$$

is a step function that takes on the values ± 1 except at the $2n_j$ discontinuity points and has average value 0 on $[0, 2\pi)$.

Suppose that k is odd. Let $\sigma = \sigma_1 + \sigma_2 + \dots + \sigma_k$. Then σ is a step function with average value 0, and σ takes on odd integer values almost everywhere.

Suppose that there exists no value of θ such that $\sigma_j(\theta) = -1$ for at least $(k+1)/2$ of the j 's. Then for all θ , at most $(k-1)/2$ of the values $\sigma_j(\theta)$ is -1 , but this implies that the average of σ on $[0, 2\pi)$ is greater than 0, which is a contradiction. Hence, there is some value of theta such that at least $(k+1)/2$ of the values $\sigma_j(\theta)$ are -1 . Taking r sufficiently small, we have that at least $(k+1)/2$ of the cycles of $f_{re^{i\theta}}$ have multiplier $\lambda_j(re^{i\theta})$ with $|\lambda_j(re^{i\theta})| < 1$. Thus $(k+1)/2 < 2d-2$, so that

$$k < k+1 \leq 4d-4.$$

If k is even, set $\sigma = \sigma_1 + \dots + \sigma_{k-1}$ and repeat the argument. □

6.5 Density of Repelling Periodic Points

Theorem 6.5.1. *The Julia set for any rational map f with degree $d \geq 2$ is equal to the closure of its repelling periodic points.*

Proof. Since the Julia set of f contains no isolated points, we can exclude finitely points from consideration without affecting the argument.

Let z_0 be any point in $J(f)$ such that z_0 is not a critical point of f and not a fixed point of f . (Since f is a rational map, it has at most finitely many fixed points and at most finitely many critical points). Then there exist distinct points $z_1, z_2 \in f^{-1}(z_0)$ which are both distinct from z_0 and local inverses φ_1 and φ_2 of f so that $\varphi_i(z_0) = z_i$ and $f \circ \varphi_i(z_0) = z_0$ for $i = 1, 2$. Let U be any neighbourhood of z_0 that is contained in the domain of φ_1 and the domain of φ_2 .

Claim: there exists $z \in U$ and $n \in \mathbb{N}$ such that $f^n(z) \in \{z, \varphi_1(z), \varphi_2(z)\}$.

Proof of claim: Suppose not. Then for all $z \in U$, $f^n(z)$ avoids the three values $z, \varphi_1(z)$ and $\varphi_2(z)$. (We cannot apply Montel's Theorem at this moment since the points that the family $f^n|_U$ avoids depend on the point z . This is easily overcome.) Let

$$g_n(z) = \frac{(f^n(z) - \varphi_1(z))(z - \varphi_2(z))}{(f^n(z) - \varphi_2(z))(z - \varphi_1(z))}.$$

Then $g^n|_U$ avoids the values 0, 1 and ∞ . Hence $\{g^n|_U\}$ is a normal family. It is easy to see that this implies that $f^n|_U$ is a normal family. Since $U \cap J(f) \neq \emptyset$, this is impossible \square

Thus we see have that either $f^n(z) = z$ or $f^{n+1} = f(f^n(z)) = f(\varphi_i(z)) = z$ (for one of $i = 1$ or 2). Hence $z \in U$ is a periodic point. We have proved that z_0 is an accumulation point of periodic points. By Theorem 9.1 at most finitely many periodic points of f are non-repelling. Hence z_0 is an accumulation point of repelling periodic points. Since all repelling periodic points are contained in the Julia set, we have the repelling periodic points are dense in the Julia set. \square

Corollary 6.5.2. *If U is an open set that intersects $J(f)$ then for n sufficiently big $f^n(U \cap J(f)) = J(f)$.*

Proof. There exists a repelling periodic point $z_0 \in U$ of say period p , so using the local behaviour of f^p near z_0 , we can find a neighbourhood V of z_0 such that $f^p(V) \supset V$. But now

$$V \subset f^p(V) \subset f^{2p}(V) \subset \dots$$

We know from before that

$$J(f) = J(f^p) \subset \bigcup_{n=1}^{\infty} f^{np}(V).$$

By compactness of $J(f)$, we have that there exists N such that $J(f) \supset \bigcup_{n=1}^N f^{np}(V) = f^{Np}(V)$. The result now follows from the total invariance of the Julia set. \square

Corollary 6.5.3. *If $U \subset \widehat{\mathbb{C}}$ is any open set that intersects the Julia set $J(f)$, then **no** sequence of iterates f^{n_i} can converge uniformly on compact subsets of U .*

Proof. Suppose that a sequence of functions f^{n_i} converges uniformly on compact subsets of U to a function g . Let $z_0 \in U \cap J(f)$. The since g is continuous, for any $\varepsilon > 0$, there exists a neighbourhood U' of z_0 such that for any $z \in U'$, we have $|g(z) - g(z_0)| < \varepsilon$. But then for all i large enough $|f^{n_i}(z) - g(z_0)| < 2\varepsilon$. In other words

$$f^{n_i}(U') \subset D(z_0, 2\varepsilon)$$

for all large i . This contradicts the previous corollary. \square

6.6 Classification of periodic Fatou components

Theorem 6.6.1. *Suppose that $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ is a rational mapping of the Riemann sphere with degree at least 2. Assume that f maps a Fatou component U onto itself, then there are four possibilities:*

- U is an immediate basin of attraction for an attracting fixed point.
- U is the immediate basin of an attracting petal for a parabolic fixed point which has multiplier $\lambda = 1$
- U is a Siegel disk; that is, U is simply connected and $f : U \rightarrow U$ is conjugate to an irrational rotation.
- U is a Herman ring; that is, U is conformally isomorphic to an annulus, \mathbb{A}_r and f is conjugate to an irrational rotation.

Proof. The Fatou component U is a hyperbolic Riemann surface, so by Theorem ??, either U contains an attracting fixed point, $\{f^n\}$ is compactly divergent, f is an automorphism of finite order, or f is conjugate to an irrational rotation of \mathbb{D} , $D \setminus \{0\}$ or an annulus, \mathbb{A}_r . Since f has degree at least 2, no iterate of f can be the identity. This also implies that $J(f)$ has no isolated points, so that U cannot be conformally isomorphic to the punctured disk. So we need to show that if $\{f^n\}$ is compactly divergent that all orbits $z_0 \rightarrow z_1 \rightarrow z_2 \rightarrow \dots$ converge to the same boundary point, which is a parabolic fixed point with multiplier 1. \square

Lemma 6.6.2. *Suppose that f is as in Theorem 6.6.1. Suppose that U is a fixed Fatou component of f and that $\{f^n\}$ is compactly divergent on U . Then all orbits $z_0 \rightarrow z_1 \rightarrow \dots$ under $f|_U$ converge to the same point $\widehat{z} \in \partial U$ and \widehat{z} is fixed by f . Moreover, the convergence is uniform on compact subsets of U .*

Proof. Let $p : [0, 1] \rightarrow U$ be a continuous path connecting z_0 and z_1 . Label this path by $p[0, 1]$. Extend the path inductively by the formula: $p[n, n+1] = f(p[n-1, n])$. Let $\delta = \text{diam}_U(p[0, 1])$. Then by the Schwarz-Pick Lemma for all n , $\text{diam}_U p[n, n+1] \leq \delta$. Let \widehat{z} be any accumulation point of $z_0 \mapsto z_1 \mapsto z_2 \mapsto \dots$. Since $\{f^n\}$ is compactly divergent, $\widehat{z} \in \partial U$. Let V be any neighbourhood of \widehat{z} . There exists a neighbourhood W of \widehat{z} so that if $p[n, n+1] \cap W \neq \emptyset$, then $p[n, n+2] \subset V$. Since this is true for any neighbourhood of \widehat{z} , we see that \widehat{z} must be a fixed point of f . Since f is a rational mapping of the Riemann sphere, we know that f has only finitely many fixed points, so in particular f has at most finitely many fixed points on ∂U . We will show that the set of fixed points of f on ∂U must be connected, from which it follows that \widehat{z} is the unique accumulation point on ∂U of orbits in U . We can express the set of accumulation points of the orbit $z_0 \mapsto z_1 \mapsto \dots$ as $\bigcap_{t>0} \overline{p[0, t]}$. This is an intersection of compact connected sets, so it must be connected. Indeed the same argument shows that every sequence of orbits must accumulate to the same fixed point. We leave it to the reader to check that convergence is uniform on compact subsets. \square

Lemma 6.6.3 (Snail Lemma). *Let V be a neighbourhood of 0. Suppose that there exists a path $p : [0, \infty) \rightarrow V \setminus \{0\}$ so that $f(p(t)) = p(t + 1)$ and so that $p(t) \rightarrow 0$ as $t \rightarrow \infty$. Then either $|f'(0)| < 1$ or $f'(0) = 1$.*

Proof. Omitted. For a proof see, for example, Dynamics in One Complex Variable by John Milnor. \square

6.7 Dynamics of polynomials

In this section we assume that f is a polynomial on \mathbb{C} of a given degree $d \geq 2$. We will suppose that f is monic, i.e., the leading term will be z^d . Notice this is possibly by simply conjugating f with a suitable dilation.

Definition 6.7.1. *The filled Julia set K of f is the set of points with bounded orbit.*

Proposition 6.7.2. *The filled Julia set is compact, with connected complement, equal to the basin of infinity. The boundary of K is equal to J . The interior of K is the union of all bounded components of the Fatou set. All bounded connected components of the Fatou set are simply connected.*

Proof. First of all, ∞ is a superattracting fixed point. There exists thus a R such that if $|z| > R$ then $|f(z)| > 2|z|$. To be more precise, notice that $f(z)/z^d \rightarrow 1$ as $z \rightarrow \infty$. Thus, for $|z| > r_0$, we have $|f(z)/z^d - 1| < 1/2$, which implies

$$|f(z)| > |z^d|/2 > 2|z|.$$

So, every point z such that $|z| > r_0$ belongs to the basin Ω of the point at infinity. A basin is open, and so $K = \widehat{\mathbb{C}} \setminus \Omega$ is compact. This implies $\partial K = \partial\Omega$, which is equal to J by the Lemma below.

Let us show that Ω is connected. Let U a bounded component of the Fatou set and let us prove that U is not contained in Ω . Since the boundary of U is contained in J (and thus in K), we have $|f^n(z)| < r_0$ for every $z \in \partial U$, and thus for every $z \in U$ by the maximum principle. This implies that U is contained in K , and the unique unbounded Fatou component coincides with the basin Ω .

Finally, let us prove that every bounded component U of the Fatou set is simply connected. Let γ be a simple closed curve in U and let V be the bounded component of $\mathbb{C} \setminus \gamma$. Again by the maximum principle, we have that V is contained in K . In particular, V cannot contain any point of $J = \partial K$. This proves that V is contained in the Fatou set, and thus in U . \square

We used the following lemma, valid for every rational map.

Lemma 6.7.3. *Let f be a rational map and \mathcal{A} be the basin of a periodic attracting cycle. Then the boundary $\partial\mathcal{A} = \overline{\mathcal{A}} \setminus \mathcal{A}$ is equal to the entire Julia set. Every connected component of the Fatou either coincides with some connected component of \mathcal{A} or is disjoint from \mathcal{A} .*

Proof. Let N be an open neighbourhood of a point of J . By Theorem 6.3.4, some $f^n(N)$ intersects \mathcal{A} , and so also N intersects \mathcal{A} (possibly a different component of \mathcal{A}). So, $J \subset \overline{\mathcal{A}}$. Since $J \cap \mathcal{A} = \emptyset$, we have $J \subset \partial\mathcal{A}$. Let us prove the other inclusion. If N is an open neighbourhood of a point of $\partial\mathcal{A}$, then any limit of iterates f^n must have a jump discontinuity on N , and so $\partial\mathcal{A} \subset J$.

The last assertion follows since every connected component of the Fatou set cannot intersect $J = \partial\mathcal{A}$. \square