3

Riemann surfaces

In this chapter we define and give the first properties of Riemann surfaces. These are the holomorphic counterpart of the (real) differential manifolds. We will see how the Fuchsian groups studied in the previous chapter are related to a classification of these surfaces.

The main sources for this chapter will be Milnor's "Dynamics in one complex variables" and Beardon's "The geometry of discrete groups".

3.1 Definitions

Definition 3.1.1. *A* holomorphic function *on a subset* V *of* $\mathbb{C}$ *is a map* $f : V \to \mathbb{C}$ *such that the limit*

$$f'(z) := \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

exists for every $z \in V$.

Definition 3.1.2. *A* Riemann surface *is a connected complex manifold of dimension 1.*

This means that a Riemann Surface is a connected topological space X which is Hausdorff (i.e., for every pair of points $x \neq y \in X$ there exist open neighbourhoods $U_x \ni x$ and $U_y \ni y$ such that $U_x \cap U_y \neq \emptyset$) endowed with a family of *charts*

$$\{ (U_j, \varphi_j) \colon j \in J \}$$

(called an *atlas* for X) such that

1. $\{ U_j \colon j \in J \}$ is an open cover of X;
2. each φ_j is a homeomorphism of U_j onto an open subset of $\mathbb{C}$;

3. if $U = U_i \cap U_j \neq \emptyset$, then
$$\varphi \circ \varphi_j^{-1} \colon \varphi_j(U) \to \varphi_i(U)$$
is a holomorphic map between the (open) subsets $\varphi_j(U)$ and $\varphi_i(U)$ of $\mathbb{C}$.

Notice that, in the second condition, we allow the images of the chart maps to be open subsets of $\mathbb{C}$, and not only the plane $\mathbb{C}$. This is because $\mathbb{C}$ and a proper subset of $\mathbb{C}$ are not *biholomorphic* (see below, think of the unit disc, by Liouville theorem).

Example 3.1.3. *Every open subset Ω of $\mathbb{C}$ is a Riemann surface, with an atlas given by a single chart. The chart map can be taken to be the identity from Ω to itself.*

Example 3.1.4. *The Riemann sphere is one the simplest Riemann surfaces that we can consider which is not a subset of $\mathbb{C}$. As a topological space, we consider a sphere, which can be seen as the plane $\mathbb{C}$ together with a point at infinity ∞. We can consider the two charts*
$$\begin{aligned} &U_1 = \mathbb{C} && \varphi_1(z) = z \\ &U_2 = \mathbb{C} \setminus \{ 0 \} \cup \{ \infty \} && \varphi_2(z) = 1/z. \end{aligned}$$
We only have to check that the composition $\varphi_2 \circ \varphi^{-1}$ is holomorphic on $\varphi_1(U_1 \cap U_2) \cong \mathbb{C} \setminus \{ 0 \}$. This map is given by $z \mapsto 1/z$, and the assertion follows.

Remark 3.1.5. *As in the case of real manifolds, notice that the requirement on the topological space to be Hausdorff is not redundant (it is not implied by the existence of an atlas). Consider for instance the topological space $\mathbb{C} \cup \{ \star \}$, endowed with the following topology: a subset U is open if*

1. *U is an open subset of $\mathbb{C}$; or*
2. *$\star \in U$, $0 \notin U$ and $(U \setminus \{ \star \}) \cup \{ 0 \}$ is an open subset of $\mathbb{C}$.*

(Notice that this space can be seen as the quotient of two copies of $\mathbb{C}$, with all corresponding non-zero points identified.)

It is easy to check that an atlas is given by the two open sets $\mathbb{C}$ and $(\mathbb{C} \setminus \{ \star \}) \cup \{ 0 \}$, with the identity for the first chart and the identity on $\mathbb{C}^*$ (and $\star \mapsto 0$) for the second.

However, this space is not Hausdorff, since the two points 0 and $\star$ cannot be separated by any pair of open subsets.

Definition 3.1.6. *A maps $f : X \to Y$ between two Riemann surfaces is* holomorphic *if it is holomorphic when read in the charts of the atlas*

This means the following. Let X and Y be Riemann surfaces, with respective atlases $\{ (U_j, \varphi_j) \colon j \in J \}$ and $\{ (V_k, \psi_k) \colon k \in K \}$. A continuous map $f \colon X \to Y$ is holomorphic if each map
$$\psi_k \circ f \circ \varphi_j^{-1} \colon \varphi_j(U_j \cap f^{-1}(V_k)) \to \mathbb{C}$$
is holomorphic.

Remark 3.1.7. *The continuity assumption is needed to ensure that the domain* $\varphi_j(U_j \cap f^{-1}(V_k))$ *is open.*

Remark 3.1.8. *By the last condition in the definition of a Riemann surface, it is enough to check that a function is holomorphic only with respect to sub-atlases which still provide open covers.*

Definition 3.1.9. *Two Riemann surfaces are* biholomorphic *if there exists a holomorphic map between them which is bijective.*

Exercise 3.1.10. *Check that the inverse of a bijective holomorphic map is holomorphic.*

Exercise 3.1.11. *Check the relation of biholomorphisms between Riemann surfaces is an equivalence relation.*

Exercise 3.1.12. *Let* $f\colon X \to Y$ *be a non-constant holomorphic map between two Riemann surfaces.*

1. *Prove that* f *is open.*
2. *Deduce that if* X *is compact, then* f *is surjective and thus also* S *is compact.*

The following lemma shows that, when taking a quotient by a group which acts properly discontinuosly, the quotient is still Hausdorff.

Lemma 3.1.13. *Let* D *be a subdomain (open connected subset) of the Riemann sphere and let* G *be a subgroup of the group of holomorphic automorphisms of* D*. If* G *acts properly discontinuously, then the quotient* D/G *is Hausdorff.*

We leave the following verifications as exercise.

Exercise 3.1.14. *Check that* D/G *is a topological space, with the topology induced by the quotient. The quotient map is continuous and open with respect to this topology. As* D *is connected,* D/G *is connected.*

Proof of Lemma 3.1.13. Take two (distinct) points in D/G (of the form $\pi(z_1)$ and $\pi(z_2)$). Consider the open neighbourhoods of z_1 and z_2 in D given by $A_n := \{\, z\colon d(z, z_1) < 1/n \,\}$ and $B_n := \{\, z\colon d(z, z_2) < 1/n \,\}$ (they exist for $n \geq n_0$ sufficiently large). Since the quotient is open, it is enough to prove that there exists n such that $\pi(A_n) \cap \pi(B_n) = \emptyset$. Assume this is false. This means that for every n there exists $w_n \in A_n$ and $g_n \in G$ such that $g_n(w_n) \in B_n$. We can assume that the g_n's are distinct. This implies that the compact set $K := \overline{A_{n_0} \cup B_{n_0}}$ satisfies $g_n(K) \cap K \neq \emptyset$ for infinitely many g_n's, which contradicts the fact that the action is properly discontinuous. Thus, D/G is Hausdorff. □

The following result provides a link between Moebius transformations and Riemann surfaces.

Theorem 3.1.15. *Let* D *be an open connected subset of the Riemann sphere and let* G *be a subgroup of the group of holomorphic automorphisms of* D*. If* G *acts properly discontinuously, then the quotient* D/G *admits a Riemann surface structure.*

In view of Lemma 3.1.13, one only needs to build an atlas for the quotient space. Full detail of the proof of this Theorem can be found on Beardon's book, Theorem 6.2.1. We prefer to study in detail a couple of examples instead of giving the general construction.

Example 3.1.16. *We consider the complex plane $\mathbb{C}$ and the group Γ generated by the translation $z \mapsto z+1$. Topologically, the quotient is easily seen to be homeomorphic to a cylinder. We want to build an atlas for it. We can see $\mathbb{C}/\Gamma$ as the strip $\{ z \in \mathbb{C} \colon 0 \le \operatorname{Re} z \le 1 \}$, with the points $0 + iy$ ($y \in \mathbb{R}$) identified with $1 + iy$. We consider the following two charts: the projection U_0 by the quotient of the strip $\{ z \in \mathbb{C} \colon 0 < \operatorname{Re} z < 1 \}$ and the quotient U_1 of the strip $\{ z \in \mathbb{C} \colon -1/2 < \operatorname{Re} z < 1/2 \}$. Calling π the quotient map, the chart maps φ_0, φ_1 are given by $id \circ \pi^{-1}$ (notice that π is invertible on the chosen chart). It is immediate to check that the intersections of the two chart is the union of the two strips $A := \{ z \in \mathbb{C} \colon 0 < \operatorname{Re} z < 1/2 \}$ and $B := \{ z \in \mathbb{C} \colon 1/2 < \operatorname{Re} z < 1 \}$. The image of this set by φ_1 can be seen as the union of A and B. The image under φ_1 is given by the union of A and the strip $C := \{ z \in \mathbb{C} \colon -1/2 < \operatorname{Re} z < 0 \}$. The transition maps are the identity from A to A and a translation by 1 from C to B.*

Example 3.1.17. *We consider the complex plane $\mathbb{C}$ and the quotient by the action of the group generated by the two translations $z \mapsto z+1$ and $z \mapsto z+i$. The quotient is homeomorphic to a torus. We can consider the atlas given by the projections of the four open sets*

$$
\begin{aligned}
V_0 &:= \{ z \colon -1/3 < \operatorname{Re} z < 2/3, 1/3 < \operatorname{Im} z < 4/3 \} ; \\
V_0 &:= \{ z \colon 1/3 < \operatorname{Re} z < 4/3, 1/3 < \operatorname{Im} z < 4/3 \} ; \\
V_0 &:= \{ z \colon -1/3 < \operatorname{Re} z < 2/3, -1/3 < \operatorname{Im} z < 2/3 \} ; \\
V_0 &:= \{ z \colon 1/3 < \operatorname{Re} z < 4/3, -1/3 < \operatorname{Im} z < 2/3 \} .
\end{aligned}
$$

The chart maps are again given by the inverse of the projection map, which is well defined on every V_i. We leave as an exercise to check that the transition maps are again translations.

Notice that we do not require that the elements of the group do not have fixed points. This can lead to some more singular examples. We will see later how to put an extra condition on the action to avoid this happen.

Example 3.1.18. *Consider the unit disc and the action generated by the involution $z \mapsto -z$. Topologically, what we get with the quotient is again a disc. However, notice that it is not possible to find a neighbourhood of the image of 0 where the quotient map is invertible. It is possible to give the quotient space the structure of Riemann surface of the unit disc (even if we will not detail it here).*

However notice that, from a "metric point" of view, what we got can be better identified with a cone (think of a way to compute the "total angle" of the quotient space at the image of 0). The name for this space is orbifold. *We do not enter in details here, but just notice that these spaces can be seen as Riemann Surfaces, endowed with a certain number of* singular points*, where the total angle is $2\pi/n$ for some integer n. This spaces arise when taking a quotient by an action with a fixed point.*

3.2 Coverings and classification

Our goal is to obtain a classification of Riemann surfaces, which means - if possible - finding a representative for every class of biholomorphism. We will first consider the simplest surfaces, and then see how we can get all other possible surfaces from them. The following definition is central in our study.

Definition 3.2.1. *A Riemann surface X is* simply connected *if every continuous map from S^1 to X can be continuously deformed to a constant map.*

Example 3.2.2. *The plane $\mathbb{C}$ and the Riemann sphere are simply connected. The punctured plane and a complex torus are not simply connected.*

Exercise 3.2.3. *Prove that a simply connected Riemann surface cannot be biholomorphic (and even homeomorphic) to a non simply connected one.*

This means that we can split all the Riemann surfaces in these two classes (simply connected or not) and give a classification inside of each class to get a complete classification.

We start with the simply connected ones. The following result provides the answer, and plays a central role in all the theory of Riemann surfaces. We will not give a proof here.

Theorem 3.2.4 (Uniformization)**.** *Any simply connected Riemann surface is biholomorphic to one of the following:*

1. *the unit disc $\mathbb{D}$;*
2. *the plane $\mathbb{C}$;*
3. *the Riemann sphere $\widehat{\mathbb{C}}$.*

Recall that the half plane is biholomorphic to the unit disc. This gives another model for the case 1.

Remark 3.2.5. *These three types are indeed non biholomorphic one to the other. Indeed, we cannot have non-constant holomorphic function from the disc to $\mathbb{C}$ by Liouville Theorem. Moreover, by the Exercise 3.1.12 we cannot have non-constant maps from the Riemann sphere to either the disc or the plane.*

We now turn our attention to the non simply connected surfaces. We will see that any such surface can be produced by taking a suitable quotient of a simply connected one (in some sense, we are going to provide a converse of Theorem 3.1.15). The central definition here will be the one of *covering*.

Definition 3.2.6. *A map $p : M \to N$ between topological spaces is a* covering *if it is surjective and every point of N admits a neighbourhood U which is homeomorphic to every component of $p^{-1}(U)$.*

Example 3.2.7. *Let us consider the real line $\mathbb{R}$ and the unit circle $S^1 \subset \mathbb{C}$. The map $\pi : \mathbb{R} \to S^1$ given by $x \mapsto e^{2\pi i x}$ is a covering map.*

Example 3.2.8. *Let us consider the real line $\mathbb{R}$ and the unit circle $S^1 \subset \mathbb{C}$. The map $\pi : \mathbb{R} \to S^1$ given by $x \mapsto e^{2\pi i x}$ is a covering map.*

Example 3.2.9. *Let us consider the real plane $\mathbb{R}^2$ and the cylinder $S^1 \times \mathbb{R} \subset \mathbb{C} \times R$. The map $\pi : \mathbb{R}^2 \to S^1 \times R$ given by $(x, y) \mapsto (e^{2\pi i x}, y)$ is a covering map.*

We will be mainly concerned in the case where M and N are Riemann surfaces. In this case we may talk about *holomorphic cover* to require that the projection is a holomorphic map. The following is the complex version of the Example 3.2.9.

Example 3.2.10. *Let us consider the complex plane $\mathbb{C}$ and the punctured plane $\mathbb{C}^*$. The map $\pi : \mathbb{C} \to \mathbb{C}^*$ given by $z \mapsto e^{2\pi i z}$ is a covering map.*

The following theorem ensures that every topological space admits a simply connected covering. We state it only in the case we will need.

Theorem 3.2.11. *Every Riemann surface X admits a (unique, up to biholomorphism) covering $p : \widetilde{X} \to X$, where $\widetilde{X}$ is simply connected.*

Definition 3.2.12. *The covering above is called the* universal covering *of X.*

In the following, we will need the following result.

Theorem 3.2.13. *Let $f : M \to N$ be a holomorphic map between Riemann surfaces and let $\pi_M : \widetilde{M} \to M$ and $\pi_N : \widetilde{N} \to N$ be the universal coverings of M and N. There exists a holomorphic map $F : \widetilde{M} \to \widetilde{N}$ such that $\pi_N \circ F = \pi_M \circ f$.*

The map F is usually called the *lifting* of the map f.

The idea that we want to make precise now is the following: in the case of the covering π from $\mathbb{R}$ to S^1, we can consider the automorphisms p of $\mathbb{R}$ that satisfy $\pi \circ p = \pi$. This means that they permute the elements in the fibers of π. In this case, we can see that this group is isomorphic to $\mathbb{Z}$ and given by the translation by an integer multiple of $2\pi i$. We can see the space S^1 as the quotient of the space $\mathbb{R}$ by the action of this group of translation, which we can write as $S^1 = \mathbb{R}/(2\pi i\mathbb{Z})$. We will see that studying a cover is essentially the same as studying the group of self maps of the covering space permuting the elements in the fibers.

Definition 3.2.14. *Given a holomorphic covering $p : M \to N$ between Riemann surfaces, a* deck transformation *is a biholomorphic map $\varphi : M \to M$ such that $p \circ \varphi = p$.*

Exercise 3.2.15. *The deck transformations of a covering form a group.*

For what we will need in this course, we can define the fundamental group of M as the group of deck transformations of its universal cover.

Since we want to interpret coverings as quotients of actions, it is important that the orbit of a point by the deck transformations is all the fiber of the image of the point. We are thus led to the following definition.

Definition 3.2.16. *A covering $p : M \to N$ is* normal *if for every $x, y \in M$ there exists a unique deck transformation φ such that $\varphi(x) = y$.*

The uniqueness in the definition above is needed, for instance, to avoid the presence of deck transformations with fixed points, giving rise to the phenomenon of orbifold sketched in Example 3.1.18.

Remark 3.2.17. *Given a normal covering $p\colon M \to N$, we can see N as a quotient of M by the group of the deck transformations.*

The following result gives the relation between normal coverings and properly discontinuous actions.

Proposition 3.2.18. *A group Γ of biholomorphisms of a Riemann surface M gives rise to a normal covering if and only if*

1. *Γ acts properly discontinuously (see Definition 2.2.5);*
2. *Γ acts freely (i.e., every non-identity element of Γ does not have any fixed point in M).*

The following is then a consequence of Theorem 3.1.15 (we just avoid the possibility of the fixed points).

Corollary 3.2.19. *Let D be an open connected subset of the Riemann sphere and let G be a subgroup of the group of holomorphic automorphisms of D. If G acts freely and properly discontinuously, then the quotient D/G admits a Riemann surface structure which is induced by the one on D and such that the quotient is a holomorphic covering.*

The following is a combination of Theorems, 3.2.11 3.2.4 and Corollary 3.2.19 (admitting the fact that every universal covering is normal).

Theorem 3.2.20. *Every Riemann surface is biholomorphic to a quotient $\widetilde{S}/\Gamma$, where $\widetilde{S}$ is $\mathbb{D}, \mathbb{C}$ or $\widehat{\mathbb{C}}$ and Γ is a group of automorphisms of $\widetilde{S}$ which acts freely and properly discontinuously.*

In the following sections we will study Riemann surfaces accordingly to their universal cover.

Exercise 3.2.21. *Let M and N be two Riemann surfaces. Prove that, if they are biholomorphic, their universal cover is the same.*

3.3 Different types of Riemann surfaces

3.3.1 When the covering space is the sphere

Lemma 3.3.1. *The group of holomorphic automorphisms of the Riemann sphere is given by the maps of the form*

$$z \mapsto \frac{az+b}{cz+d}$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$.

Proof. It is easy to check that all complex Moebius transformations as in the statement are bihomorphisms of the Riemann sphere, and that they form a group. We only need to prove that there are no other biholomorphisms. Let g by any such biholomorphisms. Up to composing with suitable complex Moebius transformations we can assume that $g(0) = 0$ and $g(\infty) = \infty$. The quotient $g(z)/z$ is a bounded holomorphic function from $\mathbb{C}^*$ to itself (see Exercise 3.3.2). Changing coordinate by $z = e^w$, we see that the composition $w \mapsto g(e^w)/e^w$ is a bounded holomorphic function on all $\mathbb{C}$. By Liouville Theorem, this function is constant. This implies that $g(z) = cz$ for some $c \neq 0$, and the assertion follows. □

Exercise 3.3.2. *Justify the assertion in the proof that the function $g(z)/z$ from $\mathbb{C}^*$ to itself is bounded.*

In order to find the Riemann surfaces which are covered by the Riemann sphere, we need to find the subgroup of the group of Moebius Transformations *with complex coefficients* which acts freely and properly discontinuously. It is immediate to see that every element (different from the identity) has a fixed point. This means that the only possible subgroup giving rise to a Riemann surface by quotient is the trivial group. We thus have the following result.

Proposition 3.3.3. *The only Riemann surface covered by the Riemann sphere is the Riemann sphere itself.*

3.3.2 When the covering space is the plane

We want to study the surfaces that arise as quotients of $\mathbb{C}$ by a suitable subgroup of its holomorphic isometries. We will need the following property.

Exercise 3.3.4. *Every biholomorphism of $\mathbb{C}$ extends to a uniquely defined biholomorphism of the Riemann sphere.*

Lemma 3.3.5. *The group of holomorphic automorphisms of $\mathbb{C}$ is given by the maps of the form $z \mapsto az + b$, with $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$.*

Proof. By Exercise 3.3.4, the group of biholomorphisms of $\mathbb{C}$ is a subgroup of the biholomorphisms of the Riemann sphere. The assertion follows since biholomorphims of $\mathbb{C}$ are characterized by the fact of fixing the point at infinity. □

In order to study the Riemann surfaces covered by the plane we have to find the subgroups of the affine group which act freely and properly discontinuously.

First of all, notice that every affine map $z \mapsto az + b$ with $a \neq 1$ has a fixed point. We can thus only find translations of the form $z \mapsto z + b$, with $b \in \mathbb{C}$, in our subgroup. Since we want the subgroup to be discrete, we have three cases:

1. the subgroup is trivial, and the quotient surface is the plane $\mathbb{C}$;

2. if the subgroup has one generator, the quotient surface is isomorphic to the cylinder $\mathbb{C}/\mathbb{Z}$ (this surface was studied in Example 3.1.16).

   Notice that the cylinder is also isomorphic to the punctured plane $\mathbb{C}^*$ via the isomorphisms
$$z \mapsto \exp(2\pi i z).$$

3. the subgroup has two generators and is thus isomorphic to $\mathbb{Z} \times \mathbb{Z}$. Notice that the two elements must be linearly independent *over* $\mathbb{R}$. We get a complex torus as a quotient. Let $z \mapsto z + a$ and $z \mapsto z + b$ be the two generators. The group can be identified with the *lattice* $\{ \lambda a + \mu b \}$ (and *linearly independent* over $\mathbb{R}$ means that a and b are linearly independent over $\mathbb{R}$, i.e., that the lattice is not contained in a line). The quotient will be also denoted by $\mathbb{C}/\Lambda$, where Λ is the lattice.

Exercise 3.3.6. *The goal of this exercise is to study when two complex tori are biholomorphic.*

1. *Let Λ be a lattice of $\mathbb{C}$. Prove that there exists $\tau \in \mathbb{H}$ such that $\mathbb{C}/\Lambda$ is isomorphic to $\mathbb{C}/\Lambda_\tau$, where $\Lambda_\tau := (\mathbb{Z} + \tau\mathbb{Z})$.*

2. *Let $\tau \in \mathbb{H}$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ (see Exercise 2.1.12) and $\tau' = g \cdot \tau := \frac{a\tau+b}{c\tau+d}$. Prove that the tori $\mathbb{C}/\Lambda_\tau$ and $\mathbb{C}/\Lambda_{\tau'}$ are biholomorphic*

3. *Let $\tau, \tau' \in \mathbb{H}$ and let $f \colon \mathbb{C}/\Lambda_\tau \to \mathbb{C}/\Lambda_{\tau'}$ be a biholomorphism. Prove that there exists $g \in SL_2(\mathbb{Z})$ such that $\tau' = g \cdot \tau$.*

Solution. Let $\Lambda = \mathbb{Z}\lambda_1 + \mathbb{Z}\lambda_2$, with λ_1 and λ_2 linearly independent on $\mathbb{R}$. Let $\tau := \pm\lambda_2/\lambda_1$ (choose the sign in such a way that $\tau \in \mathbb{H}$). Then the multiplication by λ_1 gives a biholomorphisms between $\mathbb{C}/\Lambda_\tau$ and $\mathbb{C}/\Lambda$.

For the second part, we have $\mathbb{C}/\Lambda_{\tau'} = \mathbb{C}/((a\tau + b)\mathbb{Z} + (c\tau + d)\mathbb{Z})$. We have $(a\tau + b)\mathbb{Z} + (c\tau + d)\mathbb{Z} = (b + d)\mathbb{Z} + (a + c)\tau\mathbb{Z}$. Since $ad - bc = 1$, we have that a and c are relatively prime, and the same is true for b and d. This implies that $(b + d)\mathbb{Z} + (a + c)\tau\mathbb{Z} = \mathbb{Z} + \tau\mathbb{Z}$, and the assertion follows.

Finally, denote by $p : \mathbb{C} \to \mathbb{C}/\Lambda_\tau$ and $p' : \mathbb{C} \to \mathbb{C}/\Lambda_{\tau'}$ the quotient maps. By Theorem **??**, the map f can be lifted to a map $F : \mathbb{C} \to \mathbb{C}$ satisfying $p' \circ F = f \circ p$. Without loss of generality, we can assume that $F_0 = 0$. This lift F must be of the form $F(z) = \alpha z$, for some $\alpha \neq 0$ (otherwise the map f would have a critical point).

By lifting $g = f^{-1}$, we get a map $G : \mathbb{C} \to \mathbb{C}$ satisfying $g \circ p' = p \circ G$. We must have $F(\Lambda_\tau) \subseteq \Lambda_{tau'}$ and similarly $G(\Lambda_{\tau'}) \subseteq \Lambda_\tau$, which implies $\Lambda_{\tau'} = \alpha\Lambda_\tau$. So, we can write $\tau' = \alpha(a\tau + b)$ and $1 = \alpha(c\tau + d)$. This implies $\tau' = \frac{a\tau+b}{c\tau+d}$. Since the application is invertible, we have $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where the matrix must be in $GL_2(\mathbb{Z})$. We only have to check that the determinant is 1. First of all, it is positive. Indeed $\operatorname{Im}\tau' = \frac{ad-bc}{|c\tau d|^2} \operatorname{Im}\tau$ and both $\tau, \tau' \in \mathbb{H}$. Then, the determinant must be an invertible integer, which gives the assertion. □

3.3.3 When the covering space is the Poincaré disc (or the hyperbolic plane)

Every Riemann surface not studied in the previous two sections can be seen as the quotient of the unit disc (or the hyperbolic plane) by the action of a Fuchsian group.

Exercise 3.3.7. *Let us consider two real numbers* $0 < r < R < +\infty$. *Define the* annulus $A_{r,R}$ *as*

$$A_{r,R} := \{ z \in \mathbb{C} \colon r < |z| < R \} .$$

1. *Prove that the universal covering of* $A_{r,R}$ *is the unit disc.*

2. *Prove that* $A_{r,R}$ *is not biholomorphic to* $\mathbb{D}^*$.

Remark 3.3.8. *We will not prove the following stronger statement: Two annuli* $A_{r,R}$ *and* $A_{r',R'}$ *are biholomorphic if and only if* $rR' = r'R$.

We can also consider the case when $R = +\infty$. *We have two cases: if* $r \neq 0$, *the annulus* $A_{r,\infty}$ *is conformally equivalent to the annulus* $A_{0,1/r}$ *via the map* $1/z$. *We are thus in the case of the exercise above. If* $r = 0$, *we are considering the annulus* $A_{0,\infty} = \mathbb{C}^*$. *This is an infinite cylinder, whose cover is the complex plane, by the exponential map (see previous section).*

Solution of Exercise 3.3.7. The first statement follows from the Uniformization Theorem (and the content of the previous two sections). A more direct way to exclude that the universal covering is $\mathbb{C}$ or $\widehat{\mathbb{C}}$ is the following: if π is the quotient map from the universal covering $\widetilde{S}$ to $A_{r,R}$, the image of π is contained in the ball B_{2R}. Since we do not have non-constant holomorphic maps on $\mathbb{C}$ with bounded image, it follows that $\widetilde{S} = \mathbb{D}$.

For the second assertion, assume there is a a biholomorphism φ between $\mathbb{D}^*$ and $A_{r,R}$. Since the image of φ is bounded, the map φ can be extended to 0. By continuity, $\varphi(0) \in \overline{A}_{r,R}$. We have two cases:

1. $\varphi(0) \in A_{r,R}$. This means that there exists a $z \in A_{0,r}$ such that $\varphi(z) = \varphi(0)$. This (since φ in open) implies that there exist x_1, x_2 near 0 and z such that $\varphi(x_1) = \varphi(x_2)$, which gives a contradiction.

2. $\varphi(0) \in \partial A_{r,R}$. In this case we directly contradict the fact that φ is an open mapping, by considering the image of a neighbourhood of 0.

□

3.4 Ramified coverings

In this section we study more in detail the phenomenon introduced in 3.1.18. Our motivation is the following: we will be interested in studying the iteration of holomorphic self maps of Riemann surfaces. In general, these maps do not provide a covering. For instance, any point where the derivative vanishes cannot admit any neighbourhood sent injectively to its image. We thus want to generalize the definition of covering that we stated above.

Recall from above that a holomorphic map $\pi : M \to N$ is a *covering* if every point in N has a neighbourhood U which is *evenly covered*, i.e., such that each connected component of $\pi^{-1}(U)$ maps to U by a biholomorphism.

Definition 3.4.1. *A continuous map $f : M \to N$ between two topological spaces is* proper *if the inverse image of any compact set is a compact subset.*

Exercise 3.4.2. *Let $f : M \to N$ be a continuous map between Riemann surfaces. Prove that the following are equivalent.*

1. *f is proper.*
2. *f is closed (i.e., the image of a closed set is a closed set) and the preimage of every point is compact*
3. *f maps "the boundary to the boundary": if $x_n \in M$ form a diverging sequence (i.e., it leaves every compact set), then the same is true for the sequence $f(x_n)$.*

Solution. Assume f is proper and x_n is a diverging subsequence. If the image sequence is not divergent, we have an accumulation point. Take a compact set containing this point. The preimage of this set must be compact, but also contains a diverging sequence, contradiction.

On the other hand, assume diverging sequences have diverging images. Let K be a compact subset of N. In particular, every sequence in K is not divergent. If the preimage of K were not compact, it would contain a diverging sequence (necessarily mapping to a non diverging one), contradiction.

Let us prove that a proper map satisfies the condition of the second point. Let C be a closed subset of M. If C is compact, its image is compact by continuity, and so also closed. If C is not compact, assume y_n are points in $f(C)$, of the form $y_n = f(x_n)$ for $x_n \in C$. If the sequence x_n is divergent, the image y_n is divergent, and thus does not have accumulation points in N. Let us thus assume that x_n is not divergent, with an accumulation point $x_0 \in M$. Let y_0 be the accumulation point of y_0. Since C is closed, we have $x_0 \in C$, which implies that $y_0 \in f(C)$ as desired. The second condition is trivially verified, since points are compact.

Finally, let us prove that a closed map having compact preimages of points is proper. Let K be a compact subset of N and let $\{ U_i \}_{i\in I}$ be an open cover of $f^{-1}(K)$. We want to fine a finite subcover. Fix any $y \in K$ and notice that the U_i form a cover of $f^{-1}(y)$, too. Since $f^{-1}(y)$ is compact, there is a finite subset I_y of I such that $\{ U_i \}_{i\in I_y}$ is a cover of $f^{-1}(y)$. Denote by U_y the finite union $\cup_{i\in I_y}$. Then $M \setminus U$ is closed and, since f is closed, $f(M \setminus U)$ is closed. Let $V_y := N \setminus f(M \setminus U)$, which is open and contains y. So, we have $K \subset \cup_{y\in K} V_y$. Since K is compact, we can extract a finite subcover form the V_y's, say $K \subset V_{y_1} \cup \cdots \cup V_{y_k}$. Let $\overline{I} := \cup_j I_{y_j}$, which is a finite set. We conclude because now $f^{-1}(K) \subset f^{-1} \cup_{j=1}^{k} V_{y_j} \subset \cup_{i\in\overline{I}} U_i$. □

Exercise 3.4.3. *Let f be a continuous map from a compact space to an Hausdorff space. Prove that f is both closed and proper.*

Definition 3.4.4. *A holomorphic map $\pi : M \to N$ between Riemann surfaces is a* branched (or ramified) covering *if every point in N has a connected neighbourhood U such that π is a proper map from every connected component of $\pi^{-1}(U)$ to U.*

Any point in M whose image do not admit an evenly covered neighbourhood is called a ramification point. The following proposition gives a description of a branched covering near a ramification point.

Proposition 3.4.5. *Let $f : M \to N$ be a branched covering. Let $p \in M$. There exist open neighbourhoods U and U' of o and $f(p)$ respectively, and open neighbourhoods V, V' of 0 in $\mathbb{C}$, and biholomorphisms $g : U \to V$ and $g' : U' \to V'$ sending p and $f(P)$ respectively to 0, such that the map $g' \circ f \circ g^{-1} : V \to V'$ is equal to $z \mapsto z^{e_p}$ for some $e_p \geq 1$.*

A ramification point is a point p such that $e_p > 1$.

Proposition 3.4.6. *The set of ramification points does not have accumulation points in M.*

Proof. It immediately follows from the fact that the ramification points are the points where the derivative of f is zero. □

Proposition 3.4.7. *Let $f : M \to N$ be a branched covering map between Riemann surfaces. Then the cardinality (with multiplicity) of the preimage of every point does not depend on the point.*

Proof. Denote by R the ramification points. By definition, f induces a well defined covering from $M \setminus (f^{-1}f(R))$ to $N \setminus f(R)$. In particular, the fibers of this restriction have constant cardinality. The assertion then follows by the local description of the map f near the ramification points. □

Definition 3.4.8. *The* degree *of a branched covering is the number of preimages of any point.*

Proposition 3.4.9. *Any proper holomorphic map between Riemann surfaces is a branched holomorphic covering.*

Example 3.4.10. *The map $z \mapsto z^2$ gives a branched covering of the disc over itself. The origin is a ramification point. Notice that this map is proper form the disc to itself, every point has 2 preimages counting multiplicities, and the origin is the only point whose two preimages coincide.*

Example 3.4.11. *Let f be a rational map on the Riemann sphere, i.e., let f be given by $f(z) = \frac{p(z)}{q(z)}$, where p and q are polynomials without common factors. Since the Riemann sphere is compact, f is proper map and thus a branched covering. We will study extensively the iterations of rational maps in the second part of this course.*

Exercise 3.4.12. *Let $f(z) = \frac{p(z)}{q(z)}$ (where p, q are polynomials without common factors) be a rational map on the Riemann sphere. Prove that the degree d is the maximum of the degrees of p and q.*

3.5 Maps between Riemann surfaces and Montel Theorem

In this section we will consider two Riemann surfaces M and N and study the set of holomorphic maps $\mathrm{Hol}(M, N)$ from M to N. We will apply later the general results to the case of self maps of a Riemann surfaces, by restricting our attention to the set of iterates of this map.

The set $\mathrm{Hol}(M, N)$ can be endowed with a natural topology, the one of the *uniform convergence on compact subsets*. Without entering into the details (that we shall omit, but for which one can refer to Milnor's book), it is possible to put a topology on $\mathrm{Hol}(M, N)$ in such a way that a sequence of maps f_n converges to f in this topology if and only if f_n converge to f locally uniformly, i.e., uniformly on every compact subset of M.

The following standard result from complex analysis motives this choice of topology. Notice that, since we require the convergence on compact subsets, the statement can be easily deduced from the analogous one valid for subsets of $\mathbb{C}$.

Theorem 3.5.1. *Let $f_n \in \mathrm{Hol}(M, N)$ a sequence of maps which converges locally uniformly to a map f. Then f is holomorphic.*

This proves that the space $\mathrm{Hol}(M, N)$ is closed with respect to this convergence.

Remark 3.5.2. *It is possible to prove that the space $\mathrm{Hol}(M, N)$ endowed with this topology is actually* metrizable*, i.e., there exists a distance on $\mathrm{Hol}(M, N)$ inducing this topology. We do not enter in these details, but just notice that this implies that the space $\mathrm{Hol}(M, N)$ is Hausdorff.*

Definition 3.5.3. *Given M, N Riemann surfaces, a family $\mathscr{F}$ of maps in $\mathrm{Hol}(M, N)$ is* normal *if one of the following occurs:*

1. *every sequence f_n in $\mathscr{F}$ admits a converging (locally uniformly) subsequence ; or*
2. *every sequence f_n in $\mathscr{F}$ admits a subsequence f_{n_k} which diverges locally uniformly: for every $K \subset M$ and $K' \subset N$ compact subsets there exists k_0 such that $f_{n_k}(K) \cap K' = \emptyset$ for every $k \geq k_0$.*

Notice that the limits of the subsequence need not be an element of $\mathscr{F}$.

Example 3.5.4. *The sequence $f_n(z) = z/n$ is normal on $\mathbb{C}$. The same is true for the sequence $g_n(z) = z + n$. However, notice that the sequence g_n is not normal when we consider the maps as self maps of $\widehat{\mathbb{C}}$!*

Example 3.5.5. *Consider the set of all rotations of the unit disc. This set is normal.*

Recall (Theorem 3.2.13) that holomorphic maps between Riemann surfaces can be lifted to the universal coverings. This implies that $\mathrm{Hol}(M, N)$ consists only in constant functions if M is the Riemann sphere and N is covered by the plane of the disc, and when M is covered by the plane and N by the disc.

We now study more in detail the set $\mathrm{Hol}(\mathbb{D}, \mathbb{D})$. Our goal is to show that all $\mathrm{Hol}(\mathbb{D}, \mathbb{D})$ is indeed a normal family.

The following is a fundamental result.

Theorem 3.5.6 (Schwartz's Lemma). *If $f : \mathbb{D} \to \mathbb{D}$ is a holomorphic map with $f(0) = 0$, then the derivative at the origin satisfies $|f'(0)| \leq 1$. If equality holds then f is a rotation about the origin (i.e., $f(z) = cz$ for some constant $c = f'(0)$ on the unit circle. On the other hand, if $|f'(0)| < 1$, then $|f(z)| < |z|$ for all $z \neq 0$.*

Sketch of proof. Consider the function $q(z) = f(z)/z$. This function is is holomorphic in $\mathbb{D}$ and satisfies $f'(0) = q(0)$.

The key step in the proof is an application of the Maximum Modulus Principle. Take any $r, 0 \leq r < 1$ and $z_0 \in \mathbb{D}$ with $|z_0| = r$. Then

$$|q(z_0)| = \frac{|f(z_0)|}{|z_0|} < \frac{1}{|z_0|} = \frac{1}{r}.$$

Since f is holomorphic on $\mathbb{D}$, f is holomorphic on $\mathbb{D}_r = \{z \in \mathbb{C} : 0 \leq |z| < r\}$. So by the Maximum Modulus Theorem, we have that for any $z \in \mathbb{D}_r$, $|q(z)| < 1/r$. Now we let $r \to 1$. Then the disks $\mathbb{D}_r$ exhaust $\mathbb{D}$, and the upper bound on each $\mathbb{D}_r$ converges to 1. So that $|q(z)| \leq 1$ for any $z \in \mathbb{D}$. Applying the Maximum Modulus Principle again, we see that if for any $z \in \mathbb{D}$, $|q(z)| = 1$, then q is constant. Notice that this is what occurs if $|f'(0)| = 1$. In this case, $|q(z)| = 1$, so $q(z) = c$ is constant, $c \in \partial\mathbb{D}$, and we have that $f(z) = cz$ is a rotation. Alternatively, $|q(z)| < 1$ for all $z \in \mathbb{D}$, so that $|f(z)| < |z|$ for all $z \in \mathbb{D} \setminus \{0\}$. □

We will admit the following Theorem, which is essentially a generalization of Schwartz Lemma to surfaces whose universal covering is the unit disc.

Theorem 3.5.7 (Pick). *If $f : S \to S'$ is a holomorphic map between hyperbolic surfaces, then exactly one of the following three statements is valid:*

1. *f is a conformal isomorphism from S onto S', and it maps S with its Poincaré metric isometrically onto S' with its Poincaré metric;*
2. *f is a covering map but is not one-to-one. In this case, it is locally but not globally a Poincaré isometry. Every smooth path $\sigma : [0, 1] \to S$ of length l in S maps to a smooth path $f \circ \sigma$ of the same length l in S', and it follows that $d_{S'}(f(p), f(q)) \leq d_S(p, q)$ for every $p, q \in S$. Here equality holds whenever p is sufficiently close to q, but strict inequality will hold, for example, if $f(p) = f(q)$ with $p \neq q$;*
3. *in all other cases, f strictly decreases all nonzero distances. In fact, for any compact set $K \subset S$ there is a constant $C_K < 1$ such that $d_S(f(p), f(q)) \leq C_K d_S(p, q)$ for every $p, q \in K$ and so that every smooth path in K with arclength l (using the Poincare metric for S) maps to a path of Poincare arclength $\leq C_K l$ in S'.*

One key consequence of the Schwarz-Pick Theorem is that any family $\mathcal{F}$ of holomorphic maps $f : S \to S'$ between two hyperbolic Riemann surfaces is *equicontinuous*: for any $z \in S$ and $\varepsilon > 0$, if $d_S(z, z') < \varepsilon$, then $d_{S'}(f(z), f(w)) \leq d_S(z, w) < \varepsilon$.

Theorem 3.5.8 (Montel). *Let S, S' be hyperbolic Riemann surfaces. Then every family of holomorphic maps from S to S' is normal.*

Proof. Let $\{f_n\}$ be a sequence of functions in $\mathrm{Hol}(S, S')$. Assume that $\{f_n\}$ is not compactly divergent. Then there exist compact sets $K_1 \subset S$ and $K_2 \subset S'$ such that for infinitely many n, $f^n(K_1) \cap K_2 \neq \emptyset$. After choosing a subsequence, we can assume that $f_n(K_1) \cap K_2 \neq \emptyset$ for all n.

Fix any $z_0 \in S$ and $w \in K_2$. Let $\delta_1 = \mathrm{diam}_S(K_1)$ and $\delta_2 = \mathrm{diam}_{S'}(K_2)$. By the Schwarz-Pick Theorem, we have that

$$d_{S'}(f_n(z), w) \leq \min_{z \in K_1} \{d_S(z, z_0)\} + \delta_1 + \delta_2.$$

This implies that the sequence $\{f_n(z_0)\}$ lies within a compact neighbourhood of $w \in S'$. The Schwarz-Pick Theorem implies that $\{f_n\}$ is equicontinuous. Thus the Arzela-Ascoli Theorem implies that $\mathrm{Hol}(S, S')$ is a normal family. □

We recall here the basic Ascoli-Arzela Theorem we used above (to be precise, we should restrict to charts to be able to apply it directly, but the same proof of Ascoli-Arzela Theorem works in our context).

Theorem 3.5.9 (Arzela-Ascoli). *If a sequence $\{f_n\}$ defined on a compact subset of $\mathbb{C}$ is bounded and equicontinuous, then it has a uniformly convergent subsequence.*

The following result combine the above Corollary with the fact the we do not have non constant holomorphic maps from the Riemann surface or the complex plane to a hyperbolic surface.

Corollary 3.5.10. *Let M be a Riemann surface, and $\mathscr{F}$ be a collection of maps from S to $\widehat{\mathbb{C}}$ such that $f(S) \subset \widehat{\mathbb{C}} \setminus \{a, b, c\}$ for every $f \in \mathscr{F}$ and three disctint points $a, b, c \in \widehat{\mathbb{C}}$. Then $\mathscr{F}$ is a normal family.*