4

Generalities about dynamics on Riemann surfaces

In this chapter we start studying the dynamics of a self-map of Riemann surface. We will decompose the Riemann surface into two subsets, one stable for the dynamics and the other one chaotic. We then study periodic orbits.

4.1 Definitions and basic properties

We consider here a holomorphis self map f of a Riemann surface S. We denote by f^n the n-fold iterate of f.

Definition 4.1.1. *The* Fatou set $\mathcal{J}$ *is the maximal open set where the sequence* $\{ f^n \}$ *is locally normal. The* Julia set $\mathcal{J}$ *is the complement of the Fatou set.*

It is immediate from the definition that the Fatou set is open (and so the Julia set is closed).

Example 4.1.2. *Let $f(z) = z^2$ be a self map of $\mathbb{C}$. The iterates of f are of the form $f^n(z) = z^{2^n}$. It is easy to see that this family is normal on the interior of the unit disc (with every subsequence converging to the constant map 0). On the other hand, the complement of the closed disc is in the Fatou set, too. Indeed, here it is immediate to verify that the family of iterates diverges on compact subsets.*

FInally, we consider any point on the unit circle. For any neighborhood of this point, any limit map of the sequence must have a jump discontinuity on $\partial\mathbb{D}$. The family is thus not normal there. So, we have $\mathcal{J} = \partial\mathbb{D}$ and $\mathcal{F} = \mathbb{C} \setminus \partial\mathbb{D}$.

Notice that, if we consider the same map on $\widehat{\mathbb{C}}$, the decomposition is the same (we just have that the sequence is not divergent on compact subsets on the complement of the closed unit disc - since now the Riemann surface is compact - but has the constant map ∞ as limit).

Lemma 4.1.3. *The Fatou and Julia set are completely invariant:*

1. $\mathcal{F} = f(\mathcal{F}) = f^{-1}(\mathcal{F})$;
2. $\mathcal{J} = f(\mathcal{J}) = f^{-1}(\mathcal{J})$.

Proof. Since the Julia set is the complement of the Fatou set, (1) holds if and only if (2) holds, so it is enough to prove (1). Suppose that $z \in \mathcal{F}(f)$. Then there exists a neighbourhood V of z such that $\{f^n\}$ forms a normal family on V. The set $f^{-1}(V)$ is a neighbourhood of the set $f^{-1}(z)$. Consider the family of iterates $\{f^n|_{f^{-1}(V)}\}$. Take any subsequence $\{f^{n_j}_{f^{-1}(V)}\}$, we can assume that $n_j > 1$ for all j. Since $\{f^n_V\}$ is a normal family, $\{f^{n_j-1}|_V\}$ possesses a subsequence, $\{f_V^{n_{j_k}-1}\}$ that converges on compact subsets of V to a meromorphic function g, but now $\{f^{n_{j_k}}\}$ converges uniformly on compact subsets of $f^{-1}(V)$ to $f \circ g$. Hence $\{f^n_{f^{-1}(V)}\}$ is a normal family. So all preimages of z belong to $\mathcal{F}(f)$, and thus $f^{-1}\mathcal{F}(f) \subseteq \mathcal{F}(f)$.

For the reverse inclusion (which also gives $f(\mathcal{F}(f)) \subseteq \mathcal{F}(f)$), we will use that f is open. Let $z \in \mathcal{F}(f)$. If the sequence $\{ f^n \}$ diverges near z, the same is true for the sequence $\{f^{n-1}\}$ near $f(z)$, and the assertion follows. Let us the fix a subsequence f^{n_j}. We look for a converging subsequence near $f(z)$. Since $z \in \mathcal{F}(f)$, we can assume that the subsequence of $\{f^{n_j+1}\}$ has a converging subsequence $\{ f^{n_{j_k}+1} \}$near z. More precisely, for every ε there exists a δ such that $d(f^{n_{j_k}+1}(z), f^{n_{j_k}+1}(z')) < \varepsilon$ if $d(z, z') < \delta$. Take $N := B(z, \delta)$. Since f is open, $f(N)$ is an open neighbourhood of $f(z)$. Moreover, f is surjective from N to $f(N)$. So, given any $w' \in f(N)$, there exists $z' \in N$ such that $f(z') = w'$. So, for every $w' \in f(N)$, we have $d(f^{n_{j_k}}(f(z), f^{n_{j_k}}w')) = d(f^{n_{j_k}+1}(z), f^{n_{j_k}+1}(z')) < \varepsilon$. So the sequence is equicontinuous and the assertion follows.

So, $\mathcal{F}(f) \subseteq f^{-1}(\mathcal{F}(f))$, which implies $\mathcal{F}(f) = f^{-1}\mathcal{F}(f)$ by the first part. The equality $\mathcal{F}(f) = f(\mathcal{F}(f))$ follows by applying f to the previous one (since f is surjective). □

Lemma 4.1.4. *For every $k \geq 1$, we have $\mathcal{J}(f) = \mathcal{J}(f^k)$ (and similarly for $\mathcal{F}$).*

Proof. This statement is equivalent to proving that $F(f^k) = F(f)$, and that is what we will prove. Suppose that $x \in F(f)$, then there exists a neighbourhood V of x such that $\{f^n\}$ forms a normal family on V. It is immediate that $\{f^{kn}\}$ forms a normal family on V too.

So suppose that $x \in F(f^k)$. Then $\{f^{kn}\}_{n=0}^{\infty}$ is a normal family. We can express

$$\{f^n\}_{n=k}^{\infty} = \{f^{kn}\}_{n=1}^{\infty} \cup \{f^{kn+1}\}_{n=1}^{\infty} \cup \{f^{kn+2}\}_{n=1}^{\infty} \cup \cdots \cup \{f^{kn+k-1}\}_{n=1}^{\infty}.$$

Since $\{f^{kn}\}_{n=1}^{\infty}$ forms a normal family on V, each $\{f^{kn+i}|V\}_{n=1}^{\infty} = \{f^i \circ f^{kn}|V\}_{n=1}^{\infty}$ $i = 1, 2, \ldots, k-1$, is a normal family too. Hence $\{f^n|_V\}$ is a normal family. □

4.2 Periodic points

A point z_0 is called *periodic* if there exists a $p \in \mathbb{N}$ such that $f^p(z_0) = z_0$. For convenience set $z_i = f^i(z_0)$, then

$$f : z_0 \to z_1 \to z_2 \to \cdots \to z_{p-1} \to z_p = z_0.$$

If $z_0, z_1, \ldots, z_{p-1}$ are all distinct, so that p is minimal so that $f^p(z_0) = z_0$, then p is called the *period* of z_0.

A quantity associated to a periodic point is its *multiplier*,

$$\lambda = Df^p(z_0) = f'(z_0)f'(z_1)\ldots f'(z_{p-1}).$$

If $|\lambda| < 1$, then z_0 is called *attracting*. It will sometimes be convenient to distinguish two different types of attraction: z_0 is called *geometrically attracting* if $0 < |\lambda| < 1$ and *super attracting* if $\lambda = 0$. If $|\lambda| > 1$, then z_0 is called *repelling*; and if $|\lambda| = 1$, then z_0 is called neutral.

Exercise 4.2.1. *Prove that the multiplier does not depend on the chart with respect to which it is computed.*

Let's immediately justify the names attracting and repelling. We say that a periodic point z_0 of period p is *topologically attracting* if there exists a neighbourhood U of z_0, such that the family of iterates $\{f^{np}|U\}_{n=1}^{\infty}$ converges uniformly to the constant function $g(z) = z_0$ on U.

Lemma 4.2.2. *Let z_0 be a periodic point of period p. Let $\lambda = Df^p(z_0)$ be the multiplier of the periodic cycle. Then z_0 is topologically attracting if and only if $|\lambda| < 1$.*

Before proving this, let's state an immediate corollary:

Corollary 4.2.3. *If z_0 is periodic point with multiplier less than 1, then z_0 is in the Fatou set of f.*

Proof. The previous lemma implies that if z_0 is a periodic point with multiplier less than one, then z_0 is topologically attracting, and the definition of topologically attracting implies that $\{f^n\}$ is a normal family in a neighbourhood of z_0. □

Proof of Lemma 4.2.2. Let $g = f^p$. Then z_0 is a fixed point of g. For convenience, we will assume that $z_0 = 0$. First suppose that $|\lambda| < 1$. We will show that 0 is a topologically attracting fixed point of g. Then the Taylor series of g at 0 has the form

$$g(z) = \lambda z + O(z^2),$$

so there exist constants $r_0 > 0$ and $C > 0$ such that

$$|g(z) - \lambda z| \leq C|z^2|, \quad \text{for } |z| < r_0.$$

Choose c so that $|\lambda| < c < 1$, and choose r, $0 < r < r_0$ so that $|\lambda| + Cr < c$. Then whenever $|z| < r$, we have that

$$|g(z)| < |\lambda z| + C|z^2| < c|z|.$$

Hence $|g^n(z)| \leq c^n|z| < c^n r$. So that as $n \to \infty$ this converges uniformly to 0.

On the other hand, suppose that 0 is a topologically attracting fixed point of g. Then there exist $\varepsilon_0 > 0$ and an iterate g^n such that $g^n(\mathbb{D}_{\varepsilon_0})$ is a proper subset of $\mathbb{D}_{\varepsilon}$. The Schwarz Lemma implies that $Dg^n(0) = \lambda^n$ satisfies $|\lambda|^n < 1$.e wo So $|\lambda| < 1$. □

We say that a periodic point z_0 of period p is *topologically repelling* if there is a neighbourhood U of z_0 so that for every $z \in U, z \neq z_0$, there exists some $n \in \mathbb{N}$ such that $f^n(z) \notin U$. We call such a neighbourhood U a *isolating neighbourhood of* z_0.

Exercise 4.2.4. *If $|\lambda| > 1$, then z_0 is topologically repelling.*

Lemma 4.2.5 (Topologically repelling periodic points). *Let z_0 be a periodic point of period p, and let $\lambda = Df^p(z_0)$ be its multiplier. Then z_0 is topologically repelling if and only if $|\lambda| > 1$.*

Proof. We will prove that if z_0 is topologically repelling, then $|\lambda| > 1$. The converse is left as an exercise.

If z_0 is topologically repelling, then it cannot be topologically attracting. So by Lemma 4.2.2, $|\lambda| \geq 1$. let $g = f^p$. We can find a compact isolating neighbourhood N of z_0 such that N is mapped homeomorphically onto a compact neighbourhood $g(N)$ of z_0. Let $N_k = N \cap g^{-1}(N) \cap \cdots \cap g^{-k}(N)$. Then N_k is a compact neighbourhood of z_0 that consists of points for which the first k forward images all belong to N. Then

$$N \supset N_1 \supset N_2 \supset \dots,$$

and $\cap N_k = \{z_0\}$, since N is an isolating neighbourhood. By compactness, it follows that $\mathrm{diam}(N_k) \to 0$ as $k \to \infty$, but it follows from the construction that

$$f(N_k) = N_{k-1} \cap f(N),$$

moreover, since $\mathrm{diam}(N_k) \to 0$, for k large, $N_{k-1} \subset f(N)$. Thus for large k, $f(N_k) = N_{k-1}$. Let U_k be the connected component of the interior of N_k that contains z_0. Then there exists $n > 0$ such that f^{-n} maps U_k onto U_{k+n}, biholomorphically, and U_{k+n} is strictly smaller than U_k. Hence, by the Schwarz lemma $|\lambda^{-n}| < 1$, which gives us that $|\lambda| > 1$. □

Lemma 4.2.6. *Suppose that z_0 is a repelling periodic point for f. Then $z_0 \in J(f)$.*

Proof. Suppose that z_0 is a fixed point of f. Since z_0 is repelling, the multiplier $\lambda = f'(z_0)$ satisfies, $|\lambda| > 1$. But then $Df^k(z_0) = \lambda^k$ grows exponentially. Hence the family $\{f^n\}$ cannot converge uniformly to a holomorphic function in any neighbourhood of z_0. If z_0 is a periodic point of period p, then it is a fixed point for f^p, and the statement follows from the Iteration Lemma. □

We say that a neutral periodic point z_0 is *parabolic* if its multiplier λ is a root of unity, but no iterate of f is the identity map.

Lemma 4.2.7. *Parabolic periodic points are contained in the Julia set.*

Proof. For convenience, we consider the case that $z_0 = 0$ is a fixed point of f. Then for some $m > 0$, $f^m(z) = z + a_q w^q + a_{q+1} w^{q+1} + \dots$, where $q \geq 2$ and $a_q \neq 0$. Then f^{mk} has Taylor series

$$z \mapsto z + k a_q w^q + \dots.$$

Thus the q-derivative of f is $q! k a_q$, which diverges to ∞ as $k \to \infty$. Again, it follows from the Iteration Lemma that $z_0 \in J(f)$. □

Understanding the local dynamics near neutral periodic points is more complicated than in the attracting and repelling cases. The remaining periodic points, those with multiplier one that are not parabolic are called *irrationally indifferent*. Whether an irrationally indifferent periodic point is contained in the Fatou set or the Julia set depends on the arithmetic properties of the multiplier. We will return to this later in the course.