2
Fuchsian groups

In the previous chapter we introduced and studied the elements of $\mathrm{Mob}(\mathbb{H})$, which are the real Moebius transformations. In this chapter we focus the attention of special subgroups of this group. The exposition is again essentially taken from Walkden's lecture, in particular (subsets of) chapters 12 and 23. We refer to those notes for a more complete exposition.

2.1 Definitions and discreteness

We start with a general definition.

Definition 2.1.1. *A* topological group *is a group endowed with a topology for which the operation is continuous.*

We will use only *metric groups*, i.e., metric spaces with an operation of group which is continuous with respect to the topology induced by the distance.

Example 2.1.2. $\mathbb{R}$*, with the sum operation, is a metric group.* $\mathbb{R}^*$ *is a metric group with the operation of product.*

Our group of interest, $\mathrm{Mob}(\mathbb{H})$, admits a structure of metric group. This is the content of the next result.

Before stating it, let us introduce a distance on $\mathrm{Mob}(\mathbb{H})$. Recall that every element $\gamma \in \mathrm{Mob}(\mathbb{H})$ is of the form $\gamma(z) = \frac{az+b}{cz+d}$, where we can assume that $ad - bc = 1$ (we will always assume this in the sequel). Roughly speaking, we would like to say that two elements are close if the corresponding coefficients are close. Identifying Moebius transformations with points in $\mathbb{R}^4$, we may use as distance the standard one on $\mathbb{R}^4$. We just have a problem to fix: remember that we can multiply all the coefficients of γ by any

$\lambda \neq 0$ and still obtain the same Moebius transformation. Having required that $ad - bc = 1$ implies that the only possible λ we may take are actually 1 and -1. We thus consider the following distance on the (normalized) elements of $\mathrm{Mob}(\mathbb{H})$: if $\gamma(z) = \frac{az+b}{cz+d}$ and $\gamma'(z) = \frac{Az+B}{Cz+D}$ are two normalized elements, we define

$$d_{\mathrm{Mob}}(\gamma, \gamma') = \min\left(\|(a,b,c,d) - (A,B,C,D)\|, \|(a,b,c,d) - (-A,-B,-C,-D)\|\right)$$

where $\|\cdot\|$ denotes the standard Euclidean metric on $\mathbb{R}^4$.

We leave the verification that the above function is indeed a distance as an exercise. Notice that the definition of distance easily extends to non normalized elements, by consider the distance of two normalizations.

Remark 2.1.3. *The above is an example of a* quotient *of a metric space.*

Proposition 2.1.4. $\mathrm{Mob}(\mathbb{H})$ *is a metric group.*

We will never make use of the precise distance we have put on $\mathrm{Mob}(\mathbb{H})$. What we will need, is just to be able to say if an element is *isolated* with respect to a certain subset.

Definition 2.1.5. *Let (X, d) be a metric space and Y a subset of X. A point $y \in Y$ is isolated in Y if there exist $\delta > 0$ such that $d(y, y') \geq \delta$ for every $y' \in Y$ different from y.*

Definition 2.1.6. *Let (X, d) be a metric space and Y a subset of X. We say that Y is* discrete *if every point of Y is isolated in Y.*

Definition 2.1.7. *A* Fuchsian group *is a discrete subgroup of* $\mathrm{Mob}(\mathbb{H})$.

Example 2.1.8. *Since any finite subset of a metric space is discrete, any finite subgroup of* $\mathrm{Mob}(\mathbb{H})$ *is a Fuchsian group.*

Exercise 2.1.9. *Prove that every finite Fuchsian group is conjugated to a group of rotations (see Exercise 1.2.20).*

Example 2.1.10. *$\mathbb{Z}$ is a discrete metric space with the usual distance $d(n, m) = |n - m|$, and a discrete subset of the metric space $\mathbb{R}$.*

Example 2.1.11. *The subgroup of translations $z \mapsto z + m$, with $m \in \mathbb{Z}$, is a Fuchsian group. The group of all translations (by any real number) is not a Fuchsian group.*

Exercise 2.1.12. *The* modular group $PSL_2(\mathbb{Z})$ *is the subgroup of* $\mathrm{Mob}(\mathbb{H})$ *given by*

$$PSL_2(\mathbb{Z}) := \left\{ \gamma \in \mathrm{Mob}(\mathbb{H}) \colon \gamma(z) = \frac{az+b}{cz+d}, a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}.$$

Check that $PSL_2(\mathbb{Z})$ is a Fuchsian group.

The following result gives a criterion to check whether a group is discrete.

Proposition 2.1.13. *Let G be a topological group. The following are equivalent:*

1. *G is discrete;*

2. *the identity element of G is isolated.*

Proof. We have to prove that, if the identity element id is isolated, the same is true for every element $g \in G$. This follows from the fact that the image of an isolated point by an homeomorphism (as any element of G) is isolated. □

Corollary 2.1.14. *Let Γ be a subgroup of $\mathrm{Mob}(\mathbb{H})$. The following are equivalent:*

1. *Γ is a discrete subgroup of $\mathrm{Mob}(\mathbb{H})$ (i.e., Γ is a Fuchsian group);*

2. *the identity element of Γ is isolated.*

We recall that a group G is said to *act* of a set X if there exists a map $G \times X \to X$.

Example 2.1.15. *$\mathbb{Z}$ acts on $\mathbb{R}$ by translations. More generally, the group of isometries (and also all the group of endomorphisms) of a metric space acts on the space itself.*

Example 2.1.16. *Let X be a topological space and $f \colon X \to X$ a homeomorphism. The the group generated by f (the positive and negative iterates of f) acts on X. Notice that this can be seen as an action of $\mathbb{Z}$ on X, given by $n \cdot x := f^{\circ n}(x)$.*

Definition 2.1.17. *Let G be a group acting on a set X. Let $x \in X$. The* orbit *$G(z)$ of z under G is the set of all elements of X of the form $g \cdot x$, for some $g \in G$.*

Remark 2.1.18. *A subgroup Γ of $\mathrm{Mob}(\mathbb{H})$ induces an action on the boundary $\partial\mathbb{H}$. Even if the orbits for Γ on $\mathbb{H}$ are discrete, this does not imply that the orbits for the induced action on the boundary is discrete, as the next exercise shows.*

Exercise 2.1.19. *Prove that the action on $\partial\mathbb{H}$ induced by the modular group (see Exercise 2.1.12) admits non discrete orbits.*

The following result gives an important equivalent characterization of Fuchsian groups. The proof of this result will be the core of the next section.

Theorem 2.1.20. *Let Γ be a subgroup of $\mathrm{Mob}(\mathbb{H})$. Then the following are equivalent:*

1. *Γ is a Fuchsian group;*

2. *for each $z \in \mathbb{H}$, the orbit $\Gamma(z)$ is a discrete subset of $\mathbb{H}$.*

2.2 Proper discontinuous actions

In this section we are going to study a particular kind of action of a group of homeomorphisms of a metric space. In what follows, G will thus be a group of homeomorphisms of a metric space (X, d).

Definition 2.2.1. *Let G be a group of homeomorphisms acting on a metric space (X, d). The* stabilizer *of an element $x \in X$ is the subset of elements $g \in G$ such that $g \cdot x = x$.*

Exercise 2.2.2. *For every $x \in X$, the stabilizer of x is a subgroup of X.*

Exercise 2.2.3. *Find the orbit and the stabilizer of* 0 *under the action of* $\mathrm{Mob}(\mathbb{D})$.

Example 2.2.4. *The stabilizer of a point $x \in \mathbb{R}^2$ in the group of (orientation preserving) isometries is given by the rotation about x.*

Definition 2.2.5. *Let G be a group acting on a metric space (X, d). We say that the action is* properly discontinuous *if for all $x \in X$ and for all non-empty compact subset $K \subset X$ the set $\{ g \in G \colon g \cdot x \in K \}$ is finite.*

The following result gives several equivalent characterizations of a properly discontinuous action. We need to restrict to the following class of metric spaces.

Definition 2.2.6. *A metric space is* proper *if every closed ball $C_\varepsilon(x) = \{y \in X \colon d(x, y) \leq \varepsilon\}$ is compact*

Exercise 2.2.7.
1. *Prove that $\mathbb{R}^n$, the hyperbolic half plane $\mathbb{H}$, the Poincaré disc and the group $\mathrm{Mob}(\mathbb{H})$ are proper metric spaces.*
2. *Prove that the set of continuous real function on $[0, 1]$ is a metric space with the distance given by $d(f, g) := \sup_{x\in[0,1]} |f(x) - g(x)|$. Prove that this metric space is not proper.*

Basically every metric space we will consider in this course is proper.

Proposition 2.2.8. *Let G be a group of homeomorphisms acting on a metric space (X, d). Then the following are equivalent:*

1. *G acts properly discontinuously on X;*
2. *for all $x \in X$,*
   a) *the orbit $G(x)$ is a discrete subset of X, and*
   b) *the stabiliser of x is a finite subgroup of G.*
3. *each point $x \in X$ has a neighbourhood U containing x such that $\gamma(U) \cap U \neq \emptyset$ for only finitely many $g \in G$.*

Proof. We will only prove some the equivalence of the first two conditions, leaving the rest for exercise (we refer as usual to Walkden's notes for a complete proof).

Assume the action is properly discontinuous and that the orbit of a point x is not discrete. There exists a sequence $\gamma_n \in G$ and a point x_0 such that $\gamma_n(x) \to x_0$. We can assume that these points are all distinct (which implies that the γ_n's are distinct, too). Consider a closed ball C at x_0. Since the metric space is proper, this ball is compact. Since $\gamma_n(x) \to x_0$, we have that $\gamma_n(x) \in C$ for $n \geq N$. There are thus infinitely many $\gamma \in \Gamma$ such that $\gamma(x)$ belongs to the compact set C, which gives a contradiction.

We check now that, if the action is properly discontinuous, the stabilizers are finite. Let $x \in X$ and let K be the (compact) set $K = \{ x \}$. The stabilizer of x is then equal to

$\{ \gamma \in \Gamma \colon \gamma(x) \in K \}$. Since the action is properly discontinuous and K is compact, this last set is finite.

Let us prove the reverse implication. Let $x \in X$ and K be a compact. We want to prove that the set $\{ \gamma \in \Gamma \colon \gamma(x) \in K \}$ is finite.

Let us start proving that the points in the orbit which belong to K are finite. If this were not the case, we would find a limit point in K (since K is compact) and this would contradict the assumption that the orbits are discrete.

By the previous analysis, it is enough to exclude that infinitely many elements in the orbit of x can coincide. This means proving that, given y in the orbit of x, the set $A := \{ \gamma \colon \gamma(x) = y \}$ is finite. Fix a $\overline{\gamma}$ such that $\overline{\gamma}(x) = y$. We then have $\overline{\gamma}^{-1} \circ \gamma(x) = x$, which means that $\overline{\gamma}^{-1} \circ \gamma$ belongs to the stabilizer of x, for every $\gamma \in A$. Since all these elements are distinct and the stabilizer is finite, the assertion follows. □

Theorem 2.2.9. *Let Γ be a subgroup of $\mathrm{Mob}(\mathbb{H})$. Then Γ is a Fuchsian group if and only if Γ acts properly discontinuously on $\mathbb{H}$.*

In the proof of this Theorem, we will need the following two lemmas.

Lemma 2.2.10. *Let $z_0 \in \mathbb{H}$ and K be a compact subset of $\mathbb{H}$. Then*

$$E := \{ \gamma \in \mathrm{Mob}(\mathbb{H}) \colon \gamma(z_0) \in K \}$$

is a compact in $\mathrm{Mob}(\mathbb{H})$.

We do not give the proof of this lemma here, but we refer to Walkden's notes (Lemma 24.2.2)

Lemma 2.2.11. *Let Γ be a subgroup of $\mathrm{Mob}(\mathbb{H})$ that acts properly discontinuously on $\mathbb{H}$. Suppose that some $p \in \mathbb{H}$ is fixed by some element of Γ. Then there exists a neighbourhood W of p such that no other point of W is fixed by an element of Γ (different from the identity).*

Proof. Assume by sake of contradiction that:

1. $\gamma(p) = p$ for some $p \in \mathbb{H}$ and $\gamma \in \Gamma$ different from the identity;
2. there exists a sequence $p_n \to p$ and elements $\gamma_n \in \Gamma$, different from the identity, such that $\gamma_n(p_n) = p_n$. We can assume without loss of generality that

$$d_{\mathbb{H}}(p_n, p) < \varepsilon \text{ for some positive small } \varepsilon, \text{ for every } n. \tag{2.1}$$

Notice that all the γ_n's are necessarily elliptic, with the p_n's as unique fixed points. In particular, they are all distinct. Let now B be the closed (hyperbolic) ball of radius 3ε centred at p. By definition of properly discontinuous action and since B is a compact subset, we have that $\gamma_n(p) \in B$ for only finitely many n. Without loss of generality (up to starting from some N), we can assume that $\gamma_n(p) \notin B$ for every n. In particular, this implies that

$$d_{\mathbb{H}}(\gamma_n(p), p) > 3\varepsilon \text{ for every } n. \tag{2.2}$$

This gives a contradiction with the triangle inequality. Indeed, we must also have

$$d_{\mathbb{H}}(\gamma_n(p), p) \leq d_{\mathbb{H}}(\gamma_n(p), \gamma_n(p_n)) + d_{\mathbb{H}}(\gamma_n(p_n), p) = d_{\mathbb{H}}(p, p_n) + d_{\mathbb{H}}(p_n, p) \leq 2\varepsilon$$

where we used the facts that γ_n is an isometry and $\gamma_n(p_n) = p_n$. The assertion follows. □

We can now prove Theorem 2.2.9

Proof of Theorem 2.2.9. Suppose Γ is a Fuchsian group. Let $z \in \mathbb{H}$ and $\emptyset \neq K \subset \mathbb{H}$ be a compact subset. We want to prove that the set $A := \{ \gamma \in \Gamma \colon \gamma(z) \in K \}$ is finite. This set is equal to $\{ \gamma \in \mathrm{Mob}(\mathbb{H}) \colon \gamma(z) \in K \} \cap \Gamma$ By Lemma 2.2.10 the first set is compact. The second is discrete by definition. So, since $\mathrm{Mob}(\mathbb{H})$ consists of isometries, we have A is finite and one implication is proved. (notice that the fact that we are considering isometries is important here, see Exercise 2.2.12).

Let us prove the reverse implication. Lemma 2.2.11 implies that there exists a point $p \in \mathbb{H}$ that is not fixed by any element of Γ (apart from the identity). By Proposition 2.1.13, if Γ were not discrete there must exist pairwise distinct elements $\gamma_n \in \Gamma$ converging to the identity. Thus $\gamma_n(p) \to p$, but $\gamma_n(p) \neq p$. It is easy to see that no neighbourhood of p as in the definition of properly discontinuous action can be found. The assertion follows. □

Exercise 2.2.12. *The goal of this exercise is to prove that the fact that we work in a group of isometries is important in the proof of the above result.*

1. *Give an example of a metric space X, a compact subset K of X and a discrete subset D of X such that $K \cap D$ is infinite.*

2. *Give an example of a metric space X, a group Γ of homeomorphisms of X, a point $x \in X$ and a compact subset K of X such that $\Gamma(x)$ is discrete but $K \cap \Gamma(x)$ is infinite.*

3. *Assume now that Γ as in the previous point is a group of isometries and the orbit of x is discrete. Show that there exists $\varepsilon > 0$ such that, for all $\gamma \in \Gamma$, we have $B_\varepsilon(\gamma(x)) \cap \Gamma(x) = \{ \gamma(x) \}$.*

4. *Conclude that if Γ acts by isometries, $\Gamma(x)$ is discrete and K is compact then $K \cap \Gamma(x)$ is finite.*

Solution of Exercise 2.2.12. For the first part, it is enough to consider the real line with the standard Euclidean distance, the compact subset $K = [0, 1]$ and the discrete set $D = \{ 1/n \colon n \in \mathbb{N} \}$. Notice that the definition of discrete set, indeed, excludes the possibility of accumulations point *in D*, but not in the whole metric space.

For the second part, we consider the Moebius transformations given by $z \mapsto 2^m z$, for some $m \in \mathbb{Z}$. It is easy to see that this is a subgroup Γ of $\mathrm{Mob}(\mathbb{H})$. We consider the action induced on the boundary $\partial\mathbb{H}$. Every element in Γ induces an homeomorphism on $\partial\mathbb{H}$. We thus consider $\partial\mathbb{H}$ as metric space, Γ as group and $K = [0, 1]$ as compact subset. The orbit of 1 is given by the points of the form 2^m, which form a discrete set. However, they accumulate the point $0 \in [0, 1]$, giving the desired example. Notice here that, even if every element of Γ is an isometry of $\mathbb{H}$, they are not isometries of the boundary. Notice also that this is an example of a group acting propertly discontinuously on $\mathbb{H}$, but not on $\partial\mathbb{H}$.

The idea for the third part is not see that, since now the group is given by isometries, every metric property we can prove locally near a point is indeed true near every point of its orbit. Since $\Gamma(x)$ is discrete, we have an ε such that $B_\varepsilon(x) \cap \Gamma(x) = \{ x \}$. We claim that this ε satisfies the requirement, i.e., $B_\varepsilon(\gamma(x)) \cap \Gamma(x) = \{ \gamma(x) \}$ for every $\gamma \in \Gamma$. This follows since γ is an isometry from $B_\varepsilon(x)$ to $B_\varepsilon(\gamma(x))$. If y is a point in the orbit of x and contained in $B_\varepsilon(\gamma(x))$ then $\gamma^{-1}(y)$ is a point in the orbit of x and contained in $B_\varepsilon(x)$. The only possibility is $\gamma^{-1}(y) = x$, which means that $y = \gamma(x)$, as desired.

For the last part, notice that the third part implies that $d(\gamma_1(x), \gamma_2(x)) \geq \varepsilon$ for every $\gamma_1 \neq \gamma_2 \in \Gamma$. If we had an infinite number of (distinct) γ's such that $\gamma(x) \in K$ (and are different), we would have a converging subsequence in K. This convering subsequence would be a Cauchy sequence, contradicting the uniform bound on the distance between points in the orbit. □

We can now prove Theorem 2.1.20.

Proof. By Theorem 2.2.9, being a Fuchsian group is equivalent to act properly discontinuously. By Proposition 2.2.8, this last property is equivalent to have discrete orbits and finite stabilizers. We thus only need to prove that, if we start with a subgroup of $\mathrm{Mob}(\mathbb{H})$, the assumption on the finiteness of the stabilizer does not need to be checked to prove that the action is properly discontinuous.

More precisely, it is enough to prove the following statement: if a subgroup of $\mathrm{Mob}(\mathbb{H})$ has discrete orbits, the stabilizer of every point is finite. We shall work with the disk model, since elements with fixed points have a simpler expression in this model.

Indeed, let $z_0 \in \mathbb{D}$ and γ be any element of the stabilizer of z_0. In particular, γ has a fixed point in the interior of $\mathbb{D}$, and is thus an elliptic element (or the identity). By Proposition 1.2.19, every element in the stabilizer of z_0 is thus conjugated to a rotation about $0 \in \mathbb{D}$ (by means of a map g sending z_0 to 0). Let us denote by S the subgroup of the rotations $\gamma_\theta(z) = e^{2\pi i\theta}z$ about 0 conjugated to the stabilizer of z_0 and assume, by sake of contradiction, that S is infinite. If this is the case, the set of associated angles is dense in $[0,1]$, and we can find a sequence of angles θ_j such that γ_{θ_j} belongs to S and $\theta_j \to \overline{\theta}$, for some $\overline{\theta} \in [0,1]$ such that $\gamma_{\overline{\theta}} \in S$. This implies that, for every $z \in \mathbb{D}$, we have $\gamma_{\theta_j}(z) \to \gamma_{\overline{\theta}}(z)$. Take any $z \neq 0$ To conclude, it is enough to prove that the orbit of $g^{-1}(z)$ under Γ is not discrete. Consider the elements $\gamma_j := g^{-1} \circ \gamma_{\theta_j} \circ g \in \Gamma$ (actually, they belong also to the stabilizer of z_0) and $\overline{\gamma} := g^{-1} \circ \gamma_{\overline{\theta}} \circ g$. We have

$$\gamma_j(g^{-1}z) = g^{-1} \circ \gamma_{\theta_j} \circ g(g^{-1}z) = g^{-1} \circ \gamma_{\theta_j}(z) \to g^{-1} \circ \gamma_{\overline{\theta}}(z) = g^{-1} \circ (g \circ \overline{\gamma} \circ g^{-1})(z) = \overline{\gamma} \circ g^{-1}(z).$$

This proves that the orbit of $g^{-1}(z)$ is not discrete, giving the desired contradiction. □

Corollary 2.2.13. *Let Γ be a Fuchsian group and let $z \in \mathbb{H}$. Then $\Gamma(z)$ has no limit points inside $\mathbb{H}$.*

2.3 Fundamental domains and Dirichlet polygons

In this section we give an idea of how we can build new surfaces starting with $\mathbb{H}$, by means of the action of a Fuchasian group.

Our idea is the following: consider the complex plane $\mathbb{C}$ and the action of a a group of isometries generated by two independent translations (we will come back to this example more in detail in the next chapter). Given a point $z_0 \in \mathbb{C}$, we want to identify a natural neighbourhood of z_0 with the following property: the plane $\mathbb{C}$ will be exactly (with some care on the boundaries...) the union of all the images of this neighbourhood. One way is to consider the points for which z_0 minimizes the distance among all the points of the orbit of z_0. It is not difficult to see that this is a parallelogram in $\mathbb{C}$, and considering a decomposition by $\mathbb{C}$ by such parallelograms we see that (apart from the points on the boundaries), every parallelogram contains exactly one point of every orbit. We call this a *fundamental domain*. Consider now a parallelogram as above. We can identify pairs of sides (the opposite ones) by the existence of an element sending one to the other. If we consider this parallelogram with the points on the sides identified by these maps, we get what is usually called a *complex torus*. This is a so called *Riemann surface*, i.e., a topological manifold, that, locally, seems like the original plane $\mathbb{C}$.

Let us now come back to $\mathbb{H}$. In order to state things precisely, we need the following definition.

Definition 2.3.1. *A geodesic is a curve that locally minimizes distances.*

On $\mathbb{C}$, endowed with the Euclidean distance, geodesics are precisely the straight lines.

Proposition 2.3.2. *The geodesics for the hyperbolic distance on $\mathbb{H}$ are the vertical lines and the circles centered at points of the real line.*

Notice that, for every two points, there exists a unique *geodesic arc* connecting them.

Definition 2.3.3. *A* polygon *is a convex closed subset of $\mathbb{H}$ such that the boundary is given by arcs of geodesics, whose number is locally finite. We say that a polygon is finite if it has a finite number of edges.*

Here by *convex* we mean the following: for every couple of points in the set, the unique geodesic arc joining them is contained in the set.

Definition 2.3.4. *A* fundamental polygon *for a Fuchsian group Γ is a polygon $P \subset \mathbb{H}$ such that the orbit of every $z \in \mathbb{H}$ intersects P in at least one point, and intersects the interior of P in at maximum one point.*

The set of all *translated copies* of P gives a tessellation of $\mathbb{H}$: the union $\cup_{g\in\Gamma} g(P) = \mathbb{H}$ and $g(\mathring{P}) \cap g'(\mathring{P}) = \emptyset$ for every $g \neq g'$.

Theorem 2.3.5. *Let $\Gamma < \mathrm{Mob}(\mathbb{H})$ be a Fuchsian group. Let $z_0 \in (H)$ a point which is not fixed by any element of Γ apart from the identity. Then the set of points*

$$P := \{\, z \in \mathbb{H} \colon d_{\mathbb{H}}(z, z_0) = d_{\mathbb{H}}(z, \Gamma(z_0)) \,\}$$

is a polygon for Γ.

Proof. First of all, notice that, since Γ is Fuchsian, the orbit of z_0 is discrete and thus P contains a neighbourhood of z_0. Then, for every pair of points $x \neq y$, notice that the region of space defined by $A_{x,y} := \{ z \in \mathbb{H} \colon d_{\mathbb{H}}(z,x) \leq d_{\mathbb{H}}(z,y) \}$ is convex. Thus, P can be see as an intersection $P = \cap_{g\in\Gamma} P_g$, where $P_g = A_{z_0, g(z_0)}$. This proves that P is convex. It is not difficult to see that the boundary of $A_{x,y}$ is a geodesic, so that the boundary of P is given by geodesics.

Let us check that their number is locally finite. This follows by the fact the action is properly discontinuous. Indeed, let us fix an hyperbolic ball of radius $r > 0$. We need to prove that only finite $A_{z_0,g(z_0)}$ intersects this ball. This is true since only the $A_{z_0,g(z_0)}$ corresponding to elements g such that $d_{\mathbb{H}}(z_0, g(z_0)) < 2r$ (which are finite) can intersect P. □

The following refinement is actually true.

Theorem 2.3.6. *The polygon above is a fundamental polygon. Moreover, if Γ is finitely generated, then the polygon P is finite.*

Definition 2.3.7. *Polygons constructed as in Theorem 2.3.5 are called* Dirichlet fundamental polygons.

The following provides a way to build the "new surface" starting from a Dirichlet polygon. It provides the pairing for the sides which allows one to identity points.

Theorem 2.3.8. *Let $\Gamma < \mathrm{Mob}(\mathbb{H})$ be a Fuchsian group and P be a Dirichlet fundamental polygon. If P is finite, then Γ is finitely generated. More precisely, there exists a decomposition of the oriented boundary of P in an even number of geodesic arcs $\delta_1, \dots, \delta_{2n}$, an involution without fixed points of $\{ 1, \dots, 2n \}$ and a generating family $g_1, \dots, g_{2n}$ for Γ such that*

$$g_i(\delta_i) = \delta_{\sigma(i)}^{-1} \text{ and } g_{\sigma(i)=g_i^{-1}}$$

for $i = 1, \dots, 2n$.

Exercise 2.3.9. *Prove the the region of the plane bounded by the two vertical line $\mathrm{Re}\, z = -1/2$ and $\mathrm{Re} = 1/2$ and the circle of radius 1 centred at 0 is a fundamental polygon for the action of the subgroup $PSL_2(\mathbb{Z})$. Find a decomposition of the sides and an involution as in Theorem 2.3.8.*