1
Basics and Moebius transformations

In this first chapter we review basics notions about distances and we introduce the hyperbolic metric on the complex half plane. We then study the group of isometries of this space, the real Moebius transformations, and give a basic classification of these maps. We will basically follow (a subset of) the notes "Hyperbolic geometry" by Charles Walkden.

1.1 Distances

Let us start recalling the definition of *distance* on a metric space.

Definition 1.1.1. *A* metric space *is a couple* (X, d) *where* X *is a set and* d *is a function* $d : X \times X \to \mathbb{R}$ *satisfying*

1. $d(x, y) = d(y, x) \forall x, y \in X$;
2. $d(x, y) = 0 \iff x = y$;
3. $d(x, y) \leq d(x, z) + d(z, y) \forall x, y, z \in X$.

The function d *is called the* distance *on* X.

Example 1.1.2. $\mathbb{R}$ *is a metric space, with the usual distance* $d(x, y) = |y - x|$.

Example 1.1.3. *The set* $\mathbb{C}$ *of complex numbers is a metric space with the distance* $d(x, y) = |y - x|$.

A way to define a distance on a space, that will be useful later, is via the lenght of curves. Let us first give the idea on the complex plane $\mathbb{C}$.

Definition 1.1.4. *An* elementary path *σ in the complex plane $\mathbb{C}$ is the image of a C^1 map $\sigma : [a, b] \to \mathbb{C}$, where $[a, b] \subset \mathbb{R}$ is an interval. The map σ is called a* parametrization *of the path.*

Definition 1.1.5. *Let $f : \mathbb{C} \to \mathbb{R}$ be a continuous function. The* integral of f along an elementary path *σ is*

$$\int_\sigma f := \int_a^b f(\sigma(t)) \, |\sigma'(t)| \, dt,$$

where $|\sigma|(t) := \left((\operatorname{Re}\sigma'(t))^2 + (\operatorname{Im}\sigma'(t))^2\right)^{1/2}$

Remark 1.1.6. *The integral of f along a given path does not depend on the particular parametrization.*

Indeed, let $\sigma : [a, b] \to \mathbb{C}$ and $\gamma : [c, d] \to \mathbb{C}$ be two parametrization of the path. We assertion follows by considering the change of variables given by $\gamma \circ \sigma^{-1}$ in the integral defining the length with respect to the parametrization σ.

In order to define distances by means of the length of curves, we will need to slightly generalize our definition of path.

Definition 1.1.7. *A* path *is the image of a continuous map $\sigma : [a, b] \to \mathbb{C}$, where $[a, b] \subset \mathbb{R}$ is an interval such that there exist a partition $a = t_0 < t_1 < \cdots < t_n = b$ such that the restriction of σ to every $[t_i, t_{i+1}]$ gives an elementary path.*

Definition 1.1.8. *The integral of a continuous function on a path is the sum of the integrals on the elementary paths composing it.*

Proposition 1.1.9. *Let X be an open and path connected subset of $\mathbb{C}$ and g a continuous positive function on X. Then the map*

$$d(x, y) := \inf_\sigma \int_\sigma g(\sigma(t)) \, |\sigma'(t)| \, dt$$

is a distance on X.

Notice that in proving the triangle inequality we use the generalized definition of path.

Exercise 1.1.10. *Prove that the function d as above satisfies the triangle inequality. Conclude that it is a distance function.*

Definition 1.1.11. *Given a path σ connecting a and b, the* length *of σ with respect to the distance (or metric) induced by g is given by*

$$\operatorname{length}_g = \int_a^b g(\sigma(t)) \, |\sigma'(t)| \, dt.$$

We will now define the *hyperbolic half plane* by considering the function $g(z) = \frac{1}{\operatorname{Im}(z)}$ on the upper half plane.

Definition 1.1.12. *Let $\mathbb{H}$ denote the upper half plane, i.e., the subset of $\mathbb{C}$ given by $\{ z \colon \operatorname{Im}(z) > 0 \}$. The* hyperbolic distance *on $\mathbb{H}$ is the one induced as in Proposition 1.1.9 with $g(z) = \frac{1}{\operatorname{Im}(z)}$.*

The hyperbolic length (denoted by $\operatorname{length}_{\mathbb{H}}$) of a path σ in $\mathbb{H}$ is thus given by

$$\operatorname{length}_{\mathbb{H}} = \int_\sigma \frac{1}{\operatorname{Im}(\sigma(t))} |\sigma'(t)| \, dt.$$

Exercise 1.1.13. *Compute the length of the segment joining the points ai and bi in $\mathbb{H}$, with $a, b > 0$.*

Exercise 1.1.14. *Compute the length of the segment joining the points ai and $b + ai$ in $\mathbb{H}$, with $a, b > 0$.*

Definition 1.1.15. *An isometry of a metric space X is a bijective map $g : X \to X$ satisfying*

$$d(g(x), g(y)) = d(x, y)$$

for every $x, y \in X$.

Example 1.1.16. *The identity is an isometry for every metric space. Translations are isometries for the euclidean metric on $\mathbb{C}$.*

We now want to study all the isometries of the hyperbolic half plane. We will see that the set of all these isometries satisfies all the requirements to be a *group*. We first recall the definition.

Definition 1.1.17. *A* group *is a couple $(G, \cdot)$ where G is a set and $\cdot \colon G \times G \to G$ is a binary operation satisfying the following properties:*

1. *$(g \cdot h)l = g \cdot (h \cdot l) \forall g, h, l \in G$(associativity);*
2. *there exists an element $e \in G$ such that $e \cdot g = g \cdot e = g \forall g \in G$ (identity);*
3. *for every $g \in G$ there exist and element $h \in G$ such that $g \cdot h = h \cdot g = e$. This element will be usually denoted by g^{-1} (existence of inverse).*

Example 1.1.18. *The set of integer numbers $\mathbb{Z}$ is a group under the operation of sum. The set of natural numbers is not a group under the same operation.*

Definition 1.1.19. *A subgroup H of G is a subset of G which is still a group.*

Proposition 1.1.20. *Given a metric space (X, d), the set of all isometries of (X, d) is a group under the operation of composition.*

We will not give a precise definition of *orientation preserving* transformation of $\mathbb{C}$.

Remark 1.1.21. *The set of all isometries of $\mathbb{C}$ preserving the orientation is a group under the operation of composition.*

We now want to study the isometries of $\mathbb{H}$.

Definition 1.1.22. *A* Moebius transformation (with real coefficients) *is map of the form*

$$\gamma(z) = \frac{az+b}{cz+d},$$

where $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$. We shall denote by $\mathrm{Mob}(\mathbb{H})$ the set of real Moebius transformations.

Remark 1.1.23. *We may define also Moebius transformations with complex coefficients. Here we only focus on the real case, and therefore call them simply Moebius transformation for simplicity.*

Exercise 1.1.24. *Every Moebius transformation is a self map of the upper half plane.*

Exercise 1.1.25. *Every Moebius transformation is a bijection. Find its inverse.*

Proposition 1.1.26. *The set $\mathrm{Mob}(\mathbb{H})$ is a group under composition.*

Proof. We will just check that the composition of two Moebius tranformation is a Moebius transformation, leaving the rest for exercise. Let γ_1, γ_2 be given by

$$\gamma_1(z) = \frac{az+b}{cz+d} \text{ and } \gamma_2(z) = \frac{Az+B}{Cz+D}.$$

Then

$$\gamma_1 \circ \gamma_2(z) = \frac{a\gamma_2(z)+b}{c\gamma_2(z)+d} = \frac{a\frac{Az+B}{Cz+D}+b}{c\frac{Az+B}{Cz+D}+d} = \frac{a(Az+B)+b(Cz+D)}{c(Az+B)+d(Cz+D)}$$
$$= \frac{(aA+bC)z+(aB+bD)}{(cA+dC)z+(cB+dD)}$$

We just have to check that $(aA+bC)(cB+dD) - (aB+bD)(cA+dC) > 0$. But we have

$$(aA+bC)(cB+dD) - (aB+bD)(cA+dC) = (ad-bc)(AD-BC)$$

and the conclusion follows since by assumption both $(ad - bc)$ and $(AD - BC)$ are positive. □

The above proof suggests a practical way to compute the composition of Moebius transformations. Consider the group $\mathcal{M}$ of real 2×2 matrices with positive determinant (with the usual operation of composition). Given a Moebius transformation $\gamma = \frac{az+b}{cz+d}$ as above, we can associate the matrix given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The condition that $ad - bc > 0$ can be read as a positivity condition on the determinant of the matrix. Conversely, given a 2×2 real matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with positive determinant, the map $\gamma(z) := \frac{az+b}{cz+d}$ is a Moebius transformation. The computation above shows that the operation of composition between

Moebius transformations corresponds to the operation of composition of the associated matrices. Notice that the correspondence is well defined from the matrices to the Moebius transformations while it is not well defined in the opposite direction. The reason is that we can have different representations of the same Moebius map. For instance, let $\lambda \neq 0$. Then the two maps

$$z \mapsto \frac{az+b}{cz+d} \text{ and } z \mapsto \frac{\lambda az + \lambda b}{\lambda cz + \lambda d}$$

give indeed the same map.

In order to solve this ambiguity, we introduce the following definition.

Definition 1.1.27. *A Moebius transformation* $\frac{az+b}{cz+d}$ *is* normalized *if* $ad - bc = 1$.

Once we restrict to normalized Moebius transformations, we are naturally led to consider the following subgroup of matrices.

Definition 1.1.28. *The* special linear group of $\mathbb{R}^2$ *is the group given by*

$$SL_2(\mathbb{R}) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R}, ad - bc = 1\}.$$

Exercise 1.1.29. *Check that* $S_2(\mathbb{R})$ *is indeed a group with the operation of composition.*

Matrices in $SL_2(\mathbb{R})$ correspond to normalized Moebius transformations.

We still have an ambiguity when considering normalized Moebius transformations: we may multiply all the coefficient by $\lambda = -1$. This is solved by considering the group of matrices $PSL_2(\mathbb{R})$ given by $SL_2(\mathbb{R})$, where we *identify* the elements A and $-B$. This group is called the *projective special linear group*. You can check that now the correspondence between normalized Moebius transformations and elements in $PSL_2(\mathbb{R})$ is well defined.

Definition 1.1.30. *We will denote by* $\mathrm{Isom}(\mathbb{H})$ *the group of isometries of* $\mathbb{H}$ *which are homeomorphisms. Similarly, we will denote by* $\mathrm{Isom}^+(\mathbb{H})$ *the group of orientation preserving isometries of* $\mathbb{H}$ *which are homeomorphisms.*

Proposition 1.1.31. $\mathrm{Mob}(\mathbb{H}) \subset \mathrm{Isom}(\mathbb{H})$. *More precisely,* $\mathrm{Mob}(\mathbb{H})$ *is a subgroup of* $\mathrm{Isom}(\mathbb{H})$.

Proof. We have to prove that every Moebius transformation preserves the hyperbolic distance, i.e., that given any Moebius transformation $\gamma(z) = \frac{az+b}{cz+d}$, we have $d_{\mathbb{H}}(\gamma(z), \gamma(z')) = d_{\mathbb{H}}(z, z')$ for every $z, z' \in \mathbb{H}$. We will just write d for $d_{\mathbb{H}}$ for simplicity. In order to prove the assertion it is enough to prove that, given any path $\sigma : [a, b] \to \mathbb{C}$ between σ between z and z', we have

$$\text{length}(\gamma \circ \sigma) = \text{length}(\sigma).$$

We will use the following two identity, which are proved by a straightforward computation

$$|\gamma'(z)| = \frac{ad - bc}{|cz + d|^2} \text{ and } \operatorname{Im} \gamma(z) = \frac{ad - bc}{|cz + d|^2} \operatorname{Im}(z).$$

We then have, by the chain rule

$$\begin{aligned}\text{length}(\gamma\circ\sigma) &= \int_a^b \frac{|(\gamma\circ\sigma)'(t)|}{\text{Im}(\gamma\circ\sigma)(t)}dt = \int_a^b \frac{|(\gamma'(\sigma(t))|\,|\sigma'(t)|}{\text{Im}(\gamma\circ\sigma)(t)}dt \\ &= \int_a^b \frac{ad-bc}{|c\sigma(t)+d|^2}\,|\sigma'(t)|\,\frac{|c\sigma(t)+d|^2}{ad-bc}\,\frac{1}{\text{Im}(\sigma(t))}dt \\ &= \int_a^b \frac{|(\sigma)'(t)|}{\text{Im}(\sigma)(t)}dt = \text{length}(\sigma)\end{aligned}$$

□

Remark 1.1.32. *The reflection with respect to the imaginary axis is an isometry of $\mathbb{H}$ (and $\mathbb{C}$), but is not a Moebius tranformation. More generally, every isometry that does not preserve the orientation cannot be a Moebius transformation.*

It can be proved that Moebius transformations are actually all orientation preserving bijective isometries of the hyperbolic plane. We will not detail the proof of the following result.

Proposition 1.1.33. $\text{Mob}(\mathbb{H}) \cong \text{Isom}^+(\mathbb{H})$

We now introduce another space, the *Poincaré disc* endowed with a distance, in the same way we did for the hyperbolic half plane. We start by noticing that the map

$$h(z) := \frac{-z+1}{-iz+1} \tag{1.1}$$

is a holomorphic invertible map between $\mathbb{D}$ and $\mathbb{H}$.

Exercise 1.1.34. *Prove the above statement.*

Definition 1.1.35. *Let $\mathbb{D}$ denote the unit disc in $\mathbb{C}$. The* Poincaré distance *on the disc is the one induced as in Proposition 1.1.9 by the function $g(z) := \frac{1}{\text{Im}(h(z))}$.*

Notice that, *by definition*, the distance of two points in the Poincaré disc is the same as the distance between their images under h in the hyperbolic plane $\mathbb{H}$. This allows one to easily characterise the group of (orientation preserving) isometries $\text{Isom}^+(\mathbb{D})$ of the Poincaré disc. Indeed, let γ be an element of $\text{Isom}^+(\mathbb{H}) = \text{Mob}(\mathbb{H})$. If we consider the map $h^{-1}\circ\gamma\circ h$, it is easy to check that this is an isometry of $\mathbb{D}$. Indeed, denoting $d_{\mathbb{H}}$ the hyperbolic distance on $\mathbb{H}$ and by $d_{\mathbb{D}}$ the induced distance on $\mathbb{D}$, we have

$$\begin{aligned} d_{\mathbb{D}}(h^{-1}\circ\gamma\circ h(z), h^{-1}\circ\gamma\circ h(w)) &= d_{\mathbb{H}}(\gamma\circ h(z),\gamma\circ h(w)) \text{ by definition of } d_{\mathbb{D}} \\ &= d_{\mathbb{H}}(h(z),h(w)) \text{ since } \gamma \text{ is an isometry on } \mathbb{H} \\ &= d_{\mathbb{D}}(z,w) \text{ again by definition of } d_{\mathbb{D}}.\end{aligned}$$

Viceversa, for every isometry δ of $\mathbb{D}$, the map $h\circ\delta\circ h^{-1}$ is an isometry of $\mathbb{H}$ (by essentially the same proof).

Exercise 1.1.36. *Let $\gamma \in \mathrm{Mob}(\mathbb{H})$. Show that $h^{-1} \circ \gamma \circ h$ is a map of the form*

$$z \mapsto \frac{\alpha z + \beta}{\overline{\beta} z + \overline{\alpha}},$$

with $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 - |\beta|^2 > 0$.

Definition 1.1.37. *We denote by* $\mathrm{Mob}(\mathbb{D})$ *the set of maps as above. They are the* Moebius transformations *of* $\mathbb{D}$.

Exercise 1.1.38. *Check that* $\mathrm{Mob}(\mathbb{D})$ *is a group with the operation of composition.*

Corollary 1.1.39. $\mathrm{Mob}(\mathbb{D}) = \mathrm{Isom}^+(\mathbb{D})$.

The map h provides in some sense a "dictionary" between $\mathbb{H}$ and $\mathbb{D}$.

1.2 Classification of Moebius transformations

In this section we want to give a rough classification of Moebius transformation. Since we mainly have in mind dynamical application, the first interest is to identify the points which are left fixed by these maps. This can be done by a straightforward computation.

In the following, it will be useful to consider the *boundary* $\partial\mathbb{H}$ of $\mathbb{H}$, given by the real line and a point at infinity. Analogously, the boundary $\partial\mathbb{D}$ of $\mathbb{D}$ can be identified with the points on the unit circle.

Exercise 1.2.1. *Every Moebius transformation induces a bijective map on the boundary.*

In the case of the half-plane $\mathbb{H}$, the image of the point ∞ under the map $\gamma(z) = \frac{az+b}{cz+d}$ is given by $\frac{a}{c}$ (in particular, it is ∞ if $c = 0$). The preimage of ∞ is the point $-\frac{d}{c}$ (again, it is ∞ if $c = 0$).

Proposition 1.2.2. *Let γ be an element of* $\mathrm{Mob}(\mathbb{H})$ *different from the identity. Then γ has either*

1. *two fixed points in $\partial\mathbb{H}$ and none in $\mathbb{H}$;*

2. *one fixed point in $\partial\mathbb{H}$ and none in $\mathbb{H}$;*

3. *no fixed points in $\partial\mathbb{H}$ and one in $\mathbb{H}$.*

Proof. Let $\gamma(z) = \frac{az+b}{cz+d}$ be a Moebius transformation, and assume it is not the identity.

Let us first consider the case when ∞ is fixed. By the discussion above, this is equivalent to have $c = 0$ (notice that this implies that $d \neq 0$). The map γ is thus given by $\gamma(z) = \frac{a}{d}z + \frac{b}{d}$. The point $\frac{b}{d-a}$ is thus a (possibly different) fixed point (which is still ∞ if $a = d$). In any case, notice that this second fixed point necessarily lies on $\partial\mathbb{H}$. We thus have the first or the second possibility of the statement.

Suppose now that ∞ is not fixed (which is the same to say that $c \neq 0$). If z_0 is a fixed point, we can write

$$z_0(cz_0 + d) = az_0 + b.$$

It follows that z_0 can take (at most) two values: both reals and different (which means that we have two solutions in the boundary), both real and coincident (which means we have a unique fixed point on the boundary) or two non-real complex conjugates. Since exactly one of the two lies in $\mathbb{H}$, this gives the last case of the statement. $\square$

Corollary 1.2.3. *If an element of* $\mathrm{Mob}(\mathbb{H})$ *fixes three points, this element is the identity.*

Definition 1.2.4. *Let* $\gamma \in \mathrm{Mob}(\mathbb{H})$. *We say that*

1. γ *is hyperbolic if it has two fixed points in* $\partial\mathbb{H}$ *and none in* $\mathbb{H}$*;*
2. γ *is parabolic if it has one fixed point in* $\partial\mathbb{H}$ *and none in* $\mathbb{H}$*;*
3. γ *is elliptic if it has one fixed point in* $\mathbb{H}$ *and none in* $\partial\mathbb{H}$.

Exercise 1.2.5. *Give an example of a Moebius transformation of each type.*

Definition 1.2.6. *Let* $\gamma_1, \gamma_2 \in \mathrm{Mob}(\mathbb{H})$ *be two Moebius transformations of* $\mathbb{H}$. *We say that* γ_1 *and* γ_2 *are conjugate in* $\mathrm{Mob}(\mathbb{H})$ *if there exists a Moebius transformation* $g \in \mathrm{Mob}(\mathbb{H})$ *such that* $\gamma_1 = g^1 \circ \gamma_2 \circ g$.

Exercise 1.2.7. *Conjugacy is an equivalence relation.*

Exercise 1.2.8. *If two elements are conjugated, they have the same number of fixed points. Hence, they are of the same type (as in Definition 1.2.4).*

In the case of matrices, there are a number of invariants (with respect to the classes of conjugation) that we can consider, i.e., numbers associated to every matrix which do not depend on the particular elements in the class chosen to compute them. Basic examples are the determinant and the trace.

Exercise 1.2.9. *Prove that the determinant and the trace are invariant under conjugation in the group of invertible* 2×2 *matrices with complex entries.*

Given the Moebius transformation $\gamma(z) = \frac{az+b}{cz+d}$, we have seen that it is meaningful to associate to it the matrix given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Remember that by definition we have $ad - bc \neq 0$. We can actually multiply all the entries and assume that $ad - bc = 1$. This does not change the transformation. We thus see that considering the determinant of the associated matrix is not very meaningful. On the other hand, we are going to see that something very related to the trace $a + d$ will be of great use in the classification. However, notice again that multiplying all the entries by -1 does not change the elements, while the trace of the associated matrix would change sign. To solve this problem, we are thus led to consider the trace of the square of the associated matrix.

Definition 1.2.10. *Let $\gamma \in \mathrm{Mob}(\mathbb{H})$ be a Moebius transformation of $\mathbb{H}$ with $\gamma(z) = \frac{az+b}{cz+d}$, where $ad - bc = 1$. The* trace *of γ is given by $\tau(\gamma) = (a+d)^2$.*

When we will talk of the trace of a Moebius transformation, we will always mean the trace of (any) associated normalized element.

Exercise 1.2.11. *Check that the trace of a normalized Moebius transformation is the square of the trace of any associated matrix.*

The following result gives the desired invariance of the trace.

Proposition 1.2.12. *The trace is invariant under conjugation.*

Exercise 1.2.13. *Prove the above proposition.*

We can now see the relation between the trace and the classification by means of the fixed points given above.

Proposition 1.2.14. *Let $\gamma \in \mathrm{Mob}(\mathbb{H})$ be a Moebius transformation of $\mathbb{H}$ different from the identity, given by $\gamma(z) = \frac{az+b}{cz+d}$ and such that $ad - bc = 1$. Then:*

1. *γ is parabolic if and only if $\tau(\gamma) = 4$;*
2. *γ is elliptic if and only if $\tau(\gamma) \in [0, 4)$;*
3. *γ is hyperbolic if and only if $\tau(\gamma) \in (4, \infty)$.*

Proof. Let us first consider the case when ∞ is not a fixed point (i.e., when $c \neq 0$). A direct computation shows that is in this case the fixed points are given by the formula

$$z_0 = \frac{a - d \pm \sqrt{(a-d)^2 + 4bc}}{2c}.$$

The number and type (real or not) of the solution thus depends on the sign of the quantity $(a-d)^2 + 4bc$, which is seen to be equal to $\tau(\gamma) - 4$. the assertion follows in this case.

If $c = 0$, ∞ is a fixed point and the other one is given by $\frac{b}{d-a}$. Thus, ∞ is the only fixed point if and only if $d = a$, which implies $ad = ad - bc = ad - 0 = 1$. Thus we have $a = d = 1$ or $a = d = -1$. In both cases, $\tau(\gamma) = 4$. If on the other hand $a \neq d$ there are two fixed points on the boundary and $\tau(\gamma) = (a+d)^2 \geq 4ad = 4$ (since $c = 0$), where the inequality is a consequence of the arithmetic-geometric inequality. □

We end this section by looking for *normal forms* for elements in the different conjugacies classes, i.e., elements in every class which has a particularly easy form.

We start with the parabolic case.

Exercise 1.2.15. *We call a* translation *a Moebius transformation of the form $\gamma(z) = z + b$.*

1. *Check that any translation is parabolic.*
2. *Prove that if $b > 0$ then γ is conjugated to $z \mapsto z + 1$.*

3. *Prove that if $b < 0$ then γ is conjugated to $z \mapsto z - 1$.*
4. *Prove that $z \mapsto z + 1$ and $z \mapsto z - 1$ are not conjugated (as real Moebius transformations).*

The next result shows that, when considering real Moebius transformation, the two normal forms above are actually the only ones we need to understand every parabolic element.

Proposition 1.2.16. *Let $\gamma \in \mathrm{Mob}(\mathbb{H})$ different from the identity. Then the following are equivalent:*

1. *γ is parabolic;*
2. *the trace of a normalization of γ is equal to 4;*
3. *γ is conjugate to a translation;*
4. *γ is conjugate either to the translation $z \mapsto z + 1$ or to the translation $z \mapsto z - 1$.*

Proof. The first two assertions are equivalent by Proposition 1.2.14 and the last two by Exercise 1.2.15.

Assume that γ is conjugate either to the translation $z \mapsto z + 1$ (the other case is completely analogous). Since this map as ∞ as the only fixed point, it is parabolic. Since the type is invariant by conjugation, γ is parabolic, too.

To conclude, it is enough to prove that any parabolic transformation γ is conjugated to some translation. We work in two steps:

1. Let ζ the unique fixed point of γ (which belongs to $\partial\mathbb{H}$). Let g be a Moebius transformation that maps ζ to ∞ (for example $g = \frac{1}{z-\zeta}$). Then $g\gamma \circ g^{-1}$ is a Moebius transformation fixing (only) ∞.
2. We claim that any Moebius transformation β fixing only ∞ is a translation.

The first assertion is immediate. Let us prove the second claim. Write β as $\beta(z) = \frac{az+b}{cz+d}$. Since ∞ is fixed, we have $c = 0$. The other fixed point is given by $\frac{b}{d-a}$, which implies (since the only fixed point is ∞) that $a = d$. Thus $\beta(z) = z + \frac{b}{a}$, and the assertion follows. □

We can give an analogous description for hyperbolic and elliptic maps. We will not give the details, but leave them for exercise (all details can be found in [**?**, Chapter ???]).

Exercise 1.2.17. *Prove that two dilations $z \mapsto k_1 z$ and $z \mapsto k_2 z$ are conjugated if and only if $k_1 = k_2$ or $k_1 = 1/k_2$.*

Proposition 1.2.18. *Let $\gamma \in \mathrm{Mob}(\mathbb{H})$. Then the following are equivalent:*

1. *γ is hyperbolic;*

2. *the trace of a normalization of* γ *is larger than* 4*;*

3. γ *is conjugate to a dilation, i.e.* γ *is conjugate to a Moebius transformation of* $\mathbb{H}$ *of the form* $z \mapsto kz$*, for some* $k > 0$*.*

To study elleptic maps, it is more convenient to work with the disc model. We first need to define the trace for Moebius transformations of the disc. We can do this in two ways. We can define it to be the trace of the corresponding element of $\mathrm{Mob}(\mathbb{H})$ by the map h given in (1.1). Or, we can define it as the square of $\alpha + \overline{\alpha}$, where α and $\overline{\alpha}$ are the diagonal coefficients of a normalization of the element. It is possible to prove that these two definitions coincide. Thus, an element of $\mathrm{Mob}(\mathbb{H})$ is parabolic, elliptic or hyperbolic (these notions are defined by means of the fixed points in the same exact way as on $\mathrm{Mob}(\mathbb{H})$) if the trace is respectively equal to 4, smaller or larger than 4.

Proposition 1.2.19. *Let* $\gamma \in \mathrm{Mob}(\mathbb{D})$ *be a Mobius transformation of* $\mathbb{D}$*. The following are equivalent:*

1. γ *is elliptic;*

2. $\tau(\gamma) \in [0, 4)$*;*

3. γ *is conjugate to a rotation* $z \mapsto e^{i\theta} z$*.*

Exercise 1.2.20. *Let* $h : \mathbb{D} \to \mathbb{H}$ *as in* (1.1)*. Let* $\gamma(z) = e^{i\theta} z \in \mathrm{Mob}(\mathbb{D})$*. Then* $h \circ \gamma \circ h^{-1}$ *is an element of* $\mathrm{Mob}(\mathbb{H})$*. Prove that* $h \circ \gamma \circ h^{-1}$ *is of the form*

$$z \mapsto \frac{\cos(\theta/2) z \sin(\theta/2)}{-\sin(\theta/2) z + \cos(\theta/2)}.$$

Maps of this form are called rotations *of* $\mathbb{H}$*.*