5

Local dynamics of holomorphic maps

In this chapter we study the local dynamics of a holomorphic map near a fixed point. The setting will be the following. We have a holomorphic map defined in a neighbourhood of the origin and of the form

$$f(z) = \lambda z + a_2 z^2 + a_3 z^3 \dots \tag{5.1}$$

We want to find a change of coordinates φ near the origin such that $\varphi^{-1} \circ f \circ \varphi(z) = \lambda z$ (or the first non zero term if $\lambda = 0$). This is called the *linearization problem*. Notice that, if we have a *periodic* point, i.e., a point which is fixed by some iterate of the map, by considering this iterate we can reduce the dynamical study to that of fixed points.

5.1 Attracting and repelling points

In this section, we study the local dynamics at a geometrically attracting or repelling periodic point.

Theorem 5.1.1 (Koenigs). *Let f be a holomoprhic map on a neighbourhood of 0, fixing the origin. Assume the multiplier λ at 0 satisfies $|\lambda| \neq 0, 1$. Then there exists a local holomorphic change of coordinate $w = \varphi(z)$, with $\varphi(0) = 0$, so that $\varphi \circ f \circ \varphi^{-1}$ is the linear map $w \mapsto \lambda w$ for all w in some neighbourhood of 0. Furthermore φ is unique up to multiplication by some non-zero constant.*

Proof. We will do the proof in the case when $|\lambda| < 1$. To prove the theorem when $|\lambda| > 1$, apply the same proof to the local inverse of f near 0.

We start proving the existence of the map φ. Consider the sequence of maps

$$\varphi_n(z) = \frac{f^n(z)}{\lambda^n}.$$

Suppose that φ_n converges uniformly to a holomorphic function φ in a neighbourhood of 0. Then $\varphi(0) = 0$, and moreover for all n:

$$\varphi_n \circ f(z) - \lambda\varphi_{n+1}(z) = \frac{f^{n+1}(z)}{\lambda^n} - \lambda\frac{f^{n+1}(z)}{\lambda^{n+1}} = 0.$$

Hence for the limiting map $\varphi(z)$, we have

$$\varphi \circ f(z) = \lambda\varphi(z),$$

as required. So we need only show that φ_n converges uniformly in a neighbourhood of 0.

Choose a constant c such that $c^2 < \lambda < c < 1$. Choose a neighbourhood $\mathbb{D}_r$ of the origin so that $|f(z)| < c|z|$ for $z \in \mathbb{D}_r$. Thus for any starting point $z_0 \in \mathbb{D}_r$ we have

$$|f^n(z_0)| < c^n r.$$

By Taylor's Theorem we have that there exists a constant $C > 0$ so that

$$|f(z) - \lambda z| \leq C|z|^2, \; z \in \mathbb{D}_r.$$

Hence

$$|f^{n+1}(z) - \lambda f^n(z)| \leq C|f^n(z)|^2 \leq Cr^2c^{2n}.$$

But now

$$|\varphi_{n+1}(z) - \varphi_n(z)| = |\frac{f^{n+1}(z)}{\lambda^{n+1}} - \frac{f^n(z)}{\lambda^n}| \leq \frac{1}{\lambda^{n+1}}Cr^2c^{2n} = \frac{Cr^2}{\lambda}\Big(\frac{c^2}{\lambda}\Big)^n,$$

which converges geometrically to 0, since $c^2 < \lambda$. Thus the φ_n converge uniformly in $\mathbb{D}_r$.

Notice that each $\varphi_n'(0) = 1$, so that φ is a local conformal isomorphism.

Let us now prove that the map φ constructed as above is the only possibility. Suppose that ψ is a second linearization of f at 0. Let us show first that

$$\lambda\psi \circ \varphi^{-1}(w) = \psi \circ \varphi^{-1}(\lambda w)$$

Let z be a point in a neighbourhood of 0 so that

$$\varphi \circ f(z) = \lambda\varphi(z) = \lambda w.$$

Then $\psi \circ \varphi^{-1}(\lambda w) = \psi \circ \varphi^{-1}(\varphi \circ f(z)) = \psi \circ f(z) = \lambda\psi(z) = \lambda\psi \circ \varphi^{-1}(w)$.

Now, $\psi \circ \varphi^{-1}$ is a holomorphic function defined in a neighbourhood of 0, so we can express it as its power series: $\psi \circ \varphi^{-1}(w) = b_1 w + b_2 w^2 + b_3 w^3 + \ldots$, but now, $\lambda\psi \circ \varphi^{-1}(w) = \psi \circ \varphi^{-1}(\lambda w)$ gives up

$$\lambda b_1 w + \lambda b_2 w^2 + \lambda b_3 w^3 + \cdots = \lambda b_1 w + \lambda^2 b_2 w^2 + \lambda^3 b_3 w^3 + \ldots.$$

Since λ is neither 0 nor a root of unity, this implies that

$$b_2 = b_3 = b_4 = \cdots = 0,$$

so that $\psi \circ \varphi^{-1}(w) = b_1 w$, equivalently, $\psi(z) = b_1\varphi(z)$. □

Assume now that f is not only locally defined near the origin, but is the local expression of a holomorphic map $f : S \to S$ at a fixed point z_0.

Definition 5.1.2. *The* total basin of attration of z_0 $\mathcal{A} = (z_0)$ *is the set of all points* $p \in S$ *such that* $\lim_{n\to\infty} f^n(p)$ *exists and is equal to* z_0.
The immediate basin of z_0 *is the connected component* $\mathcal{A}_0$ *of* $\mathcal{A}$ *containing* z_0.

We leave the proof of the following consequence of Theorem 5.1.1 as an exercise.

Corollary 5.1.3 (Global linearization). *Let* $\mathcal{A}$ *be the basin of attraction of an attracting periodic point* z_0. *There is a holomorphic map* $\varphi : \mathcal{A} \to \mathbb{C}$, *such that* $\varphi \circ f(z) = \lambda\varphi(z)$, *for all* $z \in \mathcal{A}$ *and* φ *maps a neighbourhood of* z_0 *to a neighbourhood of 0. Furthermore,* φ *is unique up to multiplication by a constant.*

We now further specialize our study to the case where $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a rational map. Let $\widehat{z}$ be an attracting fixed point.

There exists a holomorphic map $\psi : (\mathcal{A}, \widehat{z}) \to (\mathbb{C}, 0)$ that conjugates f with multiplication by λ. Since ψ is a local conformal isomorphism at $\widehat{z}$ there exists $\varepsilon > 0$ and a mapping $\psi_\varepsilon : \mathbb{D} \to \mathcal{A}_0$ that is a local inverse to ψ.

Lemma 5.1.4 (Finding a critical point). *This local inverse* $\psi_\varepsilon : \mathbb{D}_\varepsilon \to \mathcal{A}_0$ *extends to some maximal open disk* $\mathbb{D}_r$. *Furthermore,* ψ_r *extends homeomorphically to* ∂D_r *and the image* $\psi_r(\partial\mathbb{D}_r)$ *contains a critical point of* f.

Proof. Step 1. There exists some largest $r > 0$, *so that* φ_ε *extends analytically to* $\mathbb{D}_r$. If it was possible to extend ψ_ε infinitely far in every direction by analytic continuation, we would obtain an entire function ψ who range omits $J(f)$, but this is impossible by Picard's Little Theorem (which says that the range of such a function can omit at most one point).

Step 2. φ *is well-defined and holomorphic* $\psi(\mathbb{D}_r) \subset \mathcal{A}_0$ *and* ψ *extends homeomorphically to* $\overline{\mathbb{D}}_r$. By Step 1, there exists a largest r such that ψ_ε extends to a holomorphic mapping $\psi : \mathbb{D}_r \to \mathcal{A}_0$, so we need only show that ψ extends homeomorphically to $\overline{\mathbb{D}}_r$. Take any $w \in \overline{\mathbb{D}}_r$, then $\lambda w \in \overline{\mathbb{D}}_{\lambda r} \Subset \mathbb{D}_R$. Moveover f is locally invertible, so that we can extend ψ to $\overline{\mathbb{D}}_R$ by the formula, $\psi(w) = f^{-1}(\lambda w)$.

Step 3. ∂U *contains a critical point of* f. First, consider any point $w_0 \in \partial\mathbb{D}_r$. If $\psi(w_0)$ is not a critical point of f, we can extend ψ to a neighbourhood of w: Since $\psi(w_0)$ is not a critical point of f, there exists a local inverse g of f near $\psi(w_0)$, now we can extend ψ to a neighbourhood of w_0 by the formula

$$\psi(w) = g(\psi(\lambda w)).$$

If no point in $\partial\mathbb{D}_r$ is a critical point of of f then since $\partial\mathbb{D}_r$ is compact, we can extend ψ to some larger disk $\mathbb{D}_{r'}$, but this contradicts Step 1. So $\psi(\partial\mathbb{D}_r)$ contains a critical point of f. □

Theorem 5.1.5. *If* f *is a rational map of degree at least 2, then the immediate basin of attraction of any attracting cycle contains a critical point.*

Proof. Super attracting cycles contain critical points so their immediate basins of attraction contain critical points to. So suppose that $z_0 \mapsto z_1 \mapsto \ldots \mapsto z_{p-1} \mapsto z_p = z_0$ is a geometrically attracting periodic orbit with period p. Then $f^p(z_0) = z_0$, so that z_0 is an attracting fixed point of f^p. By Lemma 5.1.4 there exists a critical point $\widehat{c}$ of f^p in the immediate basin of attraction of z_0, but now $Df^p(\widehat{c}) = f'(\widehat{c})f'(f(\widehat{c}))f'(f^2(\widehat{c}))\ldots f'(f^{p-1}(\widehat{c})) = 0$. Hence some point in the orbit $\widehat{c} \mapsto f(\widehat{c}) \mapsto \ldots \mapsto f^{p-1}(\widehat{c})$ must be a critical point of f. Thus there is a critical point in the immediate basin of attraction of the cycle $z_0 \mapsto z_1 \mapsto \ldots \mapsto z_p$. □

5.2 Superattracting points

Theorem 5.2.1 (Bottcher). *Suppose that $f(z) = a_n z^n + a_{n+1}z^{n+1} + \ldots$. There exists a local holomorphic change of coordinates $w = \varphi(z)$, with $\varphi(0) = 0$, which conjugates f to the n-th power map $w \mapsto w^n$ in some neighbourhood of 0. Furthermore φ is unique up to multiplication by some $(n-1)$-st root of unity.*

Proof. Sketch of the proof of existence: show that the sequence of functions

$$\varphi_k(z) = (f^k(z))^{1/n^k} = z(1 + n^{k-1}b_1 z + \ldots)^{1/n^k} = z(1 + \frac{b_1}{n}z + \ldots)$$

converges uniformly. Notice that the expression on the right gives an explicit choice of the n^k-th root.

It is easy to argue as in the proof of Koenigs' Linearization formula that a uniform limit of the φ_k satisfies the conjugacy relation.

First observe that

$$\frac{z_{k+1}^{1/n^{k+1}}}{z_k^{1/n^k}} = \frac{z_k^n(1 + O(z_k))^{1/n^{k+1}}}{z_k^{1/n^k}}$$
$$= (1 + O(z_k))^{1/n^{k+1}} = 1 + O(z_k/n^{k+1}.)$$

As $k \to \infty$ that converges to 1 superexponentially quickly.

Now,

$$\varphi(z) = \lim_{k\to\infty} z_k^{1/n^k} = z \prod_{k=0}^{\infty} \frac{z_{k+1}^{1/n^{k+1}}}{z_k^{1/n^k}}$$
$$= z \prod_{k=0}^{\infty} (1 + O(\frac{z_k}{n^{k+1}})),$$

which is a convergent infinite product.

The proof of uniqueness is left as an exercise. □

Notice that every polynomial map has a superattracting fixed point at ∞. Thus the above theorem applies in this situation. We will make use of this theorem more in detail in the sequel.

5.3 Parabolic points

In this section we study the dynamics near a *parabolic* fixed point, i.e., a fixed point where the multiplier is a root of unity. The following elementary results shows that in this case, in general, we have no hope to conjugate the map to its linear part.

Proposition 5.3.1. *Let f be a holomorphic map fixing the origin and such that there exists q such that $(f'(0))^q = 1$. Then f is linearizable at the origin if and only if f^q is equal to the identity.*

Proof. Suppose that there exists a map φ defined in a neighbourhood of the origin and such that $\varphi \circ f^q \circ \varphi^{-1} = z$. This immediately implies that $f^q(z) = \varphi^{-1} \circ \varphi(z) = z$, and one implication follows.

Conversely, assume $f^q(z) = z$. We need to build a conjugacy for f. Let $\lambda := f'(0)$ as usual. We consider the map

$$\psi(z) := \frac{1}{q} \sum_{j=0}^{q-1} \lambda^{-j} f^j.$$

It is immediate to check that $\psi(z)$ is defined and invertible in a neighbourhood of the origin and that $\psi \circ f \circ \psi^{-1}(z) = \lambda z$. This completes the proof. □

By considering an iterate of the system, we can assume that the multiplier is equal to 1. We thus have to study the local dynamics of a map of the form

$$f(z) = z + a_{n+1} z^{n+1} + \cdots = z(1 + a_{n+1} z^n + \dots), a_{n+1} \neq 0.$$

Notice, in particular, that this time the system is not invertible near the fixed point. This is source of difficulty with respect to the other cases.

There are n complex numbers v such that

$$n a_{n+1} v^n = 1,$$

and n complex numbers v such that

$$n a_{n+1} v^n = -1.$$

We label these $2n$ vectors by $0 \leq j < 2n$ so that

$$v_j = e^{i\pi \frac{j}{n}} v_0, \quad \text{and} \quad n a_{n+1} v_j = (-1)^j.$$

The even indexed numbers are called the *repulsion vectors* for f at 0, and the odd indexed numbers are called the *attraction vectors* for f at 0.

Proposition 5.3.2. *Let f be a holomorphic map defined on a neighbourhood of 0 with*

$$f(z) = z + a_{n+1} z^{n+1} + \dots, \ a_{n+1} \neq 0,$$

with attraction and repulsion vectors v_j defined as above. Then if an orbit $z_n = f^n(z_0)$ converges to 0, then either

- *the sequence is eventually identically equal to 0 or*
- $\lim_{k\to\infty} k^{1/n} z_k$ *exists and is equal to one of the attracting vectors* v_j, j *odd.*

Similarly, if an inverse orbit

$$z_0 \hookleftarrow z_{-1} \hookleftarrow z_{-2} \hookleftarrow \dots$$

of f^{-1} *tends to 0, then either*

- *the sequence is eventually identically 0 or*
- $lim_{k\to\infty} k^{1/n} z_k$ *exists and is equal to one of the repulsion vectors* v_j, j *even.*

Finally, each of the attraction and repulsion vectors occurs as a limit.

Associated to each direction v_j, there is a "petal" P_j, which is an open neighbourhood of points which are attracted to the origin tangentially to the attraction vector (or repelled tangentially to the repulsion vector).

Theorem 5.3.3 (Parabolic Flower Theorem). *If* $\widehat{z}$ *is a fixed point of multiplicity* $n+1 \geq 2$, *there exist simply connected petals* P_j, $j = 1, \dots, 2n$, *attractign or repelling depends on* j *odd or even. The petals ca be chosen so that their union with* $\{\,\widehat{z}\,\}$ *is an open neighbourhood of* $\widehat{z}$. *When* $n > 1$, *each* P_j *intersects each of its immediate neighboroods in a simply connected region but is disjoint from all remaining petals.*

When $\lambda = 1$, each (attracting) petal is invariant for the dynamics $F : P \to P$ (the repelling petals are invariant for the local inverse). Inside of each petal, it is possible to linearize the dynamics.

Proposition 5.3.4 (Parabolic Linearization). *Let* $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ *be a rational map with degree at least 2. Assume that* $f(\widehat{p}) = \widehat{p}$ *and* $f'(\widehat{p}) = 1$. *Suppose that* P *is a petal for* $\widehat{p}$. *Then there is a mapping* $\alpha : P \to \mathbb{C}$ *such that*

$$\alpha(f(z)) = 1 + \alpha(z), z \in P \cap f^{-1}(P).$$

This proposition tells us that the dynamics inside a petal is conjugate with dynamics of translation by 1.

Sketch of proof. Assume that $n = 1$ and $\widehat{p} = 0$, so that

$$f(z) = z + az^2 + \dots, \quad a \neq 0.$$

First conjugate f by $z \mapsto az$, so that we can assume that

$$f(z) = z + z^2 + \dots,$$

now move 0 to infinity by conjugating f with $z \mapsto -1/z$, to obtain

$$g(z) = z + 1 + b/z + \dots.$$

When z is very big, this is evidently like translation by 1. To get the precise conjugation, one sets

$$\varphi_n(z) = g^n(z) - n - b \log n,$$

and proves that φ_n converges uniformly to a function φ in a neighbourhood of ∞, and that for the limiting map φ, $\varphi \circ f(z) = \varphi(z) + 1$. □

- When $f(z) = z + a_{n+1}z^{n+1} + \dots, a_{n+1} \neq 0$, f is conjugate to

$$z \mapsto z + 1 + \frac{b_1}{z^{1/p}} + \frac{b_2}{z^{2/p}} + \dots .$$

- If $f(z) = \lambda z + a_{n+1}z^{n+1} + \dots$, where λ is a root of unity, then for some p, $f^p(z) = z + \dots$ near zero, and the petals (fixed by f^p) are permuted in disjoint cycles of order p by f.
- Just as for attracting periodic points, the conjugacy inside the petal can be extended to the entire basin of attraction of a parabolic point.
- The basin of attraction is contained in the Fatou set of f.
- The boundary of the basin of attraction, which contains the parabolic point, is contained in $J(f)$.

Arguing as in the case of immediate basins of attraction for attracting cycles one can show:

Theorem 5.3.5. *Every parabolic basin of attraction contains a critical point of f.*

This implies the following bound on the number of attracting and parabolic cycles.

Theorem 5.3.6. *For a rational map of degree at least 2, the number of attracting cycles plus the number of parabolic cycles is at most $2d - 2$ (the number of critical points of the rational map).*

Corollary 5.3.7. *For each attracting and repelling basin, the quotient manifold P/f is conformally isomorphic to the cylinder $\mathbb{C}/\mathbb{Z}$.*

5.4 Indifferent points

Only sketched, not in the exam.