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Thermodynamics and Bifurcations in Several Complex Variables

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Abstract

This habilitation thesis covers a large part of the research I have carried out since the end of my PhD. The main topic is the study of dynamical systems in several complex variables, mostly by means of pluripotential theory techniques. In the first part I describe my works related to the characterization of stability and bifurcation in families of endomorphisms of $\mathbb{P}^k(\mathbb{C})$. In the second part, I describe the results related to the statistical study of endomorphisms of $\mathbb{P}^k(\mathbb{C})$ and Hénon maps. In the last part, I collect the results which are not directly related to these two main directions, or that, on the opposite, rely on both.

Résumé

Cette thèse d'habilitation couvre une grande partie des mes travaux depuis la fin de ma thèse de doctorat. Le sujet principal est l'étude des systèmes dynamiques à plusieurs variables complexes, principalement au moyen de techniques de la théorie du pluripotentiel. Dans la première partie je décris mes travaux liés à la caractérisation de la stabilité et de la bifurcation dans les familles d'endomorphismes de $\mathbb{P}^k(\mathbb{C})$. Dans la deuxième partie, je présente les résultats liés à l'étude statistique des endomorphismes de $\mathbb{P}^k(\mathbb{C})$ ainsi que des applications de Hénon. Dans la dernière partie, je rassemble les résultats qui ne sont pas directement liés à ces deux directions principales, ou qui, au contraire, s'appuient sur les deux.

Acknowledgments

First of all, I would like to thank Viêt-Anh Nguyên for his role as my mentor since my arrival in Lille, for all the valuable advice and support, and for agreeing to be my *garant* for this habilitation. I am honored to have Romain Dujardin, Mikhail Lyubich, and George Marinescu as referees for this manuscript. Thank you for your work, your comments, and all the discussions we have had over the past years. I would also like to express my gratitude to François Berteloot, Laura DeMarco, Carlangelo Liverani, and Anna Zdunik for accepting to be part of this defense committee. I am deeply honored by your presence today, as your work has been an inspiration for a large part of the results presented in this manuscript.

Writing this manuscript was an occasion for me to reflect on the seven years I spent in London and Lille. While these experiences cannot be summarized in a few lines, I want to take this opportunity to thank those people who had the greatest impact on what, in hindsight, has found its way into these pages.

Some time ago, I was reminded that there are no 'ex students', but only 'students'. I agree, and I would like to begin by thanking my PhD advisors, Marco Abate and François Berteloot. Far from feeling released from their roles on the day of my PhD defense, they have continued to be a constant source of good advice – and they still are. I am proud to be called your student and grateful that the prefix 'ex' will never be added.

I would like to thank Davoud Cheraghi and Sebastian van Strien for welcoming me in London and for their constant support since then. The opportunity to do my postdoc in your group has been invaluable, and many of the projects described in this manuscript were initiated during my time in London. I extend my gratitude to you, the DynamIC group, and Imperial College London for having made this possible.

The Laboratoire Paul Painlevé has made me feel at home since the day I arrived. I am grateful to all the members for their hearty welcome and constant support, and for the *bonne ambiance* that made working there both enjoyable and productive. I especially thank Viêt-Anh Nguyên and Volker Mayer, as well as the other members of the holomorphic dynamics group, François Bacher, Sahil Gehlawat, and Karim Rakhimov, and of the *équipe d'analyse*. I also thank Stephan de Bièvre, Benoit Fresse, Emmanuel Fricain, and Sophie Grivaux, for all the good advice and continuous support, as well as all the administrative team for their constant presence and help.

This manuscript contains a selection of results obtained in collaboration with several people. Without Marco Abate, Matthieu Astorg, François Berteloot, Maxence Brévard, Tien-Cuong Dinh, Christophe Dupont, Yan Mary He, Samuele Mongodi, Yûsuke Okuyama, Karim Rakhimov, and Johan Taflin, many of these pages would simply not exist. I thank all of you for your ideas, enthusiasm, patience, and for being constant sources of inspiration, as well as everyone else with whom I have had the opportunity to discuss and learn. I hope that many of you will soon join the list above. Among the too many not already mentioned, I would like to at least especially thank Eric Bedford, Filippo Bracci, Xavier Buff, Charles Favre, Livio Flaminio, Feliks Przytycki, and Julio Rebelo, also for their continuous support and precious advice, as well as Sébastien Biebler, Nguyen-Bac Dang, Thomas Gauthier, Lucas Kaufmann, Jasmin Raissy, Matteo Ruggiero, Gabriel Vigny, Duc Viet Vu, and all the members of the ANR project PADAWAN and the ANR-DFG project QuaSiDy.

I could not complete this page without especially thanking Tien-Cuong Dinh for his constant role as a mentor and for all the advice and support, since our first meeting in Paris in 2012. It is an honor for me to be able to learn from and collaborate with you. When I accepted your first invitation to Singapore, I could not imagine what was about to start.

Finally, I would like to thank my family and all my friends for their constant and unconditional support – whenever I need it, whatever I do, wherever I am.

Contents

Introduction

This manuscript is a survey on the main results that I obtained after my PhD thesis, which was defended in 2016. The domain is that of complex dynamics in several variables, at the interface of dynamical systems, complex analysis, and geometry. The main tools are (pluri)potential theory and the theory of (positive closed) currents.

One of my main topics of interest since my PhD thesis has been the dependence of holomorphic dynamical systems from a parameter and the study of the discontinuity of dynamically relevant objects for special values of the parameter itself. In my PhD thesis, I contributed to the generalization of the classical theory by Lyubich, Mañé-Sad-Sullivan, and DeMarco about the stability of rational maps to the more general setting of endomorphisms of $\mathbb{P}^k(\mathbb{C})$ [BBD18; Bia19a] and proved the first result on completely parabolic implosion in two complex variables [Bia19b]. Since then, and in continuation of those results, I started a systematic exploration of bifurcations [BB18a; BB18b; AB23; BO20; BR22; BB23b], in particular exhibiting new phenomena with respect to the one-dimensional theory [BT17; AB22]. I describe most of my research in this direction in Chapter 1. After an overview of the context and of the motivations, I briefly recall the main results of [BBD18; Bia19a], which set the stage for the rest of my work on bifurcations, and list a few natural open questions. A description of the main results obtained in [BT17; BB18a; AB23; AB22; BR22] is then given in Sections 1.2, 1.3, and 1.4.

The second main research direction that I pursued has been the study of statistical properties of complex dynamical systems. In 2018, together with Tien-Cuong Dinh, we started a long term project involving the systematic study of the thermodynamics of endomorphisms of projective spaces in any dimensions, with a new approach based on pluripotential theory and functional analysis. This study already led to a unified treatment of essentially all statistical properties previously known for continuous observables, in a much larger generality and with sharper statements [BD23a; BD22]. I describe these results in Section 2.2. More recently, we also started investigating the statistical properties of the measure of maximal entropy of complex Hénon maps, and of automorphisms of compact Kähler manifolds. The results already obtained in this direction are contained in [BD24; BD23b] and are described in Section 2.3.

In parallel to the main directions above, I worked on a number of – related or less related – other research projects. I summarize in Chapter 3 the results that do not fit in the two large directions above, but that still have dynamical content, or that, on the opposite, involve both the two topics.

I describe the proof of the monotonicity of the dynamical degrees for polynomial-like and horizontal-like maps in any dimension obtained in [BDR23] in Section 3.3 and the introduction of a dynamical *volume dimension* satisfying a generalized version of the Mañé-Manning formula in any dimensions [BH23] in Section 3.4. I also postpone to this chapter the description of the results of the two papers [BB23b] and [BO20]. The first is directly related to the topic of stability and bifurcation, but relies both on the paper [BR22] described in Section 1.4 and on [BD23a], described in Section 2.2. The second, although also related to bifurcations, has a different motivation, and will be then described separately here. Parts of the papers [BR22] and [BD23a] are also described in this chapter, as they are close to [BDR23] and [BB23b], respectively.

Plan of the manuscript and brief description of the results

The first two chapters of this manuscript are essentially self-contained, and each of them will start with an introduction on the general motivations and the context. On the other hand, I give below an overview of each of them. The sections below should be taken more as a road map to the reading of these chapters, also explaining how some projects emerged and the relations between parts which are a priori far in the manuscript. The reader is invited to refer to them for the precise definitions, the statements of the results, and the references. As mentioned above, Chapter 3 is of a different flavour; I will highlight below the relations between the parts of this chapter and the rest of the manuscript, both on the level of the techniques and content, and of how such projects started and developed.

Chapter 1 - Stability and bifurcations in several complex variables

The main topic of Chapter 1 is the study of *families* of dynamical systems, and more precisely of the properties that persist – or not – under a perturbation of an element of the family. After a brief introduction on the motivations and the context, Chapter 1 takes the move from Theorem 1.1.1. This result, obtained with François Berteloot and Christophe Dupont [BBD18] and contained in my PhD thesis, provides a generalization to families of endomorphisms of $\mathbb{P}^k$ in any dimension of the classical dichotomy *stability-bifurcation* due to Lyubich, Mañé-Sad-Sullivan, and DeMarco for families of rational maps. I refer to Section 1.1.1 for a presentation of this result and its context. In Section 1.1.2, I list a few natural questions that immediately arise when comparing the characterizations of stability in dimension 1 with those valid in any dimension given by Theorem 1.1.1. Addressing these questions, and in general developing a general theory of stability and bifurcation in higher dimensions, has been a main motivation for my work since my PhD thesis.

Section 1.2 presents those results which are the closest to my PhD thesis, both as content and date. I first present a work with Johan Taflin [BT17], where we gave an explicit example of a family f_λ of endomorphisms of $\mathbb{P}^2$ for which the chaotic *Julia set* of f_λ move continuously (in the Hausdorff topology) with the parameter, but at the same time the bifurcation locus coincides with the parameter space. This example shows, at the same time, two main differences between the stability-bifurcation theories in dimension 1 and higher. Although the relation between Hausdorff continuity and stability is not completely clear yet, the existence of open sets of bifurcation, even in the full family of endomorphisms of a given degree, was established independenly from [BT17] by Dujardin, Taflin, and Biebler and has emerged as a main specific feature of the higher-dimensional theory. I give more details on [BT17] and of the related results in Section 1.2.1.

In Section 1.2, I also describe an estimate of the Hausdorff dimension of the bifurcation locus, valid everywhere in the parameter space, and optimal near the so-called *isolated Lattès maps*, that I obtained with François Berteloot [BB18a]. The proof is based on a geometric *transfer* construction between the phase space and the parameter space. I give more details on this in Section 1.2.2.

In Section 1.3 I describe the two works [AB23] and [AB22], which are the result of a long term collaboration with Matthieu Astorg. The original idea of this project, started when we were both PhD students in Toulouse, was to study more precisely the dichotomy stability-bifurcation from [BBD18], as well as other dynamical properties, in families of *polynomial skew product*, i.e., polynomial endomorphisms of $\mathbb{P}^2$ of the form $(z,w) \mapsto (p(z), q(z,w))$. Astorg had a more

1-dimensional background, and these families looked like a natural ground of collaboration. Moreover, Astorg had just worked with maps of this type for the construction of wandering domains. As a result of this collaboration, we first established some decomposition formulas for the bifurcation locus and current in these families, we proved that *hyperbolicity* is preserved in stable components, and we classified a family of hyperbolic components in their parameter space, see Section 1.3.1. We also established an approximation formula for the bifurcation current, which is actually valid in any family of endomorphisms, by means of dynamically defined hypersurfaces in the parameter space, see Section 1.3.2. All the above is contained in [AB23].

Thanks to the study above, we realized that a new phenomenon had to occur in these families with respect to the one-dimensional setting: as soon as some bifurcation was created, a cascade of bifurcations had to happen. A precise way to say this is that the support of the *bifurcation current* is equal to the support of the *bifurcation measure*, which could a priori be (and indeed is, in dimension 1) a smaller set in the parameter space. Our proof, given in [AB22] and described in Section 1.3.3, is based on both an analytical criterion to ensure that a point belongs to the support of the bifurcation measure, and on a geometric construction to verify the criterion at a given parameter as soon as one simple bifurcation appears nearby. This geometric construction was the key point of [AB22], and possibly the main result in our collaboration with Astorg so far. In order to complete our arguments, we also had to use some tools coming from the theory of thermodynamic formalism. Although independent from the results in this direction that will be presented in Chapter 2, it turned out to be very useful that I had started being more interested in that direction by that time.

The last result presented in Chapter 1 is the outcome of my collaboration [BR22] with my postdoc Karim Rakhimov, who came to Lille in 2020-2022. The question was motivated by a result by Berger-Dujardin in the setting of the stabilility-bifurcation dichotomy that had been developed in parallel to [BBD18] by Dujardin and Lyubich for families of *Hénon maps*, i.e., polynomial diffeomorphisms of $\mathbb{C}^2$. More precisely, the goal was to extend the measurable holomorphic motion established in [BBD18] for almost all points with respect to the measure of maximal entropy, to almost all points with respect to all measures of – for instance - sufficiently large entropy.

I give more details on this result in Section 1.4. A point I would like to emphasize here is the following. Already in my PhD thesis, I had established a generalization of the theory of [BBD18] to the much larger setting of *polynomial-like maps of large topological degrees* [Bia19a]. More than the generalization itself, this result showed that the algebraicity of the maps was not an essential tool in the theory. Our result with Rakhimov also holds in these more general families. On the other hand, we had to put an a priori stronger assumption on the *dynamical degrees* of the elements of the family. The question of whether this assumption was really needed, or was just implied by being of *large topological degree*, was a natural one. It turned out that the correct answer is the second, as we proved with Tien-Cuong Dinh [BDR23], after exchanges motivated by the above result. The description of this result is given in Section 3.3.1, see also below for a brief introduction with further motivations. I also postpone to Section 3.3.2 the presentation of the result with Rakhimov in this more general context.

As mentioned above, the works [BO20], obtained in collaboration with Yûsuke Okuyama, and [BB23b], obtained with my PhD student Maxence Brévard, are also related to bifurcations. For the reasons given above, I decided to postpone their presentation to Chapter 3. For coherence, I will describe them below also in this introduction.

Chapter 2 - Thermodynamics and statistical properties of equilibrium states

Chapter 2 is dedicated to the second large topic of my research since my PhD thesis in 2016 – and possibly the main one since 2018. This is the study of holomorphic dynamical systems from a statistical point of view. Roughly speaking, this means the construction of natural invariant measures associated to such systems, and the study of the orbits of typical points with respect to such measures. The general goal is to establish limit theorems, paralleling those of sequences of independent random variables, for the sequence $\{g \circ f^n\}$ – where g is an *observable* and f is the dynamical system – seen as a sequence of random variables with respect to the invariant measures that we consider. For the sake of this introduction, these measures will depend on a given function φ, that we will call *weight*, and will be called *equilibrium states* associated to φ, and denoted by μ_φ. This chapter contains the two pairs of works [BD23a; BD22] and [BD24; BD23b], all obtained in collaboration with Tien-Cuong Dinh.

Following the work of Brolin, Lyubich, Freire-Lopes-Mañé, Denker-Przytycki-Urbański, Przytycki-Smirnov-Rivera-Letelier, Haydn, Zdunik, and others, the study of the existence, uniqueness, and statistical properties of equilibrium states in the setting of rational maps is now a classical subject. By (even more) classical techniques, the question of the existence of the equilibrium states can be reduced to showing the convergence $\lambda^{-n}\mathcal{L}^n_\varphi(g) \to \rho_\varphi$ for some suitable $\lambda = \lambda(\varphi)$, where $\mathcal{L}^n_\varphi$ is the n-th iterate of the (Ruelle-Perron-Frobenius) *transfer operator* $\mathcal{L}_\varphi$, defined as

$$\mathcal{L}_\varphi(g)(y) = \sum_{f(x)=y} e^{\varphi(x)} g(x).$$

And a way to approach this problem is the following. One first shows the existence of a good scaling factor λ, usually by means of some fixed point theorem, and then tries to show that the sequence $\lambda^{-n}\mathcal{L}^n_\varphi(g)$ is uniformly bounded and equicontinuous. By means of the theory of almost periodic operators, first applied to the thermodynamics formalism by Lyubich, this leads to a unique limit function $\rho = \rho_\varphi$. And a rough idea is that, the better one controls the convergence towards ρ, the more precise limit theorems one can prove for the equilibrium state associated to φ. In order to show the equicontinuity of the sequence, one has to control the regularity of the function $\mathcal{L}^n_\varphi(g)$. By its very definition, this amounts to precisely estimate the distortion of the inverse branches of f^n.

The starting point of the project that lead to [BD23a; BD22] was a paper by Urbański-Zdunik, where they proved the existence and uniqueness of the equilibrium states associated to generic endomorphisms of $\mathbb{P}^k$ (in any dimension k) and Hölder continuous weights. Their proof, somehow in the spirit of dimension 1, consists in precise estimates on the regularity of inverse branches of endomorphisms. Our first goal in [BD23a] was to give a new proof of the result by Urbański-Zdunik by means of a new method, based instead on pluripotential theory. Let us take g equal to function $\mathbb{1}$ constantly equal to 1 for simplicity. The main idea behind the method is that, instead of controlling the inverse branches of f^n, we consider the currents $dd^c\mathcal{L}^n_\varphi\mathbb{1}$. In order to show that the sequence $\mathcal{L}^n_\varphi\mathbb{1}$ is (up to some normalization) equicontinuous, we show that

$$|dd^c\mathcal{L}^n_\varphi\mathbb{1}| \lesssim (\min \mathcal{L}^n_\varphi\mathbb{1}) \cdot |dd^c h|,$$

where h is an explicit continuous functions, with controlled modulus of continuity. Thanks to more standard comparisons principles, this leads to the equicontinuity of the sequence. The comparisons principles will be given in Section 2.2.2, and a simplified version of our construction will be presented in Section 2.2.3.

The strength of our method comes from a number of features which, combined, lead to a number of consequences. One of them is that it does not rely on a fixed point theorem in order to find the scaling factor λ, but this is constructed as part of the method. This makes it very flexible, in the sense that it is very suitable to study the dependence of the estimates and of the equilibrium states from the weight φ. This point is crucial for the statistical study. Another point is that it naturally allows one to start looking for a norm with the property that, when both g and φ are bounded in this norm, the same is true for $\mathcal{L}_\varphi^n g$, and the operator $\lambda^{-1}\mathcal{L}_\varphi$ is a contraction with respect to this norm (outside of an invariant line). This is the so called *spectral gap* property. Together with the possibility of precisely controlling the dependence on φ, this gives a spectral gap for the transfer operator *and its perturbations*, possibly the strongest control that one can get on the transfer operator. Indeed, this allows one to control very precisely the iteration of the operator and deduce the most complete statistical study.

Since all the statistical properties follow from the spectral gap by means of more standard techniques, I will not detail this point in the manuscript, but I just give here a rough motivation of why this should be true. It is a general idea in probability that the statistical properties of a random variable X (with respect to a given measure ν) are related to the generating function $t \mapsto \mathbb{E}(e^{tX})$, i.e., to $t \mapsto \langle \nu, e^{tX} \rangle$. More precisely, the statistical properties of X are somehow encoded in the Taylor coefficients of the development in t of this function. In our case, the random variable X will be given by the sum $S_n(u) = \sum_{j=1}^n u \circ f^i$, for $u \colon X \to \mathbb{R}$ an observable of a given regularity. By the form of the transfer operator, we have $\langle \mu_\varphi, e^{tS_n(u)} \rangle = \langle \mu_\varphi, \lambda^{-n} \mathcal{L}^n_{\varphi+tu} \mathbb{1} \rangle$. Hence, understanding the operator $\mathcal{L}_\varphi$ – and its perturbations $\mathcal{L}_{\varphi+tu}$ – as above allows one to precisely control the function $\langle \mu_\varphi, e^{tS_n(u)} \rangle$ (and its development in t), and deduce the statistical properties of μ_φ.

With the motivations above, we then embarked in the main goal of the project, and what I consider the main result in this manuscript: the construction of the norm for which the transfer operator becomes a contraction, which is the main result of [BD22]. As the construction is quite technical, and requires the introduction of several intermediate norms, I decided to include in Section 2.2.4 an informal discussion about the properties that these intermediate norms must satisfy in order to lead to the final one. The reader can think of that section as a continuation of this introduction. For each property, I will present the main difficulty, and the idea to overcome it. In Section 2.2.5, I instead give the precise construction of the norms. Finally, in Section 2.2.6 I show the spectral gap property for the perturbed transfer operator. This section can be read by just assuming the existence of a norm satisfying the properties in Section 2.2.4.

I will just highlight here two of the main challenges, and their solutions. First, even in the simplest case $\varphi = 0$, the operator $\mathcal{L}_0 = f_*$ does not preserve any real regularity of functions. Our norm is then *dynamical*, and already takes into account the action of f in its definition. It coincides with the Hölder norm in the case of hyperbolic maps, but becomes weaker and weaker (but still dominating a log-Hölder norm) as soon as the critical set is more and more recurrent in the Julia set. A second problem is that, while the operator f_* commutes with dd^c, this is not true anymore for the operator $\mathcal{L}_\varphi$. This leads to serious issues (also described more in detail in the informal

Section 2.2.4). In order to solve them, instead of just using the operator dd^c as in Sections 2.2.2 and 2.2.3, we need to combine it with the operator $u \mapsto i\partial u \wedge \overline{\partial} u$. Obtaining comparison principles for this operator is definitely more challenging than proving those for just the operator dd^c. The establishment of such (quantitative) principles is a key point in our solution of the problem.

Section 2.3 is dedicated to the two works [BD24; BD23b]. The general motivation, part of an ongoing project, would be to develop a study similar to the above also in the setting of, for instance, Hénon maps, i.e., polynomial diffeomorphisms of $\mathbb{C}^2$. These systems display both expanding and contracting directions, so the methods developed in [BD23a; BD22] cannot be applied, and, in particular, it is not known whether these systems have in general a spectral gap on a suitable functional space. As a partial step towards this more long term goal, we established the central limit theorem for the measure of maximal entropy of every Hénon maps and Hölder continuous observables. This already solves a long-standing question in the domain. I refer to Section 2.3.1 for more details and a description of our method. In Section 2.3.2, I describe the companion paper [BD23b], where we prove a similar result in the setting of automorphisms of Kähler manifolds. The first difference with respect to [BD24] is that here we have to use the theory of *superpotentials*, as developed by Dinh-Sibony. Although we initially believed that this result would have been a simple adaptation of [BD23b], up to the technical use of superpotentials, it turned out that a number of further technical complications appeared. I describe some of them in Section 2.3.2.

Chapter 3 - Other related works

This chapter will mainly describe the four works [BB23b; BO20; BDR23; BH23], as well as some parts of [BD23a] and [BR22]. I give a quick overview of each section below, and highlight the relations of each part with the rest of the manuscript. A tool that is used several times in the results in this Chapter (as well as in Sections 1.2.2 [BB18a] and 1.3.2 [AB23]) is a linearization process along generic inverse branches of endomorphisms (with respect to invariant measures with strictly positive Lyapunov exponents) due to Berteloot-Dupont-Molino.

Section 3.1 provides a link between the domains of Chapter 1 and 2. The topic is the equidistribution of repelling points in two different but related settings: with respect to the equilibrium states as in Section 2.2, and the distribution of their holomorphic motions in stable families, as in Chapter 1 and Theorem 1.1.1. The key point in the first case, proved in [BD23a], is to get a more precise version of the Briend-Duval method for the proof of the equidistribution of repelling points with respect to the measure of maximal entropy of endomorphisms of $\mathbb{P}^k$. The constant Jacobian of the measure plays a crucial role in that proof, and we have here to develop precise estimates on the Jacobian of pull-backs of the equilibrium states by the maps f^n. A key ingredient here is the precise linearization along generic inverse branches provided by Berteloot-Dupont-Molino. The details are in Section 3.1.1.

In the work [BB23b], joint with my PhD student Maxence Brévard, we adapt this construction in the setting of families of endomorphisms. I had already adapted the work by Briend-Duval in this context in my PhD thesis (and in particular in [Bia19a]), obtaining the equidistribution of the holomorphic motions of repelling points (and, in particular, proving the existence of such motions) with respect to natural measures related to the measures of maximal entropy, in stable families of endomorphisms. The work [BB23b] closes this circle, by proving the existence of the holomorphic motions for a large set of repelling points, related to the equilibrium states, with respect to natural measures that I had introduced in my work [BR22] with Rakhimov, described in Section 1.4.

Section 3.2 is dedicated to my joint work [BO20] with Yûsuke Okuyama. This work originated from a question by Charles Favre about the degeneration of endomorphisms of $\mathbb{P}^2$ to Hénon maps, related to his recent work on degenerations of holomorphic endomorphisms to a hybrid space. We explicitly compute the *non-archimedean Lyapunov exponent* in a class of examples, and we also show that every Hénon map can be approximated by maps in the bifurcation locus of the family of the endomorphisms. The proof relies on a generalization of some estimates that Dujardin had recently developed to show that the bifurcation of saddle sets for endomorphisms of $\mathbb{P}^2$ is independent from the bifurcation of their Julia sets as in Theorem 1.1.1. We also study the limit behaviour of the individual Lyapunov exponents at the degeneration. This estimate crucially relies on an approximation formula for the individual Lyapunov exponents, which is again based on the Berteloot-Dupont-Molino linearization process along inverse branches.

As mentioned above, the motivation for the work [BDR23], joint with Tien-Cuong Dinh and Karim Rakhimov, came from my other work [BR22] with Rakhimov. In various algebraic settings, by the work of Gromov and the celebrated Khovanskii-Teissier inequalities, one can show that the sequence of the *dynamical degrees* of a systems (the numbers detecting the growth rates of the mass of positive closed currents under iteration) is log-concave. This implies that the sequence is also increasing, up to some degree of maximal value, and then decreasing after that. We showed in [BDR23], see Section 3.3.1, that this monotonicity property holds true also in non-algebraic settings, in particular for the invertible horizontal-like maps and the polynomial-like maps. In this last case, this implies that, for polynomial-like maps of large topological degree, the topological degree strictly dominates all other degrees. Up to some further technical results (in particular, the generalization of a theorem by de Thélin and Dupont on the Lyapunov exponents of measures with sufficiently large entropy) presented in Section 3.3.2, this is enough to extend the results of Section 1.4 to this larger setting.

Finally, in Section 3.4, I describe the work [BH23], joint with Yan Mary He. The motivation is the celebrated *Mañé-Manning formula* for rational maps, stating that the measure-theoretic entropy of an invariant measure is the product of its Lyapunov exponent and its Hausdorff dimension. A conjectural higher-dimensional version of this formula is due to Binder-DeMarco, and has motivated a number of results since its formulation. We showed here a variation of this formula. More precisely, given an endomorphism of $\mathbb{P}^k$, we introduced a dynamical *volume dimension* for a large class of invariant measures (essentially those for which the strong holomorphic motion established with Rakhimov [BR22] applies - and in particular all the equilibrium states as in Section 2.2), and showed that this dimension is the ratio between the entropy of the measure and the *sum* of its Lyapunov exponents. Both the construction of this dimension and the proof of the formula again rely on the Berteloot-Dupont-Molino linearization result. More motivations, consequences, and an overview of the construction are given in Section 3.4.

List of presented works

I write near each item the code with which it is referred to in the manuscript, and the Section where it is presented.

Published papers (by the date of the preprint version)

1. Fabrizio Bianchi and Johan Taflin, [BT17]
   Bifurcations in the elementary Desboves family, Section 1.2
   Proceedings of the American Mathematical Society, 145.10 (2017) 4337-4343.
2. François Berteloot and Fabrizio Bianchi, [BB18a]
   Perturbations d'exemples de Lattès et dimension de Hausdorff du lieu de bifurcation, Section 1.2
   Journal de Mathématiques Pures et Appliquées, 116 (2018), 161-173
3. François Berteloot and Fabrizio Bianchi, [BB18b]
   Lyapunov exponents and stability in projective holomorphic dynamics, Section 1.1
   Proceedings of the Simons Semester in Banach Center - Dynamical Systems, 115, Polish Acad. Sci. Inst. Math., Warsaw, 37-71, (2018).
   (Chapter in volume, peer-reviewed)
4. Matthieu Astorg and Fabrizio Bianchi, [AB23]
   Hyperbolicity and bifurcations in families of polynomial skew products, Section 1.3
   American Journal of Mathematics, 145.3 (2023), 861-898
5. Fabrizio Bianchi and Yûsuke Okuyama, [BO20]
   Degeneration of quadratic polynomial endomorphisms to a Hénon map, Section 3.2
   Indiana University Mathematics Journal, 69 (2020), 2549-2570
6. Matthieu Astorg and Fabrizio Bianchi, [AB22]
   Higher bifurcations for polynomial skew products, Section 1.3
   Journal of Modern Dynamics, 18, 69-99 (2022)
7. Fabrizio Bianchi and Tien-Cuong Dinh, [BD23a]
   Equilibrium states of holomorphic endomorphisms of $\mathbb{P}^k$ I: existence and properties, Sections 2.2 and 3.1
   Journal de Mathématiques Pures et Appliquées, 172 (2023), 162-201
8. Fabrizio Bianchi and Tien-Cuong Dinh, [BD24]
   Every Hénon map is exponentially mixing of all orders and satisfies the CLT, Section 2.3
   Forum of Mathematics, Sigma, 12 (2024), e4

Preprints

9. Fabrizio Bianchi and Tien-Cuong Dinh, [BD22]
   Equilibrium states of holomorphic endomorphisms of $\mathbb{P}^k$: spectral gap and limit theorems, Section 2.2
   To appear in **Geometric and Functional Analysis (GAFA)**
   Preprint available at arXiv:2204.02856.

10. Fabrizio Bianchi and Karim Rakhimov, [BR22]
Strong probabilistic stability in holomorphic families of endomorphisms of $\mathbb{P}^k$ *and polynomial-like maps*, Sections 1.4 and 3.3.2
To appear in **International Mathematics Research Notices (IMRN)**
Preprint available at arXiv:2206.04165.
11. Fabrizio Bianchi and Tien-Cuong Dinh, [BD23b]
Exponential mixing of all orders and CLT for automorphisms of compact Kähler manifolds, Section 2.3
Preprint, available at arXiv:2304.13335.
12. Fabrizio Bianchi, Tien-Cuong Dinh, and Karim Rakhimov, [BDR23]
Monotonicity of dynamical degrees for horizontal-like maps and polynomial-like maps, Section 3.3
To appear in the **Transactions of the American Mathematical Society**
Preprint, available at arXiv:2307.10665.
13. Fabrizio Bianchi and Maxence Brévard, [BB23b]
Holomorphic motions of weighted periodic points, Section 3.1
Preprint, available at arXiv:2307.12734.
14. Fabrizio Bianchi and Yan Mary He, [BH23]
A Mañé-Manning formula for expanding measures for endomorphisms of $\mathbb{P}^k(\mathbb{C})$, Section 3.4
Preprint, available at arXiv:2308.03013.

My previous papers [BBD18; Bia19a; Bia19b], which are contained in my PhD thesis, are referred to in the manuscript for context.

I decided not to include in this manuscript the two works [BM22; BM24] below. These papers grew from discussions with Samuele Mongodi on the applications of the theory of currents to the problem of classifying obstructions to Steinness for manifolds. Although the techniques are related to those used in many parts of this manuscript, since the content is non dynamical they will not be presented here.

15. Fabrizio Bianchi and Samuele Mongodi, [BM22]
On minimal kernels and Levi currents on weakly complete complex manifolds,
Proceedings of the American Mathematical Society, 150.9 (2022), 3927-3939.
16. Fabrizio Bianchi and Samuele Mongodi, [BM24]
Generalized Levi currents and singular loci for families of plurisubharmonic functions,
Journal of Geometric Analysis 34 (2024), article number 203.

1

Stability and bifurcations in several complex variables

1.1 Context and motivations

Given a certain class of dynamical systems, say parametrized by some parameter space Λ, a central goal is to *classify the possible dynamical behaviours* and to *describe a generic element of the class*. Roughly speaking, the main steps towards this goal may be summarized as follows.

(A1) Prove that there exists a dynamically relevant decomposition of Λ into *stability* and *bifurcation* loci. The stability region is where the global dynamics does not change under small perturbation, and the bifurcation locus is where a small change in the parameters gives rise to global changes in the dynamics;

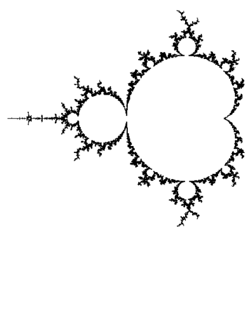
(A2) Prove that maps in the stable locus can be characterized by means of dynamically relevant notions (for instance, *hyperbolicity*). Understand whether a *generic* map is stable;

(A3) Identify all possible ways in which bifurcations occur. Describe the specific dynamical features of maps corresponding to parameters in the bifurcation locus.

One of the long term goals of my research, and the general theme of the works presented in this chapter, is to settle the above program in the setting of holomorphic endomorphisms of projective spaces in any dimension.

The one-dimensional case, i.e., that of rational maps (or simply polynomials) acting on the Riemann sphere (or $\mathbb{C}$), is nowadays quite well understood. The point **(A1)** of the program was settled by Lyubich [Lyu83b] and Mañé-Sad-Sullivan [MSS83] in the 1980s. They proved that several different definitions of stability are actually equivalent. They deduced that the stability locus is *dense* in any family of polynomials or rational maps. Thus, in this setting *a generic map is stable* (see point **(A2)**).

Let us recall a fundamental example. The simplest case we can consider is the quadratic polynomial family. We have a parameter space $\Lambda = \mathbb{C}$ and, for every $\lambda \in \Lambda$, we consider the quadratic polynomials $f_\lambda(z) = z^2 + \lambda$. In this case, the bifurcation locus is precisely the boundary of the Mandelbrot set (at right). All the white parts of the picture are *stability components*, i.e., connected components of the stability locus. Given a λ, we can associate to the polynomial f_λ its *Julia set*. This is the part of the plane where the chaotic part of the dynamics is concentrated (for instance, it is the support



of the – unique – measure of maximal entropy). Stability as described above is equivalent to the fact that the Julia set moves continuously with λ: thus, this set moves continuously when λ stays in a stability component, while it has a discontinuity if the parameter is in the bifurcation locus. *Hyperbolicity* (i.e., uniform expansion on the Julia set) propagates in stable components: as soon as a parameter is hyperbolic, all parameters in the same component satisfy the same property. Conjecturally, all white *stable* components are hyperbolic, i.e., all parameters in the same component correspond to maps which are uniformly expanding on their Julia set. This is the content of the so called *Fatou hyperbolicity conjecture*, possibly the main open question in the domain.

1.1.1 Previous works and definitions

By the classical results by Mañé-Sad-Sullivan [MSS83] and Lyubich [Lyu83b] mentioned above, the stability of a family of rational maps of degree $d \geq 2$ is determined by the stability of its repelling cycles. More precisely, the so-called λ-lemma ensures that, once one can follow holomorphically with the parameter a dense subset of the Julia set J_{λ_0} (the support of the unique measure of maximal entropy of f_{λ_0} [FLM83; Lyu82; Lyu83a]) at a given parameter λ_0, then *every* point of J_{λ_0} can be followed holomorphically. The important point here is that the motions corresponding to different points (that exist thanks to Montel theorem) do not intersect. This is a consequence of Hurwitz theorem. The *stability locus* is the (open dense) subset of the parameter space where the above condition of stability holds true. The complement of such set is the *bifurcation locus*. By a result of DeMarco [DeM03], such locus is the support of a naturally defined *bifurcation current*, given by the complex Laplacian of the function associating to every λ the Lyapunov exponent of the measure of maximal entropy of f_λ, see also [Prz85; Sib81] for the polynomial case. In particular, the (pluri)harmonicity of this function is equivalent to the stability as above. Let us recall that the use of potential-theoretic tools in holomorphic dynamics had been pioneered by Brolin [Bro65], who showed, by these means, the equidistribution of preimages of generic points with respect to the equilibrium measure of complex polynomials, a result later generalized to all rational maps in [FLM83; Lyu82; Lyu83a].

In my PhD thesis, I contributed to a generalization of the theory by Mañé-Sad-Sullivan, Lyubich, and DeMarco to families of endomorphisms of $\mathbb{P}^k$ of algebraic degree $d \geq 2$ in any dimension $k \geq 1$, see [BBD18; Bia19a] and also [BB18b] for an overview of the arguments and methods. As most of the one-dimensional techniques do not apply anymore, the main tools came from ergodic and (pluri)potential theory. Very roughly speaking, compactness of suitable spaces of currents and plurisubharmonic functions played the role of Montel theorem. And precise statistical properties of the measure of maximal entropy [BD01; DS10a; FS94] (among them, an explicit lower bound for its Lyapunov exponents [BD99]) played somehow the role of an asymptotic Hurwitz theorem. As a consequence, dynamical stability in such families (defined for instance by the vanishing of a natural bifurcation current [BB07; Pha05], or by a condition on the stability of the repelling points) is equivalent to the existence of a holomorphic motion for a *full measure subset* of the Julia set, with respect to the measure of maximal entropy.

Let us be more precise. Given a complex manifold M, a holomorphic family $(f_\lambda)_{\lambda \in M}$ of endomorphisms of $\mathbb{P}^k$ parametrized by M is a holomorphic map $f \colon M \times \mathbb{P}^k \to M \to \mathbb{P}^k$ of the form $f(\lambda, z) = (\lambda, f_\lambda(z))$ and such that every f_λ has the same algebraic degree d. We always assume that $d \geq 2$. We denote by μ_λ the equilibrium measure of f_λ (i.e., the unique measure of maximal entropy $k \log d$ of f_λ) and recall that the Julia set J_λ of f_λ is by definition the support of μ_λ.

Theorem 1.1.1 (Berteloot-Bianchi-Dupont [BBD18; Bia19a]). *Let M be an open connected and simply connected manifold and $(f_\lambda)_{\lambda\in M}$ a holomorphic family of endomorphisms of $\mathbb{P}^k$ of a given algebraic degree $d \geq 2$. The following conditions are equivalent:*

(S1) *asymptotically all the repelling points in the Julia sets move holomorphically with λ;*

(S2) *the sum $L(\lambda) = \int \log |\operatorname{Jac} f_\lambda| \mu_\lambda$ of the Lyapunov exponents of μ_λ satisfies $dd^c L \equiv 0$;*

(S3) *there are no* Misiurewicz parameters*;*

(S4) *there exists an* equilibrium lamination*;*

(S5) *there exists an* acritical equilibrium web.

Definition 1.1.2. *We say that a family is* stable *if any of the equivalent conditions in Theorem 1.1.1 locally holds.*

More generally, the *stability locus* of a holomorphic family is the maximal open set of the parameter space where the conditions in Theorem 1.1.1 (locally) hold. The *bifurcation locus* is the complement of the stability locus.

In the remaining part of this section we fix some notations and give the definitions of the conditions in Theorem 1.1.1.

Denote by $\mathcal{J}$ the set of all holomorphic maps $\gamma \colon M \to \mathbb{P}^k$ such that $\gamma(\lambda)$ belongs to J_λ for all $\lambda \in M$. We often identify a map γ with its graph Γ_γ in the product space $M \times \mathbb{P}^k$. The family $(f_\lambda)_{\lambda\in M}$ induces a dynamical system $\mathcal{F}$ on the space $\mathcal{J}$ by $\mathcal{F}\gamma(\lambda) := f_\lambda(\gamma(\lambda))$. Observe that $\mathcal{J}$ is a metric space, with the topology of local uniform convergence, and that $(\mathcal{J}, \mathcal{F})$ is a well-defined dynamical system.

The *repelling J-cycles* of f_λ are the repelling cycles of f_λ which belong to J_λ. In dimension $k = 1$, all repelling cycles are automatically repelling J-cycles and are dense in the Julia set. When $k \geq 2$ there are examples of repelling points outside of the Julia set (see for instance [FS01; HP94]). However, repelling J-cycles are still dense in the Julia set [BD99].

Definition 1.1.3. *We say that the repelling J-cycles of f_λ* move holomorphically *over an open subset $\Omega \subseteq M$ if for every $n \geq 1$ there exists a set of holomorphic maps $\gamma_{j,n} \in \mathcal{J}$ such that $\mathcal{R}_n(\lambda) = \{\gamma_{j,n}(\lambda) : 1 \leq j \leq N_d(n)\}$ for all $\lambda \in \Omega$, where $\mathcal{R}_n(\lambda) :=$ {repelling $n-$periodic points of f_λ in J_λ} and $N_d(n) = \operatorname{Card}(\mathcal{R}_n(\lambda))$ for all $\lambda \in M$.*

I introduced the following, a priori weaker, condition on the repelling cycles in [Bia19a].

Definition 1.1.4. *We say that* asymptotically all *repelling cycles move holomorphically on Ω if there exists a set $\mathcal{P} = \cup \mathcal{P}_n \subset \mathcal{J}$ with the following properties:*

1. $\operatorname{Card}(\mathcal{P}_n) = d^{kn} + o(d^{kn})$*;*
2. *every $\gamma \in \mathcal{P}_n$ is n-periodic for $\mathcal{F}$;*

3. *for all open* $\Omega' \Subset \Omega$*, we have*

$$d^{-kn} \cdot \mathrm{Card}\big(\{\gamma \in \mathcal{P}_n \colon \gamma(\lambda) \textit{ is repelling for all } \lambda \in \Omega'\}\big) \to 1 \quad \textit{for} \quad n \to \infty.$$

We denote by C_f the critical set of f, and by $\mathrm{GO}(C_f)$ the grand orbit of C_f, i.e., $\mathrm{GO}(C_f) := \cup_{n,m\geq 0} f^{-m}(f^n(C_f))$.

Definition 1.1.5. *We say that* $\lambda_0 \in M$ *is a* Misiurewicz *parameter if there exist integers* $p_0, n_0 \geq 1$ *and a holomorphic map* σ *defined on some neighbourhood of* λ_0 *such that* $\sigma(\lambda) \in \mathcal{R}_{p_0}(\lambda)$ *and* $\Gamma_\sigma \cap W \neq \emptyset$ *but* $\Gamma_\sigma \not\subseteq W$ *for some irreducible component* W *of* $f^{n_0}(C_f)$*.*

Definition 1.1.6. *A* dynamical lamination *for the family* f *is a* $\mathcal{F}$*-invariant subset* $\mathcal{L}$ *of* $\mathcal{J}$ *such that*

1. $\Gamma_\gamma \cap \Gamma_{\gamma'} = \emptyset$ *for all* $\gamma \neq \gamma' \in \mathcal{L}$*;*
2. $\Gamma_\gamma \cap \mathrm{GO}(C_f) = \emptyset$ *for all* $\gamma \in \mathcal{L}$*;*
3. $\mathcal{F} : \mathcal{L} \to \mathcal{L}$ *is* d^k*-to-*1*.*

An equilibrium lamination *or* μ_λ-measurable holomorphic motion of J_λ *is a dynamical lamination satisfying* $\mu_\lambda(\{\gamma(\lambda) : \gamma \in \mathcal{L}\}) = 1$ *for all* $\lambda \in M$*.*

For every $\lambda \in M$, we denote by $p_\lambda : \mathcal{J} \to J_\lambda$, the natural map given by $p_\lambda(\gamma) = \gamma(\lambda)$. A *web* for the family f is a $\mathcal{F}$-invariant probability measure compactly supported on $\mathcal{J}$. An *equilibrium web* for the family f is a web $\mathcal{M}$ such that $(p_\lambda)_*\mathcal{M} = \mu_\lambda$ for all $\lambda \in M$.

Definition 1.1.7. *A web* $\mathcal{M}$ *is said to be* acritical *if* $\mathcal{M}(\mathcal{J}_s) = 0$*, where*

$$\mathcal{J}_s := \{\gamma \in \mathcal{J} : \Gamma_\gamma \cap \mathrm{GO}(C_f) \neq \emptyset\}$$

If $k = 2$ or if M is an open subset of the (parameter space of the) family $\mathcal{H}_d(\mathbb{P}^k)$ of all the endomorphisms of $\mathbb{P}^k$ of a given algebraic degree $d \geq 2$, the conditions in Theorem 1.1.1 are equivalent to the motion of all the repelling J-cycles as in Definition 1.1.3, see [BBD18]. By a result of Berteloot [Ber18], **(S1)** can actually be weakened to the following, a priori even weaker, condition: there exists a function $N \colon \mathbb{N} \to \mathbb{N}$ with $\limsup_{n\to\infty} d^{-kn} N(n) > 0$ such that, for every n, $N(n)$ repelling n-periodic points move holomorphically (as repelling periodic points) over M. We refer to [Lev82; Lyu83b] for the equivalence of **(S3)** to the other conditions in dimension 1, and to another recent work by Berteloot and Brévard [BB23a] for further characterizations of stability in any dimension.

1.1.2 First questions and summary of results

Comparing Theorem 1.1.1 with the one-dimensional version by Mañé-Sad-Sullivan, Lyubich, and DeMarco, the following are possibly the first natural questions that arise:

(SQ1) is the Hausdorff continuity of the Julia sets equivalent to the other notions of stability?

(SQ2) is the stability locus dense?

(SQ3) do all repelling points move holomorphically (as in Definition 1.1.3) in any stable family?

(SQ4) does hyperbolicity propagate in stable components, i.e., is every map in a given component hyperbolic as soon as one given map in the component is?

As already mentioned above, the answer to all these questions is positive in dimension 1.

In the next Section 1.2, I will present some results giving a negative answer to **(SQ1)** and **(SQ2)** [BT17], as well as some estimates on the Hausdorff dimension of the bifurcation locus [BB18b].

Section 1.3 describes my works [AB23; AB22] with Matthieu Astorg, where we aim at a precise description of the parameter space of families of polynomial skew products. We answer positively to **(SQ4)** in this context, and we give a more precise description of the bifurcation locus, as well as a classification of hyperbolic components near the boundary of the parameter space. We also study the self-intersections of the bifurcation current, as well as their supports. We establish some equidistribution results for dynamically defined hypersurfaces of the parameter space towards the bifurcation current, which hold in general families of endomorphisms.

We conclude this section by recalling that a parallel theory to that of [BBD18; Bia19a] has been developed in the setting of polynomial diffeomorphisms of $\mathbb{C}^2$ by Dujardin and Lyubich, see [DL15; Duj22]. Stability is defined also in this setting by means of a number of equivalent conditions, among them the existence of a *branched* holomorphic motion for the Julia sets, meaning that natural motions of distinct points can a priori intersect. In [BD17] Berger and Dujardin introduced a notion of *regular point* and proved that such motion is *unbranched* at every regular point. This applies to almost every point with respect to all measures with strictly positive and negative Lyapunov exponents, and in particular to the measure of maximal entropy. This result was the inspiration for the paper [BR22], where with Karim Rakhimov we proved an analogous result in the setting of endomorphisms of $\mathbb{P}^k$. This is described in Section 1.4. Building on the techniques developed there, with Maxence Brévard we also partially addressed the question **(SQ4)** in [BB23b]. As this work also relies on results that will be explained in Chapter 2, its description is postponed to Section 3.1.2.

1.2 Hausdorff continuity of the Julia sets, open sets of bifurcation, and Hausdorff dimension of the bifurcation locus

In this section I describe the two works [BT17] and [BB18a], which were the first I wrote after the completion of my PhD manuscript, and are those more closely related to it. They address the first two natural questions **(SQ1)** and **(SQ2)** in the list in the previous section.

1.2.1 Bifurcations in the elementary Desboves family

The following theorem, obtained with Johan Taflin, gives an explicit example of a family of endomorphisms of $\mathbb{P}^2$ for which the answer to both the questions **(SQ1)** and **(SQ2)** is negative.

Theorem 1.2.1 (Bianchi-Taflin [BT17]). *The family of endomorphisms of $\mathbb{P}^2$ given by*

$$f_\lambda([x:y:z]) = \big[-x(x^3+2z^3) : y(z^3-x^3+\lambda(x^3+y^3+z^3)) : z(2x^3+z^3)\big]$$

with $\lambda \in \mathbb{C}^*$ satisfies the following properties:

1. *the Julia set of f_λ depends continuously on λ, for the Hausdorff topology;*
2. *the bifurcation locus coincides with $\mathbb{C}^*$.*

The family above is called the *elementary Desboves family* and these maps were previously studied by Bonifant-Dabija [BD02] and Bonifant-Dabija-Milnor [BDM07]. Each element of the family preserves the Fermat curve $\mathcal{C} = \{x^3 + y^3 + z^3 = 0\}$. Moreover, the following dynamical features hold for every $\lambda \in \mathbb{C}^*$:

1. f_λ preserves the line $Y = \{y = 0\}$ and $(f_\lambda)_{|Y}$ is a Lattés map g (independent of λ);
2. f_λ preserves the pencil $\mathcal{P}$ of lines passing through the point $p_0 := [0 : 1 : 0]$ and the action on $\mathcal{P}$ is conjugated to g;
3. p_0 is an attracting fixed point and the Fatou set of f_λ is equal to the basin $\mathcal{A}$ of p_0. As a consequence, for every λ we have the decomposition $\mathbb{P}^2 = \mathcal{A} \cup J_\lambda$.

We can now give a sketch of the proof of Theorem 1.2.1. The proof of the first assertion is a direct consequence of the third property above. Indeed, such property implies that the map $\lambda \mapsto J_\lambda$ is upper-semicontinuous. As such map is always lower-semicontinuous, the continuity follows.

We then need to establish that every λ is in the bifurcation locus. In order to do this, by Theorem 1.1.1 we can show that Misiurewicz parameters are dense in $\mathbb{C}^*$. This requires to first better understand the critical set of each f_λ. It is not difficult to see that this consists of 3 lines through p_0 (corresponding to the critical points of g) and a further cubic C'_λ, which actually depends on λ and is generically transverse to the elements of the pencil $\mathcal{P}$. In particular, the intersections between C'_λ and Y move with λ. A simple computation also shows that, for every λ, at least one among the two points $Y \cap \{x = 0\}$ and $Y \cap \{z = 0\}$ is repelling. We consider the preimages of such point in Y, which correspond to preimages under g and are dense in Y. As these preimages do not depend of λ while $C'_\lambda \cap Y$ does, this implies that there exist non-persistent Misiurewicz parameters for a dense set of $\lambda \in \mathbb{C}^*$. We prove this by showing that any persistent intersection would have to persist also for $\lambda = 0$ (where f_0 is still an endomorphism of $\mathbb{P}^2 \setminus \{p_0\}$), which is excluded by an explicit check. The assertion follows.

Notice that Theorem 1.2.1 does not contradict the possible equivalence of stability and Hausdorff continuity of the Julia sets in the family $\mathcal{H}_d(\mathbb{P}^k)$ of all endomorphisms of $\mathbb{P}^k$ of a given degree d. Hence, in this family, question **(QS1)** is still open. Similarly, it does not imply the existence of an open set in the bifurcation locus for the family $\mathcal{H}_d(\mathbb{P}^k)$ (a *robust bifurcation*). The existence of such an open set was first announced, essentially in parallel to [BT17] and independently from Theorem 1.2.1, by Dujardin [Duj17], who presented two different mechanisms leading to robust bifurcations, in any dimension and degree.

Let us now focus on the case $k = 2$. The methods by Dujardin give open sets in the bifurcation locus near maps of the form $(z, w) \mapsto (p(z), q(w))$, where p is a bifurcating polynomial in the family of degree d polynomials $\mathcal{P}_d(\mathbb{C})$, and q has a rather specific form. This means that these maps are in the closure of the interior of the bifurcation locus of $\mathcal{H}_d(\mathbb{P}^k)$. Dujardin then asked whether any product map of the form $(z, w) \mapsto (p(z), q(w))$ (where p or q is bifurcating in $\mathcal{P}_d(\mathbb{C})$) is contained in the closure of the interior of the bifurcation locus of $\mathcal{H}_d(\mathbb{P}^2)$. This question was positively answered by Taflin [Taf21].

1.2.2 Hausdorff dimension estimates

The Hausdorff dimension of the bifurcation locus can also be studied for generic families. Let us recall the fundamental result of Shishikura [Shi98] stating that the Hausdorff dimension of the boundary of the Mandelbrot set (which is the bifurcation locus of the quadratic family) is equal to 2. From this one can deduce [McM00b; Tan98] that the Hausdorff dimension of the bifurcation locus is always maximal for every family of rational maps. For families $(f_\lambda)_{\lambda\in M}$ of endomorphisms of $\mathbb{P}^k$, we proved with François Berteloot [BB18a] an estimate for the Hausdorff dimension of the bifurcation locus near any bifurcation parameter λ_0, which depends on the values of the Lyapunov exponents of f_{λ_0}. In particular, this dimension is always maximal near (higher-dimensional) isolated *Lattès maps*, i.e., endomorphisms of $\mathbb{P}^k$ which are suitably semi-conjugated to affine maps on $\mathbb{C}^k$ [BD05; Mil06].

Theorem 1.2.2 (Berteloot-Bianchi [BB18a]). *Let f be any holomorphic family of endomorphisms of $\mathbb{P}^k$, parametrized by M. Assume that $\lambda_0 \in M$ is such that f_{λ_0} is a Lattès map, and λ_0 is accumulated by parameters λ for which f_λ is not a Lattès map. Then the Hausdorff dimension of the bifurcation locus is maximal at λ_0.*

This is coherent with a conjecture in [Duj17], which states that Lattès maps should be contained in the closure of the interior of the bifurcation locus of $\mathcal{H}_d(\mathbb{P}^k)$. A step towards the settling of this conjecture in dimension 2 has been done by Biebler [Bie19], who showed that for every Lattès map L of degree d on $\mathbb{P}^2$ there exists an integer $n_0 = n(L)$ such that for every $n \geq n_0$ the iterate L^n belongs to the closure of the interior of the bifurcation locus of the family $\mathcal{H}_{d^n}(\mathbb{P}^2)$.

We conclude this section giving an idea of the proof of Theorem 1.2.2. Let us assume that M has dimension 1 and suppose that λ_1 is a Misiurewicz parameter. We also assume that the Lyapunov exponents $\chi_1 \leq \cdots \leq \chi_k$ of μ_{λ_1} do not satisfy any resonance relation. Then, we can show the inequality

$$\liminf_{r\to 0} \dim_H(\mathrm{Bif} \cap \mathbb{D}(\lambda_1, r)) \geq \frac{\chi_1}{\chi_k}\Big(\frac{k\log d}{\chi_k}\Big) - (2k-2), \tag{1.1}$$

where d is the algebraic degree of the family. The proof of the above estimate is based on the following classical *transfer* geometric argument. Roughly speaking, whenever a Misiurewicz bifurcation happens, one can construct a large Iterated Function System in the phase space of the Misiurewicz parameter, persisting in a neighbourhood of the parameter. Up to reducing M, we can think of it as a tube $T_0 = M \times B$, for some ball $B \subset \mathbb{P}^k$, and a collection $\mathcal{T} = \cup_{n\geq 1}\mathcal{T}_n$ of subsets of T_0, where the cardinality of $\mathcal{T}_n$ is larger than δ^n for some $d^{k-1} < \delta < d^k$, such that each element of $\mathcal{T}_n$ is mapped bijectively to T_0 by the family f^n. The intersection between the postcritical set and a repelling point at the Misiurewicz parameter implies that the postcritical set also intersects every element of $\mathcal{T}$. It is not difficult to see that every such intersection gives a new Misiurewicz parameter. Thanks to the intersections with the postcritical set one can then *transfer* the original IFS to the parameter space. The Hausdorff dimension of the bifurcation locus can be then estimated from below by that of the transferred IFS.

This strategy works particularly well in dimension 1. In higher dimension, we need to combine the above arguments with a precise linearization along inverse orbits which was established by Berteloot-Dupont-Molino [BDM08] and results by Pesin-Weiss [PW96] on the Hausdorff dimension of limit sets of contractions with controlled geometry. Theorem 1.2.2 then directly follows by applying the general estimate (1.1) to Misiurewicz parameters λ_1 arbitrarily close to λ_0, for which we have $\chi_1 \sim \chi_k \sim (\log d)/2$.

1.3 Hyperbolicity and bifurcations in families of polynomial skew products

This section describes the two papers [AB23; AB22], joint with Matthieu Astorg, in which we study and characterize hyperbolicity and bifurcations in families of polynomial skew products.

Polynomial skew products are regular polynomial endomorphisms of $\mathbb{C}^2$ of the form $f(z,w) = (p(z), q(z,w))$, for p and q polynomials of a given degree $d \geq 2$. *Regular* here means that the coefficient of w^d in q is non zero, which is equivalent to the extendibility of these maps as holomorphic self-maps of $\mathbb{P}^2$. Despite their specific forms, these maps had already provided examples of new phenomena with respect to the established theory of one-variable polynomials or rational maps, see for instance [Ast+16; Duj16] and the already mentioned [Duj17; Taf21].

We will denote in what follows by $\mathbf{Sk}(p,d)$ the family of all polynomial skew products of a given degree d over a fixed base polynomial p up to affine conjugacy, and denote by D_d its dimension. By Theorem 1.1.1, it is possible to partition the parameter space of the family $\mathbf{Sk}(p,d)$ (identified with $\mathbb{C}^{D_d}$) into the two dynamically defined *stability* and *bifurcation* loci. The first goal of [AB23] was to describe more precisely this decomposition, and in general develop in more detail the program **(A1)**-**(A3)**, in families of polynomial skew products. As part of our work, we also solved **(SQ3)** in this context.

1.3.1 Hyperbolicity and bifurcations

While many of our results apply to more general families, we mainly focus here on the family of *quadratic skew products*, i.e., polynomial skew products of (algebraic) degree 2, that are in this context the analogue of the family $z^2 + c$. By means of an affine change of coordinates, the dynamical study of this family can be reduced to that of the family

$$f_\lambda : (z,w) \mapsto (z^2 + d, w^2 + az^2 + bz + c) \tag{1.2}$$

with d and $\lambda := (a,b,c)$ as (complex) parameters. Since bifurcations due to the parameter d are of one-dimensional nature, we fix here $p(z) := z^2 + d$ and consider the parameter space $\mathbb{C}^3$ of the family $\mathbf{Sk}(p,2) := \{f_\lambda : (a,b,c) \in \mathbb{C}^3\}$.

In [AB23], we especially focused our attention on parameters near the boundary of this space, i.e., near the hyperplane at infinity, that we denote by $\mathbb{P}^2_\infty$, both from the point of view of stability and bifurcation. The first result gives a complete description of the bifurcation locus near $\mathbb{P}^2_\infty$ from both a topological and measure-theoretical point of view. We denote by J_p the Julia set of p. Given $z \in \mathbb{C}$, we set $E_z := \{ [a,b,c] \colon az^2 + bz + c = 0 \} \subset \mathbb{P}^2_\infty$ and $E := \cup_{z \in J_p} E_z$. An analogous result for quadratic rational maps had been proved in [BG15a].

Theorem 1.3.1 (Astorg-Bianchi [AB23]). *The accumulation on $\mathbb{P}^2_\infty$ of the bifurcation locus of the family* (1.2) *coincides with E. Moreover, the bifurcation current T_{bif} on $\mathbb{C}^3$ extends as a positive* *closed current $\widehat{T}_{\rm bif}$ to $\mathbb{P}^3 = \mathbb{C}^3 \cup \mathbb{P}^2_\infty$ and we have*

$$\widehat{T}_{\rm bif} \wedge [\mathbb{P}^2_\infty] = \int_{J_p} [E_z]\mu_p(z).$$

The proof of this result relies on several ingredients. The first is a decomposition for the bifurcation current (and locus), see [AB23, Theorem 3.3] for the precise statement. We then prove that special dynamically defined hypersurfaces $\mathrm{Per}^v_n(\eta)$ in $\mathbb{C}^3$ *equidistribute* towards the bifurcation current $T_{\rm bif}$ (and $\widehat{T}_{\rm bif}$), see the next Section 1.3.2 for more details. Moreover, we can precisely control the intersections of these hypersurfaces with $\mathbb{P}^2_\infty$. We thus obtain the convergences

$$\frac{1}{d^{2n}}[\mathrm{Per}^v_n(\eta)] \to \widehat{T}_{\rm bif} \quad \text{and} \quad \frac{1}{d^{2n}}[\mathrm{Per}^v_n(\eta)] \wedge [\mathbb{P}^2_\infty] \to \int_{z\in J_p} [E_z]\mu_p.$$

Theorem 1.3.1 then reduces to proving that the convergences above imply

$$\frac{1}{d^{2n}}[\mathrm{Per}^v_n(\eta)] \wedge [\mathbb{P}^2_\infty] \to \widehat{T}_{\rm bif} \wedge [\mathbb{P}^2_\infty],$$

which is a problem of intersection of currents. To do this, we exploit the theory of *horizontal positive closed currents* [Duj04; DS06c]. This requires to prove some uniform estimates on the directions at which the bifurcation locus approaches $\mathbb{P}^2_\infty$.

Once the bifurcation locus near the hyperplane at infinity was understood, we turned our attention to its complement, and in particular to the characterization of the hyperbolic components as in **(SQ3)**. Notice that, in order for those to exist, p must be hyperbolic.

The stability of a polynomial skew product as in (1.2) is determined by the behaviour of the critical points of the form $(z, 0)$ with $z \in J_p$. As is the case for polynomials, when all these points escape to infinity by iteration, the map is hyperbolic. It is however not clear a priori why such property should propagate on a stable component. In our next result not only we solved **(SQ3)** in the setting of polynomial skew products (thus giving meaning to the expression *hyperbolic components* here), but we also gave a complete classification of hyperbolic components that are analoguous to the so-called *shift locus* from dimension 1.

More precisely, let $\mathcal{D}$ be the set of parameters for which all critical points in $J_p \times \mathbb{C}$ escape, and let $\mathcal{D}' \subset \mathcal{D}$ be the subset of parameters λ for which there is an arc joining λ to $\mathbb{P}^2_\infty \backslash E$ inside $\mathcal{D}$. Set

$$\mathcal{S}_p := \Big\{ s : \pi_0(\mathring{K}_p) \to \{0,1,2\} : \sum_{U\in\pi_0(\mathring{K}_p)} s(U) \le 2 \Big\},$$

where $\pi_0(\mathring{K}_p)$ denotes the set of bounded Fatou components of p.

Theorem 1.3.2 (Astorg-Bianchi [AB23]). *Let $(f_\lambda)_{\lambda\in M}$ be a holomorphic family of polynomial skew products.*

1. *Any f_λ in a stable component containing a hyperbolic parameter is hyperbolic.*
2. *Assume that $M = \mathbf{Sk}(p, 2)$. All connected components of $\mathcal{D}'$ are hyperbolic components, and there is a natural bijection between $\mathcal{S}_p$ and the connected components of $\mathcal{D}'$.*

The condition of the base polynomial p of being hyperbolic is actually not necessary, if we replace hyperbolicity with *vertical expansion*, i.e., an expansion condition for the vertical fiber-wise dynamics, see [Jon99]. Theorem 1.3.2 holds in this case too, proving that stability preserves vertical expansion, and giving a classification of vertically expanding component.

The proof of the first item of Theorem 1.3.2 is based on a characterization of hyperbolicity (and vertical expansion) due to Jonsson [Jon99] and based on the (non) accumulation of the postcritical set on the Julia set. We showed that stability preserves this equivalent notion. The proof of the second item is topological in nature. The main issue was to exclude the possibolity that a given hyperbolic component could accumulate two distinct components of $\mathbb{P}^2_\infty \backslash E$. To prove this, we showed that the combinatorial invariants $s \in \mathcal{S}_p$ encode the isotopy class of the Julia set in $J_p \times \mathbb{C}$.

1.3.2 Equidistribution towards the bifurcation current

As mentioned above, one of our main tools in the proof of Theorem 1.3.1 is an approximation result for the bifurcation current by means of dynamically defined hypersurfaces in the parameter space. In dimension 1, the idea of seeing $T_{\rm bif}$ as a limit of currents detecting dynamically interesting parameters goes back to Levin [Lev82] (see also [Lev90]), who proved that the centres of the hyperbolic components of the Mandelbrot set equidistribute the bifurcation current, which is supported on its boundary. Versions of this result were established for any family of polynomials (and actually rational maps) in [BB09], where now currents became an essential tool, see also [Oku14], and also for the distribution of maps with a cycle of any given multiplier [BB11; BG15b; Gau16; GOV19] or with preperiodic critical points [DF08; FG15].

In our situation, in the proof of Theorem 1.3.1 we needed an equidistribution property towards $T_{\rm bif}$ of the parameters admitting a periodic point with *vertical* multiplier η. Since the same techniques allow one to prove a general result valid for any family of endomorphisms of $\mathbb{P}^k$, in any dimensions k, we proved the following more general result. It is the first equidistribution result in the parameter space for holomorphic dynamical systems in dimension larger than one.

Theorem 1.3.3 (Astorg-Bianchi [AB23]). *Let $(f_\lambda)_{\lambda\in M}$ be the family of all holomorphic endomorphisms of $\mathbb{P}^k$ of a given degree $d \geq 2$. For all $\eta \in \mathbb{C}$ outside of a polar subset, we have*

$$\frac{1}{d^{2n}}[\mathrm{Per}_n(\eta)] \to T_{\rm bif},$$

where $\mathrm{Per}_n(\eta) := \{\lambda \colon \exists z \in J_{f_\lambda}$ *of exact period n for* f_λ *and such that* $\mathrm{Jac}_z\, f_\lambda = \eta\}$.

The general strategy of the proof of Theorem 1.3.3 follows the main line of the one dimensional case and is based on techniques and tools from pluripotential theory. However, one of the difficulties we had to face here was the possible presence of infinitely many non-repelling cycles for an endomorphism of $\mathbb{P}^k$ – something which is excluded for $k = 1$ as such non-repelling cycles are known to be in finite number. We thus needed more quantitative estimates on the number of repelling cycles with small multiplier, which are related to the approximation formula for the sum of the Lyapunov exponents valid in any dimension established in [BDM08].

1.3.3 Higher bifurcations for polynomial skew products

For families of rational maps, the study of the self-intersections of the bifurcation current (which are meaningful because of the continuity of its potential) was started in [BB07], see also [Pha05; DF08; Duj11]. A geometric interpretation of the support of these currents is the following: the support of $T^k_{\mathrm{bif}} := T^{\wedge k}_{\mathrm{bif}}$ is the locus where k critical points bifurcate *independently*. Moreover, the current T^k_{bif} is known to equidistribute many kinds of dynamically defined parameters, such as maps possessing k cycles of prescribed multipliers and periods tending to infinity (see, e.g., [BB07; Gau16]). This gives rise to a natural stratification of the bifurcation locus as

$$\operatorname{Supp} T_{\mathrm{bif}} \supseteq \operatorname{Supp} T^2_{\mathrm{bif}} f \supseteq \cdots \supseteq \operatorname{Supp} T^{k_{\max}}_{\mathrm{bif}},$$

where $k_{\max}$ is the dimension of the parameter space. The inclusions above are not equalities in general, and are for instance strict when considering the family of all polynomials or rational maps of a given degree (where $k_{\max}$ is equal to $d-1$ and $2d-2$, respectively). It is worth pointing out that this stratification is often compared with an analogous stratification for the Julia sets of endomorphisms of $\mathbb{P}^k$ (given by the supports of the self-intersections of the Green current, see for instance [DS10a]). We refer to [Duj11] for a more detailed exposition.

The arguments in the proof of Theorem 1.3.3 can easily be adapted to prove a similar statement for the bifurcation currents T^k_{bif}: given generic $\eta_1, \dots, \eta_k \in \mathbb{C}$, $k \le k_{\max}$, the bifurcation current T^k_{bif} detects the distribution of skew products having k cycles of periods tending to infinity and respective vertical multipliers $\eta_1, \dots, \eta_k \in \mathbb{C}$. It is then natural to ask whether the supports of the bifurcation currents still give a natural stratification of the bifurcation locus.

The main goal of the paper [AB22] was to show that the situation in families of higher dimensional dynamical systems is completely different from the one-dimensional counterpart. Namely, we established the following result.

Theorem 1.3.4 (Astorg-Bianchi [AB22]). *Let p be a polynomial with Julia set not totally disconnected, which is neither conjugated to $z \mapsto z^d$ nor to a Chebyshev polynomial. Let $\mathbf{Sk}(p,d)$ denote the family of polynomial skew products of degree $d \ge 2$ over the base polynomial p, up to affine conjugacy, and let D_d be its dimension. Then the associated bifurcation current T_{bif} satisfies*

$$\operatorname{Supp} T_{\mathrm{bif}} \equiv \operatorname{Supp} T^2_{\mathrm{bif}} \equiv \cdots \equiv \operatorname{Supp} T^{D_d}_{\mathrm{bif}}.$$

Theorem 1.3.4 is stated for the full family $\mathbf{Sk}(p,d)$ which was introduced at the beginning of this Section 1.3. One could ask whether such a result holds for algebraic subfamilies of $\mathbf{Sk}(p,d)$: clearly, some special subfamilies have to be ruled out, such as the family of trivial product maps of the form $(p,q) : (z,w) \mapsto (p(z), q(w))$. A less obvious example in degree 3 is given by the subfamily of polynomial skew products over the base polynomial $z \mapsto z^3$ of the form

$$f_{a,b} : (z,w) \mapsto (z^3, u^3 + awz^2 + bz^3), \qquad (a,b) \in \mathbb{C}^2.$$

One can check that $f_{a,b}$ is semi-conjugated to the product map

$$g_{a,b} : (z,u) \mapsto (z^3, u^3 + au + b)$$

via the blow-up $\pi : (z,w) \mapsto (z,zw)$, that is, we have $f_{a,b} \circ \pi = \pi \circ g_{a,b}$. It follows that $\operatorname{Supp} T^2_{\mathrm{bif}}(\Lambda) \subsetneq \operatorname{Supp} T_{\mathrm{bif}}(\Lambda)$, where $\Lambda := \{f_{a,b}, (a,b) \in \mathbb{C}^2\}$.

The proof of Theorem 1.3.4 indeed uses the fact that the family $\mathbf{Sk}(p,d)$ is general enough so that it is possible to perturb a bifurcation parameter to change the dynamical behaviour of a critical point in a vertical fibre without affecting all other fibres. It would be interesting to classify algebraic subfamilies of $\mathbf{Sk}(p,d)$ that, like Λ, are degenerate in the sense that a bifurcation in one fibre implies a bifurcation in all other fibres; for such families, the conclusion of Theorem 1.3.4 would not hold. Likewise, it would be natural to try to extend Theorem 1.3.4 to other families with a similar fibred structure, see for instance [DT21]. To do this, one should first ensure that such a family is large enough in the sense above.

The proof of Theorem 1.3.4 essentially consists of two ingredients, respectively of analytical and geometrical flavours.

The first is an analytical sufficient condition for a parameter to be in the support of T^k_{bif}. This is inspired by analogous results by Buff-Epstein [BE09] and Gauthier [Gau12] in the context of rational maps, and is based on the notion of *large scale condition* introduced in [Ast+19]. It is a way to give a quantified meaning to the *simultaneous independent bifurcation of multiple critical points*, and to exploit this condition to prove the non-vanishing of T^k_{bif}.

The second ingredient is a procedure to build these multiple independent bifurcations at a common parameter starting from a simple one. The idea is to start with a parameter with a Misiurewicz bifurcation (i.e., a non-persistent collision between a critical orbit and a repelling point, see Definition 1.1.5), and to construct a new parameter nearby where two – and actually, D_d – independent Misiurewicz bifurcations occur simultaneously. Here we say that k Misiurewicz relations are *independent* at a parameter λ if the intersection of the k hypersurfaces given by the Misiurewicz relations has codimension k in $\mathbf{Sk}(p,d)$, and we denote by Bif^k the closure of such parameters.

This geometrical construction is our main technical result, and the key point of [AB22]. Together with the analytic arguments mentioned above (which give $\mathrm{Bif}^k \subseteq \operatorname{Supp} T^k_{\mathrm{bif}}$ for all $1 \le k \le D_d$) and the trivial inclusion $\operatorname{Supp} T^{D_d}_{\mathrm{bif}} \subseteq \operatorname{Supp} T_{\mathrm{bif}}$, it implies Theorem 1.3.4.

Theorem 1.3.5 (Astorg-Bianchi, [AB22]). *Let p be a polynomial with Julia set not totally disconnected, which is neither conjugated to $z \mapsto z^d$ nor to a Chebyshev polynomial. Let $\mathbf{Sk}(p,d)$ denote the family of polynomial skew products of degree $d \ge 2$ over the base polynomial p, up to affine conjugacy, and let D_d be its dimension. Then*

$$\mathrm{Bif} = \mathrm{Bif}^2 = \cdots = \mathrm{Bif}^{D_d}.$$

In order to construct the desired Misiurewicz parameter, we had to consider the motion of a sufficiently large hyperbolic subset of the Julia set near a parameter in the bifurcation locus. This hyperbolic set needed to satisfy some precise properties, and this is where the assumptions on p come into play. The construction uses tools from the thermodynamic formalism of rational maps, and more generally of endomorphisms of $\mathbb{P}^k$ (but is independent of the results presented in Section 2.2). Once the hyperbolic set is constructed, the proof proceeds by induction. We show that, given a Misiurewicz relations satisfying a given list of further properties, it is possible to construct, one by one, the extra Misiurewicz relations happening simultaneously.

Our main theorems and the method developed for their proof have a number of consequences and corollaries. Below are a few of them.

Corollary 1.3.6. *Let p be a polynomial with Julia set not totally disconnected, which is neither conjugated to $z \mapsto z^d$ nor to a Chebyshev polynomial. Near any bifurcation parameter in $\mathbf{Sk}(p,d)$ there exist algebraic subfamilies M^k of $\mathbf{Sk}(p,d)$ of any dimension $k < D_d$ such that the support of the bifurcation measure of M^k has non-empty interior in M^k.*

These families are given by the maps satisfying a given critical relation. Notice that d (and thus D_d) can be taken arbitrarily large. This result is for instance an improvement of Thereom 1.2.1, where 1-parameter families with the same property are constructed.

More strikingly, as already mentioned in Section 1.2, in [Duj17; Taf21], Dujardin and Taflin constructed open sets in the bifurcation locus in the family $\mathcal{H}_d(\mathbb{P}^k)$ of all endomorphisms of $\mathbb{P}^k$, $k \geq 1$, of a given degree $d \geq 2$. Their strategy also works when considering the subfamily of polynomial skew products (and actually these open sets are built close to this family). Combining Theorem 1.3.4 with their result we thus get the following consequence.

Corollary 1.3.7. *Let p be a polynomial with Julia set not totally disconnected, which is neither conjugated to $z \mapsto z^d$ nor to a Chebyshev polynomial. The support of the bifurcation measure in $\mathbf{Sk}(p,d)$ has non empty interior.*

Notice that it is not known whether the bifurcation locus is the closure of its interior (see the last paragraph in [Duj17]). Hence, a priori, the open sets as above could exist only in some regions of the parameter space. The last consequence of our main theorems is a uniform and optimal bound for the Hausdorff dimension of the support of the bifurcation measure, which is a generalization to this setting of the main result in [Gau12].

Corollary 1.3.8. *Let p be a polynomial with Julia set not totally disconnected, which is neither conjugated to $z \mapsto z^d$ nor to a Chebyshev polynomial. The Hausdorff dimension of the support of the bifurcation measure in $\mathbf{Sk}(p,d)$ is maximal at all points of its support.*

Notice that, in the family of all endomorphisms of a given degree, such a uniform estimate is not known even for the bifurcation locus, see Section 1.2.2 and [BB18a] for some local estimates.

It is natural to ask whether a similar result as Theorem 1.3.4 stays true in the full family of all endomorphisms of $\mathbb{P}^k$. As it is, the proof of Theorem 1.3.4 relies on the fibered structure at several points, and it is unclear how to adapt it to the general case. On the other hand, recently Gauthier, Taflin, and Vigny could establish a version of Corollary 1.3.7 in this full family. This was a step in the proof of a much deeper result on the sparsity of posticritically finite maps on $\mathbb{P}^k$ [GTV23].

1.4 Stability of expanding measures

The main goal of the work [BR22], joint with my postdoc Karim Rakhimov, was to strengthen Theorem 1.1.1 by showing that, in stable families of endomorphisms of $\mathbb{P}^k$ of algebraic degree $d \geq 2$, dynamical stability implies the existence of a well-defined local motion for *almost all points* with respect to *all measures on the Julia set with strictly positive Lyapunov exponents and not* *charging the post-critical set*. By results of de Thélin, Dinh, and Dupont [deT06; deT08; Din07; Dup12], this in particular applies to all ergodic measures whose measure-theoretic entropy is strictly larger than $(k-1)\log d$. In this case, the motion is well-defined on all the parameter space. Observe that the topological entropy of an endomorphism of $\mathbb{P}^k$ of algebraic degree d is equal to $k \log d$ [Gro03] (which is then equal to the measure-theoretic entropy of the unique measure of maximal entropy), and that $(k-1)\log d$ is a natural threshold for many dynamical phenomena. We refer to [BD23a; BD22; Dup12; SUZ14; UZ13], and in particular to Section 2.2, for large classes of examples of such measures in any dimension, and their statistical properties.

An analogous result is already known in the setting of polynomial diffeomorphisms of $\mathbb{C}^2$. As was mentioned in Section 1.1.1, a parallel theory to that of [BBD18; Bia19a] has been developed in this setting by Dujardin and Lyubich, see [DL15]. The current work was inspired by the unbranching property proved by Berger and Dujardin in [BD17] and provides a parallel to that result for families of endomorphisms of $\mathbb{P}^k$. As was already the case for [DL15] and [BBD18], our approach is completely different from the one in [BD17], since the key point here is to understand the relation between the Julia sets and the dynamics of the critical set (which does not exist in the case of diffeomorphims of $\mathbb{C}^2$). This is achieved with techniques from pluripotential theory.

In order to state the main result of [BR22], we need a more general version of some definitions in Section 1.1.1. Recall that we denote by $\mathcal{J}$ the set of all holomorphic maps $\gamma\colon M \to \mathbb{P}^k$ such that $\gamma(\lambda)$ belongs to J_λ for all $\lambda \in M$. We often identify a map γ with its graph Γ_γ in the product space $M \times \mathbb{P}^k$. The family $(f_\lambda)_{\lambda\in M}$ induces a dynamical system $\mathcal{F}$ on the space $\mathcal{J}$ by $\mathcal{F}\gamma(\lambda) := f_\lambda(\gamma(\lambda))$. Recall that dynamical laminations are defined in Definition 1.1.6.

Definition 1.4.1. *Given $\lambda_0 \in M$ and an f_{λ_0}-invariant probability measure ν supported on J_{λ_0}, a dynamical lamination $\mathcal{L}$ is said to be a* (λ_0, ν)-lamination *(or* ν-lamination *for brevity) if $\nu(\{\gamma(\lambda_0) : \gamma \in \mathcal{L}\}) = 1$.*

Recall that, for every $\lambda \in M$, we denote by $p_\lambda : \mathcal{J} \to J_\lambda$, the natural map given by $p_\lambda(\gamma) = \gamma(\lambda)$.

Definition 1.4.2. *Given $\lambda_0 \in M$ and an f_{λ_0}-invariant probability measure ν supported on J_{λ_0}, a* (λ_0, ν)-web *(or* ν-web *for brevity) is a web $\mathcal{M}$ such that $(p_{\lambda_0})_*\mathcal{M} = \nu$.*

We say that $\mathcal{J}$ is *unbranched at* (λ_0, z_0) if there exists (at most) one $\gamma \in \mathcal{J}$ with $\gamma(\lambda_0) = z_0$. It follows from [BBD18, Lemma 2.5] that, if $(f_\lambda)_{\lambda\in M}$ is stable, for every $\lambda \in M$ and $z \in J_\lambda$ there exist $\gamma \in \mathcal{J}$ such that $\gamma(\lambda) = z$. Moreover, by [BBD18, Section 4.3], such γ is unique for every $\lambda \in M$ and μ_λ-almost every $z \in J_\lambda$. Hence, $\mathcal{J}$ is unbranched at (λ, z) for every λ and μ_λ-almost every $z \in J_\lambda$.

Our main result can be stated as follows. For every $\lambda \in M$, we denote by C_{f_λ} the critical set and by $C^+_{f_\lambda} := \cup_{m\geq 0} f^m_\lambda(C_{f_\lambda})$ the postcritical set of f_λ.

Theorem 1.4.3 (Bianchi-Rakhimov [BR22])**.** *Let M be an open connected and simply connected manifold and $(f_\lambda)_{\lambda\in M}$ a stable family of endomorphisms of $\mathbb{P}^k$ of algebraic degree $d \geq 2$. Fix $\lambda_0 \in M$ and consider an ergodic f_{λ_0}-invariant probability measure ν_0 on J_{λ_0} with strictly positive Lyapunov exponents and such that $\nu(C^+_{f_{\lambda_0}}) = 0$. Then, $\mathcal{J}$ is unbranched at (λ_0, z) for ν_0-almost every $z \in J_{\lambda_0}$. Moveover, up to replacing M with a sufficiently small open neighbourhood M_{λ_0,ν_0} of λ_0,*

(S4') *there exist a ν_0-lamination;*

(S5') *there exists an ergodic acritical ν_0-web $\mathcal{M}$ and a constant $A > 0$ such that for all $\lambda \in M_{\lambda_0,\nu_0}$ the Lyapunov exponents of $(p_\lambda)_* \mathcal{M}$ are uniformly bounded from below by A.*

The main reason why we may need to reduce M to M_{λ_0,ν_0} (possibly depending on ν_0) is the fact that the smallest Lyapunov exponent of the motion of ν_0 may, a priori, become negative at some $\lambda_1 \in M$. In our construction (and more precisely in the construction of the ν_0-lamination) we need to restrict to the parameters where such an exponent stays positive.

It is proved in [BBD18] that the existence of a graph $\gamma\colon M \to \mathbb{P}^k$ whose orbit does not intersect the postcritical set implies the stability of the family, and is then equivalent to it. In particular, it follows directly that the existence of an acritical web (associated to any measure, hence in particular **(S4')**) implies that the family is stable. The implication **(S5')** $\Rightarrow$ **(S4')** follows from similar arguments as in [BBD18; Bia19a; BB18a]. The main point of Theorem 1.4.3 is the proof of **(S5')**, and in particular the positivity of the Lyapunov exponents.

The first task is to prove the existence of an acritical ν_0-web. This is done by first approximating the measure ν_0 with measures supported on the repelling points for f_{λ_0}, and using the holomorphic motions of such points (which, at least asymptotically, exist by assumption) to construct a measure $\mathcal{M}$ on $\mathcal{J}$ such that $(p_{\lambda_0})_* \mathcal{M} = \nu_0$. This measure $\mathcal{M}$ can be made invariant and ergodic by standard arguments. It is then an ergodic ν_0-web. This in particular implies that $(p_\lambda)_* \mathcal{M}$ is ergodic for all λ, so that the Lyapunov exponents as in the statement are well defined.

We then need to prove the positivity of such exponents near λ_0. By a generalization of a result by Pham [Pha05], the upper sums of such exponents are plurisubharmonic (in particular, the sum of all of them but the smallest is upper semicontinuous). We show that the sum is pluriharmonic. This gives the lower semicontinuity of the smallest exponent. Since this exponent is positive at λ_0 by assumption, this gives the desired positivity in a neighbourhood of λ_0.

Proving the pluriharmonicity of the Lyapunov sums gives a version of **(S3)** for these more general measures. In order to get it, we follow the general strategy as in [BBD18] for the proof of the pluriharmonicity of the sum of the Lyapunov exponents of the measure of maximal entropy. A delicate point is to ensure that, as soon as (the graph of) an element in the support of $\mathcal{M}$ intersects some component of the postcritical set, it has to be contained in it. This is achieved thanks to the absence of Misiurewicz parameters in a stable family, a condition that makes no reference to the measure.

Given any $\lambda_0 \in M$ and any ergodic f_{λ_0}-invariant measure ν_0 as in Theorem 1.4.3, it is possible to define a local measurable holomorphic motion of J_λ associated to the measure ν_0. In particular, by [deT08; Dup12] and [deT06; Din07], Theorem 1.4.3 applies when the measure-theoretic entropy of the measure ν_0 is strictly larger than $(k-1)\log d$. We then have the following consequence.

Corollary 1.4.4. *Let M be an open connected and simply connected manifold and $(f_\lambda)_{\lambda \in M}$ a holomorphic family of endomorphisms of $\mathbb{P}^k$ of a given algebraic degree $d \geq 2$. Assume that the family $(f_\lambda)_{\lambda \in M}$ is stable in the sense of Theorem-Definition 1.1.1.*

1. *For any ergodic f_{λ_0}-invariant probability measure ν_0 such that $h_{\nu_0}(f_{\lambda_0}) > (k-1)\log d$, the properties **(S4')** and **(S5')** hold on M;*
2. *there exists a dynamical lamination $\mathcal{L}$ satisfying $\nu(\{\gamma(\lambda) : \gamma \in \mathcal{L}\}) = 1$ for every $\lambda \in M$ and every ergodic f_λ-invariant measure ν whose measure-theoretic entropy is strictly larger than $(k-1)\log d$.*

Results analogous to Theorem 1.4.3 and Corollary 1.4.4 also hold in the much more general setting of families of polynomial-like maps of large topological degree, see [DS10a] and Section 3.3. A version of Theorem 1.1.1 in this more general setting is the main result of [Bia19a]. Theorem 1.4.3 in this more general setting has exactly the same statement and proof. On the other hand, as part of the proof of the analogue of Corollary 1.4.4, we also needed to prove a generalization of de Thélin and Dupont theorem above [deT08; Dup12], giving the strict positivity of the Lyapunov exponents of measures of sufficiently large measure-theoretic entropy for polynomial-like maps of large topological degree. As other properties of polynomial-like maps related to this one will be studied more in detail in Chapter 3, and in particular in Section 3.3.1, we postpone to Section 3.3.2 the description of these results.

We conclude this section by observing that a version of the equidistribution of repelling points (in this case, repelling graphs) towards a web $\mathcal{M}_\nu$ as above, when ν is a suitable equilibrium state, was the main goal of my work [BB23b], joint with Maxence Brévard. This is a result in the direction of **(SQ3)**. As equilibrium states will be studied in Chapter 2, and in particular in Section 2.2, the description of such result is postponed to Section 3.1.2. We also observe that it is natural to ask whether it is true that all the slices $(p_\lambda)_*\mathcal{M}$ are equilibrium states as soon as one of them has such property. This question, together with related ones, is addressed by Brévard and Rakhimov in [BR24], which is part of Brévard's PhD thesis.

2 Thermodynamics and statistical properties of equilibrium states

2.1 Motivations

Let $f \colon X \to X$ be a dynamical system displaying some chaotic behaviour (i.e., strong sensitivity of the orbits from the initial condition). A basic question would be to understand the orbits of points. But this is, essentially by definition, an impossible task for most of the orbits. A possible approach to the problem is to instead consider an *observable*, i.e., a function $u \colon X \to \mathbb{R}$, and study the sequence $u(x), u(f(x)), \dots$ of the evaluations of u along orbits. This leads to consider the sequence $U_i := u \circ f^i$ as a sequence of *random variables* on the space X. We can take here any invariant measure ν as reference. The fact that ν is invariant precisely says that the U_i's are identically distributed. On the other hand, they are clearly not independent, as they arise from a completely deterministic setting (once one fixes a point, the orbit is completely determined). A general, heuristic, goal can then be phrased as follows.

General goal: Prove that the variables U_i are *sufficiently weakly correlated* to satisfy the main properties of independent random variables: central limit theorem, laws of iterated logarithms, almost sure invariance principles, large deviations theorems...

The more general one can take ν, and the more refined statistical properties one can prove, the more one can say to understand the system from a probabilistic point of view. Doing this for all invariant (even ergodic) measures is likely an impossible task, but one could at least consider all measures having some further dynamical significance. Here is where the *thermodynamic formalism* enters into play. This theory takes its motivation from thermodynamics and statistical mechanics and has found powerful applications in the field of dynamical systems, at least since the seminal works of Sinai, Ruelle, Bowen [Sin72; Rue78b; Bow08]. Roughly speaking, it is the study of the limit distributions (or *equilibrium states*) of configuration spaces under given constraints (or, more mathematically, of the invariant measures which are maximisers of some dynamically defined functional). More rigorously, one fixes a function φ (a *weight*) of a given regularity, and defines the pressure functional $P(\varphi) = \sup_\nu (h_\nu + \langle \nu, \varphi \rangle)$, where h_ν is the metric entropy of the measure ν and the supremum is taken over all invariant measures. An equilibrium state (associated to φ) is then an invariant measure (usually denoted μ_φ) maximizing the function $P(\varphi)$. The general goal can then be split as follows.

- **(B1)** Prove the existence (and, if possible, uniqueness) of equilibrium states μ_φ for φ of a given regularity;
- **(B2)** Prove the (if possible, exponential) mixing of μ_φ (i.e., the *decay of correlations*);

(B3) Establish statistical laws with respect to μ_φ, for observables u of a given (possibly different) regularity.

The more general we can take φ and u, the better we can say to understand the system from a statistical point of view.

I describe in this chapter the two pairs of works [BD23a; BD22] and [BD24; BD23b], obtained in collaboration with Tien-Cuong Dinh. In the first two, we completed the program above in the setting of generic endomorphisms of $\mathbb{P}^k$ and (for instance) Hölder continuous weights. In the second two, we proved the Central Limit Theorem for the measure of maximal entropy of any complex Hénon map, and of any automorphism of a compact Kähler manifold with a natural condition on the dynamical degrees.

2.2 Equilibrium states of holomorphic endomorphisms of $\mathbb{P}^k$

In this section I describe the completion of the program **(B1)**-**(B3)** in the setting of endomorphisms of $\mathbb{P}^k$ and (for instance) Hölder continuous weights φ, which we obtained with Tien-Cuong Dinh [BD23a; BD22]. I consider the results in this section my main achievements since my PhD, and the main results in this manuscript. Hence, I will give more details about the proofs. In order to make the text more readable, I will make some simplifying assumptions, in order to just focus on the main arguments.

2.2.1 Results

Let $f\colon \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic endomorphism of the complex projective space $\mathbb{P}^k = \mathbb{P}^k(\mathbb{C})$, with $k \geq 1$, of algebraic degree $d \geq 2$. Denote by μ the unique measure of maximal entropy for the dynamical system $(\mathbb{P}^k, f)$ [Lyu83a; BD01; DS10a] and recall that the Julia set of f is the support $\mathrm{supp}(\mu)$ of μ. The measure μ corresponds to the equilibrium state of the system in the case without weight, i.e., when the weight is zero. The general program **(B1)**-**(B3)** above had already been studied for Hölder continuous weights using a geometric approach, in dimension 1, see, e.g., Denker-Przytycki-Urbański [Prz90; DU91a; DU91c; DPU96] and Haydn [Hay99] just to name a few, and also in higher dimensions, see Szostakiewicz-Urbański-Zdunik [UZ13; SUZ14]. In the papers [BD23a; BD22], we developed an analytic method which allowed us to obtain more general and more quantitative results. Many results are new even when $\varphi = 0$ and even for $k = 1$.

As in [BD23a; BD22], throughout this section we make use of the following technical assumption for f:

(Hf) the local degree of the iterate $f^n := f \circ \cdots \circ f$ (n times) satisfies

$$\lim_{n\to\infty} \frac{1}{n} \log \max_{a\in\mathbb{P}^k} \deg(f^n, a) = 0.$$

Here, $\deg(f^n, a)$ is the multiplicity of a as a solution of the equation $f^n(z) = f^n(a)$. Note that generic endomorphisms of $\mathbb{P}^k$ satisfy this condition, see [DS10b]. Our study still holds under a weaker condition that the exceptional set of f (i.e., the maximal proper analytic subset of $\mathbb{P}^k$

invariant by f^{-1}) is empty or more generally has no intersection with $\mathrm{supp}(\mu)$ (in particular, this condition is superfluous in dimension 1). However, this situation requires more technical conditions on the weight φ. The main result of [BD23a] is the following.

Theorem 2.2.1 (Bianchi-Dinh [BD23a]). *Let f be an endomorphism of $\mathbb{P}^k$ of algebraic degree $d \geq 2$ and satisfying the Assumption* **(Hf)** *above. Let φ be a real-valued function on $\mathbb{P}^k$ such that*

$$(\mathbf{H}\varphi) \qquad \|\varphi\|_{\log^q} < \infty \quad \textit{and} \quad \Omega(\varphi) := \max \varphi - \min \varphi < \log d$$

for some $q > 2$. Then φ admits a unique equilibrium state μ_φ, whose support is equal to the small Julia set of f. This measure μ_φ is K-mixing and mixing of all orders, and repelling periodic points of period n (suitably weighted) are equidistributed with respect to μ_φ as n goes to infinity. Moreover, there is a unique conformal measure m_φ associated to φ. We have $\mu_\varphi = \rho m_\varphi$ for some strictly positive continuous function ρ on $\mathbb{P}^k$ and the preimages of points by f^n (suitably weighted) are equidistributed with respect to m_φ as n goes to infinity.

We set

$$\|\varphi\|_{\log^q} := \sup_{a,b \in \mathbb{P}^k} |\varphi(a) - \varphi(b)| \cdot (\log^\star |a-b|)^q,$$

where $\log^\star r := 1 + |\log r|$ for every $r > 0$. We say that a function is $\log^q$-continuous if $\|\varphi\|_{\log^q} < \infty$, i.e., if its oscillation on a ball of radius r is bounded by a constant times $(\log^\star r)^{-q}$, see Section 2.2.2.1 for details. As mentioned in the introduction to this chapter, an *equilibrium state* as in the statement above is defined as follows, see for instance [Rue72; Wal00; PU10]. Given a *weight*, i.e., a real-valued continuous function, φ as above, we define the *pressure* of φ as

$$P(\varphi) := \sup \{h_\nu(f) + \langle \nu, \varphi \rangle\},$$

where the supremum is taken over all Borel f-invariant probability measures ν and $h_\nu(f)$ denotes the metric entropy of ν. An equilibrium state for φ is then an invariant probability measure μ_φ realizing a maximum in the above formula, that is,

$$P(\varphi) = h_{\mu_\varphi}(f) + \langle \mu_\varphi, \varphi \rangle.$$

On the other hand, a *conformal measure* is defined as follows. Define the *Perron-Frobenius* (or *transfer*) operator $\mathcal{L}$ with weight φ as (we often drop the index φ for simplicity)

$$\mathcal{L}g(y) := \mathcal{L}_\varphi g(y) := \sum_{x \in f^{-1}(y)} e^{\varphi(x)} g(x), \tag{2.1}$$

where $g \colon \mathbb{P}^k \to \mathbb{R}$ is a continuous test function and the points x in the sum are counted with multiplicity. A conformal measure is an eigenvector for the dual operator $\mathcal{L}^*$ acting on positive measures (the name *conformal* comes from [Pat76], where related measures are obtained on limit sets of finitely generated discrete subgroups of conformal mappings acting on the hyperbolic space, see [DU91b; DU91c]).

Notice that, in the case where φ is Hölder continuous, a part of Theorem 2.2.1 was established by Urbański-Zdunik [UZ13] (also under a genericity assumption for f), see also [Prz90; DU91a; DU91c; DPU96] for previous results in dimension $k = 1$. When φ is constant, the operator $\mathcal{L}$ reduces to a constant times the push-forward operator f_* and we get $\mu_\varphi = \mu$. For an account of the known results in this case, see for instance [DS10a] and [Bro65; FLM83; Lyu82; Lyu83a] for $k = 1$.

A reformulation of Theorem 2.2.1 is the following: given φ as in the statement, there exist a *scaling ratio* $\lambda > 0$ and a continuous function $\rho = \rho_\varphi \colon \mathbb{P}^k \to \mathbb{R}$ such that, for every continuous function $g \colon \mathbb{P}^k \to \mathbb{R}$, the following uniform convergence holds:

$$\lambda^{-n}\mathcal{L}^n g(y) \to c_g \rho \tag{2.2}$$

for some constant c_g depending on g. By duality, this is equivalent to the convergence, uniform on probability measures ν,

$$\lambda^{-n}(\mathcal{L}^*)^n \nu \to m_\varphi, \tag{2.3}$$

where m_φ is a conformal measure associated to the weight φ. The equilibrium state μ_φ is then given by $\mu_\varphi = \rho m_\varphi$, and we have $c_g = \langle m_\varphi, g\rangle$.

To prove Theorem 2.2.1, we developed in [BD23a] a new and completely different approach with respect to [UZ13] and to the previous studies in dimension 1. As we will see later, the flexibility of this method allowed for a more quantitative understanding of the convergences (2.2) and (2.3), and for the direct establishment of several statistical properties of the equilibrium states.

I will give more details on the method in Section 2.2.3, but I explain here the main idea. I just consider for simplicity the case where both of the functions g and φ are of class $\mathcal{C}^2$ (the general case requires suitable approximations of g and φ by $\mathcal{C}^2$ functions, and this is where the assumption on $\|\varphi\|_{\log^q} < \infty$ for some $q > 2$ plays its main role). Given such a function g, first we want to prove that the ratio between the maximum and the minimum of $\mathcal{L}^n g$ stays bounded with n. This allows us to define the good scaling ratio λ and to get that the sequence $\lambda^{-n}\mathcal{L}^n g$ is uniformly bounded. Next, we would like to prove that this sequence is actually equicontinuous. By means of the theory of almost periodic operators [Lyu88], which was first applied to the thermodynamical formalism in [Lyu82; Lyu83a], this would imply the existence and uniqueness of the limit function ρ.

In order to establish the above controls, we study the sequence of $(1,1)$-currents given by $dd^c\mathcal{L}^n g$. First we prove that suitably normalized versions of these currents are uniformly bounded by a common positive closed $(1,1)$-current R. This is the core of our method which replaces all controls on the distortion of inverse branches of f^n in the geometric method of [UZ13] by a unique, global, and flexible estimate. Namely, for every $n \in \mathbb{N}$ we can get an estimate of the form

$$\Big| dd^c \frac{\mathcal{L}^n g}{c_n} \Big| \lesssim \sum_{j=0}^{\infty} \Big(\frac{e^{\Omega(\varphi)}}{d}\Big)^j \frac{(f_*)^j \omega_{\mathrm{FS}}}{d^{(k-1)j}} \quad \text{with} \quad c_n := \|g\|_{\mathcal{C}^2} \langle \omega_{\mathrm{FS}}^k, \mathcal{L}^n \mathbb{1}\rangle. \tag{2.4}$$

Here, ω_{FS} denotes the usual Fubini-Study form on $\mathbb{P}^k$ normalized so that ω_{FS}^k is a probability measure. Notice that the last infinite sum gives a key reason for the assumption $\Omega(\varphi) < \log d$ made on the weight φ as the mass of the current $f_*^j\omega_{\mathrm{FS}}$ is equal to $d^{(k-1)j}$.

We then establish some general criteria, interesting in themselves, which allow one to bound the oscillation of $c_n^{-1}\mathcal{L}^n g$ in terms of the oscillation of the potentials of the current in the RHS of (2.4). This latter oscillation is actually controllable. Assumption **(Hf)** allows us to have a simple control which makes the estimates less technical but such a control exists without Assumption

(Hf). Both the criteria and the above controls are collected in Section 2.2.2. They can be assumed in Section 2.2.3 if the reader just prefers to focus on the dynamical arguments in the method.

Combining all these ingredients, the existence and uniqueness of the equilibrium state and conformal measure, as well as the equidistribution of preimages and the equality $P(\varphi) = \log \lambda$, follow from more standard arguments (which I will not recall in this text), see for instance [PU10; UZ13] and [BD23a, Section 4.2]. We also prove that the entropy of μ_φ is larger than $k \log d - \Omega(\varphi) > (k-1)\log d$, and that all the Lyapunov exponents of μ_φ are strictly positive. This also leads to a lower bound for the Hausdorff dimension of μ_φ.

We also establish the equidistribution of repelling periodic points with respect to μ_φ, which completes the proof of Theorem 2.2.1. This result is due to Lyubich [Lyu82; Lyu83a] (for $k = 1$) and Briend-Duval [BD99] (for any $k \geq 1$) when $\varphi = 0$, and is new even for $k = 1$ otherwise. Since the proof of this result uses different techniques, and is also related to my other work [BB23b], I will describe it in more details in Section 3.1.

Without extra arguments, given a continuous test function g the convergence as $n \to \infty$ in (2.2) is not uniform in g. Our next and main goal was to establish an exponential speed of convergence in (2.2). This requires to build a suitable (semi-)norm for (or equivalently, a suitable functional space on) which the operator $\lambda^{-1}\mathcal{L}$ turns out to be a contraction.

Establishing the following statement was then our main goal and is the main result of [BD22]. As far as we know, this is the first time that the existence of a spectral gap for the perturbed Perron-Frobenius operator is proved in this context even in dimension 1, except for hyperbolic endomorphisms or for weights with ad-hoc conditions (see for instance [Rue92; MS00]). Observe that, while in Theorem 2.2.1 φ is required to just be $\log^q$-continuous, here it may a priori have to be (slightly) more regular. An important and specific feature of our norms, which will be highlighted below, is their dependence on the map f.

Theorem 2.2.2 (Bianchi-Dinh [BD22]). *Let $f, q, \varphi, \rho, m_\varphi$ be as in Theorem 2.2.1 and $\mathcal{L}, \lambda$ the above Perron-Frobenius operator and scaling factor associated to φ. Let $A > 0$ and $0 < \Omega < \log d$ be two constants. Then, for every constant $0 < \gamma \leq 1$, there exist two explicit equivalent norms for functions on $\mathbb{P}^k$: $\|\cdot\|_{\diamond_1}$, depending on f, γ, q and independent of φ, and $\|\cdot\|_{\diamond_2}$, depending on f, φ, γ, q, such that*

$$\|\cdot\|_\infty + \|\cdot\|_{\log^q} \lesssim \|\cdot\|_{\diamond_1} \simeq \|\cdot\|_{\diamond_2} \lesssim \|\cdot\|_{\mathcal{C}^\gamma}\,.$$

Moreover, there are positive constants $c = c(f, \gamma, q, A, \Omega)$ and $\beta = \beta(f, \gamma, q, A, \Omega)$ with $\beta < 1$, both independent of φ and n, such that when $\|\varphi\|_{\diamond_1} \leq A$ and $\Omega(\varphi) \leq \Omega$ we have

$$\|\lambda^{-n}\mathcal{L}^n\|_{\diamond_1} \leq c, \quad \|\rho\|_{\diamond_1} \leq c, \quad \|1/\rho\|_{\diamond_1} \leq c, \quad \textit{and} \quad \left\|\lambda^{-1}\mathcal{L}g\right\|_{\diamond_2} \leq \beta \, \|g\|_{\diamond_2}$$

for every function $g \colon \mathbb{P}^k \to \mathbb{R}$ with $\langle m_\varphi, g\rangle = 0$. Furthermore, given any constant $1 < \delta < d^{\gamma/(2\gamma+2)}$, when A is small enough the norm $\|\cdot\|_{\diamond_2}$ can be chosen so that we can take $\beta = 1/\delta$.

According to this theorem, on the space of functions with bounded $\|\cdot\|_{\diamond_2}$ norm, the operator $\lambda^{-1}\mathcal{L}$ admits a spectral gap. It acts as the identity of the line spanned by ρ while its norm on the invariant hyperplane $\{g : \langle m_\varphi, g\rangle = 0\}$ is bounded by $\beta < 1$.

The construction of the norms $\|\cdot\|_{\diamond_1}$ and $\|\cdot\|_{\diamond_2}$ is quite involved and is the main achievement of this project. We use here ideas from the theory of interpolation between Banach spaces [Tri95] combined with techniques from pluripotential theory and complex dynamics. As above, I give here a rough overview (see also Section 2.2.4) and refer to Sections 2.2.5 and 2.2.6, where I will give more details on the construction and properties of the norms and on the proof of the spectral gap, respectively.

Roughly speaking, an idea from interpolation theory allows us to reduce the problem to the case where $\gamma = 1$. The definition of the above norms in this case requires a control of the derivatives of g (in the distributional sense), and this is where we use techniques from pluripotential theory. This also explains why these norms are bounded by the $\mathcal{C}^1$ norm. Note that we should be able to bound the derivatives of $\mathcal{L}g$ in a similar way. A quick expansion of the derivatives of $\mathcal{L}g$ using (2.1) gives an idea of the difficulties that one faces. The existence of these norms is still surprising to us.

Let us highlight two among these difficulties. First, the objects from complex analysis and geometry are too rigid for perturbations with a non-constant weight: the operators f_*, d, and dd^c do not commute with the operator $\mathcal{L}$. In particular, the dd^c-method developed by the Dinh and Sibony (see for instance [DS10a]) cannot be applied in this context, even for small perturbations of the weight $\varphi = 0$. Moreover, there may be critical points on the support of the measure, which cause a loss in the regularity of functions under the operators f_* and $\mathcal{L}$. Notice that we do not assume that our potential degenerates at the critical points.

Our solution to these problems is to define a new invariant functional space and norm in this mixed real-complex setting, that we call the *dynamical Sobolev space* and *semi-norm*, taking into account both the regularity of the function (to cope with the rigidity of the complex objects) and the action of f (to take into account the critical dynamics). The construction of this norm requires the definition of several intermediate semi-norms and the precise study of the action of the operator f_* with respect to them. Some of the intermediate estimates already give new or more precise convergence properties for the operator f_* and the equilibrium measure μ.

A spectral gap for the Perron-Frobenius operator and its perturbations is one of the most desirable properties in dynamics. It allowed us to obtain several statistical properties of the equilibrium state. In the present setting, we have the following result. As this follows from the spectral gap in Theorem 2.2.2 by means of standard techniques, I will not detail the proof in this manuscript. I just notice that the estimates on ρ and $1/\rho$ in Theorem 2.2.2 are needed to study the operator $\mathcal{L}$ (which leaves invariant the space generated by ρ) by means of the operator $L(\cdot) := \lambda^{-1}\rho^{-1}\mathcal{L}(\rho\,\cdot)$, which fixes constant functions.

Theorem 2.2.3 (Bianchi-Dinh [BD22]). *Let $f, \varphi, \mu_\varphi, m_\varphi, \|\cdot\|_{\diamond_1}$ be as in Theorems 2.2.1 and 2.2.2 , λ the scaling ratio associated to φ, and assume that $\|\varphi\|_{\diamond_1} < \infty$. Then the equilibrium state μ_φ is exponentially mixing for observables with bounded $\|\cdot\|_{\diamond_1}$ norm and the preimages of points by f^n (suitably weighted) equidistribute exponentially fast towards m_φ as n goes to infinity. The measure μ_φ satisfies the Large Deviation principle (LDP) for all observables with finite $\|\cdot\|_{\diamond_1}$ norm, the Almost Sure Invariant Principle (ASIP), Central Limit Theorem (CLT), Almost Sure Central Limit Theorem (ASCLT), Law of Iterated Logarithm (LIL) for all observables with finite $\|\cdot\|_{\diamond_1}$ norm which are not coboundaries, and the local Central Limit Theorem for all observables with finite $\|\cdot\|_{\diamond_1}$ norm which are not $(\|\cdot\|_{\diamond_1}, \varphi)$-multiplicative cocycles. Moreover, the pressure $P(\varphi)$ is equal to $\log\lambda$ and is analytic in the following sense: for $\|\psi\|_{\diamond_1} < \infty$ and t sufficiently small, the function $t \mapsto P(\varphi + t\psi)$ is analytic.*

In particular, all the properties in Theorem 2.2.3 hold when the weight φ and the observable are Hölder continuous and satisfy the necessary coboundary/cocycles requirements. Under such assumptions, some of the above properties were previously obtained with ad hoc arguments, see [PUZ89; DU91a; DU91c; DPU96; Hay99; DNS07; SUZ15] when $k = 1$, [UZ13; SUZ14] for mixing, CLT, LIL when $k \geq 1$, and [Dup10] for the ASIP when $k \geq 1$ and $\varphi = 0$. Note that the LDP and the local CLT are new even for $\varphi = 0$ (for all $k \geq 1$ and for all $k > 1$, respectively). The proof of Theorem 2.2.3 exploits the spectral gap established in Theorem 2.2.2 and is based on the theory of perturbed operators [Nag57; PP90; Rou83; Bro96; Gou15]. Observe in particular that the fine control given for instance by the local CLT is simply impossible to prove using weaker arguments, such as martingales, see, e.g., [Gou15, p. 163].

2.2.2 Notations and preliminary comparison principles

2.2.2.1 $\log^p$-continuous functions

Given a subset U of $\mathbb{P}^k$ or $\mathbb{C}^k$ and a real-valued function $g\colon U \to \mathbb{R}$, define the *oscillation* $\Omega_U(g)$ of g as

$$\Omega_U(g) := \sup g - \inf g$$

and its continuity modulus $m_U(g,r)$ at distance r as

$$m_U(g,r) := \sup_{x,y\in U\colon\ \mathrm{dist}(x,y)\leq r} |g(x)-g(y)|.$$

We may drop the index U when there is no possible confusion.

Definition 2.2.4. *The semi-norm $\|\cdot\|_{\log^p}$ is defined for every $p > 0$ and $g\colon \mathbb{P}^k \to \mathbb{R}$ as*

$$\|g\|_{\log^p} := \sup_{a,b\in\mathbb{P}^k} |g(a)-g(b)|\cdot(\log^\star \mathrm{dist}(a,b))^p = \sup_{r>0,a\in\mathbb{P}^k} \Omega_{\mathbb{B}_{\mathbb{P}^k}(a,r)}(g)\cdot(1+|\log r|)^p,$$

where $\mathbb{B}_{\mathbb{P}^k}(a,r)$ denotes the ball of center a and radius r in $\mathbb{P}^k$ and $\log^\star(\cdot) := 1 + |\log(\cdot)|$.

Observe, in particular, that all Hölder continuous functions satisfy the above condition for all $p > 0$. We have the following technical lemma, which will be important to deduce Theorem 2.2.1 for $\log^q$-continuous functions φ from its counterpart for smooth ones.

Lemma 2.2.5. *For every $\log^p$-continuous function $g\colon \mathbb{P}^k \to \mathbb{R}$, $p > 0$, $s \geq 1$, and $0 < \varepsilon \leq 1$, there exist continuous functions $g_\varepsilon^{(1)}$ and $g_\varepsilon^{(2)}$ such that*

$$g = g_\varepsilon^{(1)} + g_\varepsilon^{(2)}, \qquad \|g_\varepsilon^{(1)}\|_{\mathcal{C}^s} \leq c\,\|g\|_\infty\, e^{(1/\varepsilon)^{1/p}}, \qquad \textit{and} \qquad \|g_\varepsilon^{(2)}\|_\infty \leq c\,\|g\|_{\log^p}\,\varepsilon,$$

where $c = c(p,s)$ is a positive constant independent of g and ε. In particular, for every $n \geq 1$ there exist $g_n^{(1)}$ of class $\mathcal{C}^2$ and $g_n^{(2)}$ continuous such that

$$g = g_n^{(1)} + g_n^{(2)}, \qquad \|g_n^{(1)}\|_{\mathcal{C}^2} \leq c\,\|g\|_\infty\, e^{\frac{1}{2}n^{2/p}}, \qquad \textit{and} \qquad \|g_n^{(2)}\|_\infty \leq c\,\|g\|_{\log^p}\, n^{-2}.$$

2.2.2.2 Comparisons between currents and their potentials

A technical key point in the proof of Theorem 2.2.1 is based on the following general idea: if u and v are two functions on some domain in $\mathbb{C}^k$ such that $|dd^c u| \le dd^c v$, then u inherits some of the regularity properties of v. We collect in this section the precise versions of the heuristic principle above, in the forms that will be used in the proof of Theorem 2.2.1. We omit the proofs, as they are mostly elementary. All the details can be found in [BD23a, Section 2.3]

Lemma 2.2.6. *There exists a positive constant A such that, for every positive closed $(1,1)$-current S_0 on $\mathbb{P}^k$ of mass 1 and for every positive closed $(1,1)$-current S on $\mathbb{P}^k$ with $S \le S_0$, we have $\Omega(u_S) \le A + \Omega(u_{S_0})$, where u_{S_0} and u_S denote the dynamical potentials (see Section 2.2.2.3 for the definition) of S_0 and S, respectively.*

Corollary 2.2.7. *There exists a positive constant A such that for every positive closed $(1,1)$-current S_0 on $\mathbb{P}^k$ and for every continuous function $g\colon \mathbb{P}^k \to \mathbb{R}$ with $|dd^c g| \le S_0$ we have $\Omega(g) \le A\,\|S_0\| + 3\Omega(u_{S_0})$.*

The following result gives a quantitative control on the oscillation of u in terms of the oscillation of v. Notice in particular that it implies that, if v is Hölder or $\log^p$-continuous for some $p > 0$, then u enjoys the same property with possibly a loss in the Hölder exponent, but not in the $\log^p$-exponent.

Proposition 2.2.8. *Let u and v be two p.s.h. functions on $\mathbb{B}^k_3$ such that $dd^c u \le dd^c v$ and v is continuous. Then u is continuous and for every $0 < s \le 1$ there is a positive constant A (independent of u and v) such that, for every $0 < r \le 1/2$, we have*

$$m_{\mathbb{B}^k_1}(u,r) \le m_{\mathbb{B}^k_2}(v,r^s) + A m_{\mathbb{B}^k_2}(u,r^s) r^{1-s} \le m_{\mathbb{B}^k_2}(v,r^s) + A\Omega_{\mathbb{B}^k_2}(u) r^{1-s}.$$

Corollary 2.2.9. *Let v be a continuous p.s.h. function on $\mathbb{B}^k_3$. Let u be a continuous real-valued function on $\mathbb{B}^k_3$ such that $|dd^c u| \le dd^c v$. Then for every $0 < s \le 1$ we have for $0 < r \le 1/2$*

$$m_{\mathbb{B}^k_1}(u,r) \le 3 m_{\mathbb{B}^k_2}(v,r^s) + A\left(\Omega_{\mathbb{B}^k_2}(u) + \Omega_{\mathbb{B}^k_2}(v)\right) r^{1-s},$$

where A is a positive constant independent of u and v.

Corollary 2.2.10. *Let S_0 be a positive closed $(1,1)$-current on $\mathbb{P}^k$ with continuous local potentials. Let $\mathcal{F}(S_0)$ denote the set of all continuous real-valued functions g on $\mathbb{P}^k$ such that $|dd^c g| \le S_0$. Then $\mathcal{F}(S_0)$ is equicontinuous.*

2.2.2.3 Dynamical potentials

Let T denote the Green $(1,1)$-current of f. It is positive closed and of unit mass, and can be defined as

$$T := \lim_{n\to\infty} \frac{1}{d^n}(f^n)^*(\omega_{\mathrm{FS}}),$$

see for instance [DS10a, Theorem 1.16 and Definition 1.17]. Let S be any positive closed $(1,1)$-current of mass m on $\mathbb{P}^k$. As the cohomology $H^{1,1}(\mathbb{P}^k, R)$ has dimension 1, there is a unique function $u_S\colon \mathbb{P}^k \to \mathbb{R}\cup\{-\infty\}$ which is p.s.h. modulo mT (i.e., locally written as $v - mv_T$ with v, v_T p.s.h. and $dd^c v_T = T$) and such that

$$S = mT + dd^c u_S \qquad \text{and} \qquad \langle \mu, u_S\rangle = 0.$$

Locally, u_S is the difference between a potential of S and a potential of mT. We call it *the dynamical potential* of S. Observe that the dynamical potential of T is zero, i.e., $u_T = 0$.

Recall that T has Hölder continuous potentials. So, u_S is locally the difference between a p.s.h. function and a Hölder continuous one. The dynamical potential of S behaves well under the push-forward and pull-back operators associated to f. Indeed, because of the invariance properties of T, we have

$$f^*S = md\cdot T + dd^c(u_S\circ f) \qquad \text{and} \qquad f_*S = md^{k-1}\cdot T + dd^c(f_*u_S),$$

which, together with the invariance properties of μ, imply

$$u_{f^*S} = u_S\circ f \qquad \text{and} \qquad u_{f_*S} = f_*u_S.$$

We refer the reader to [DS10a] for details. We will only use currents S such that u_S is continuous.

In the rest of this section we give estimates on the dynamical potentials of the currents $(f^n)_*\omega_{\text{FS}}$. As explained in Section 2.2.1, these estimates will allow us to globally control the distortion of f^n. We always assume that f satisfies the Assumption **(Hf)** in Section 2.2.1

Define

$$\omega_n := d^{-(k-1)n}(f^n)_*\omega_{\text{FS}}.$$

Recall that f_* multiplies the mass of a positive closed $(1,1)$-current by d^{k-1}. Therefore, all currents ω_n have unit mass. We denote by u_n the dynamical potential of ω_n. In particular, u_0 is the dynamical potential of ω_{FS}. It is known that u_0 is Hölder continuous, see [Kos97; DS10a].

Observe that $d^{-1}f^*\omega_{\text{FS}}$ is a smooth positive closed $(1,1)$-form of mass 1. Therefore, there is a unique smooth function v such that

$$dd^c v = d^{-1}f^*\omega_{\text{FS}} - \omega_{\text{FS}} \qquad \text{and} \qquad \langle\mu, v\rangle = 0.$$

It is not difficult to check that

$$u_n = d^{-(k-1)n}(f^n)_*u_0 \qquad \text{and} \qquad u_0 = -\sum_{n=0}^{\infty} d^{-n} v\circ f^n.$$

We will need explicit bounds on the oscillation $\Omega(u_n)$ of u_n. These are provided in the next result. The proof is elementary, but we notice that assumption **(Hf)** is used here (in a similar way as in the proof of Lemma 2.2.12 below). Without such assumption, the estimate is more complicated, as it can diverge (in a controlled way) near the periodic critical hypersurfaces.

Lemma 2.2.11. *For every constant $A>1$, there exists a positive constant c independent of n such that $\|u_n\|_\infty \le cA^n$ and $\Omega(u_n)\le cA^n$ for all $n\ge 0$.*

As an application of this estimate and of the Hölder continuity of the Green function, we have the following lemma that can be used to study the regularity of functions $g\colon \mathbb{P}^k \to \mathbb{R}$. Although this is not the precise form that we will use later, we give the statement and proof since it contains an idea that will be used several times in the following. It also gives an idea of how the assumption **(Hf)** is used.

Lemma 2.2.12. *Let $g\colon \mathbb{P}^k \to \mathbb{R}$ be a continuous function and $0 < \beta < 1$ a constant such that*

$$|dd^c g| \le \sum_{n=0}^{\infty} \beta^n \omega_n. \tag{2.5}$$

Then, for every $q > 0$, there is a positive constant $c = c(q, \beta)$ independent of g such that

$$\|g\|_{\log^q} \le c.$$

Proof. We bound the continuity modulus $m(g,r)$ of g by means of Corollary 2.2.9. We only need to consider $0 < r \le 1/2$. For this purpose, since T has Hölder continuous local potentials, it suffices to bound the continuity modulus of the dynamical potential of the RHS of (2.5). This dynamical potential is equal to

$$u := \sum_{n=0}^{\infty} \beta^n u_n.$$

Fix a constant $1 < A < 1/\beta$. By Lemma 2.2.11, we have $\|u_n\|_\infty \lesssim A^n$. Hence, for every N, we have

$$m(u,r) \lesssim \sum_{n\le N} \beta^n m(u_n, r) + \sum_{n>N} (A\beta)^n \lesssim \sum_{n\le N} \beta^n m(u_n,r) + (A\beta)^N.$$

Applying [DS10b, Corollary 4.4] inductively to some iterate of f, we see that the Assumption **(Hf)** implies:

(H'f) for every constant $\kappa > 1$, there are an integer $n_\kappa \ge 0$ and a constant $c_\kappa > 0$ independent of n such that for all $x, y \in \mathbb{P}^k$ and $n \ge n_\kappa$ we can write $f^{-n}(x) = \{x_1, \ldots, x_{d^{kn}}\}$ and $f^{-n}(y) = \{y_1, \ldots, y_{d^{kn}}\}$ (counting multiplicity) with the property that

$$\mathrm{dist}(x_j, y_j) \le c_\kappa \, \mathrm{dist}(x,y)^{1/\kappa^n} \quad \text{for } j = 1, \ldots, d^{kn}.$$

By definition, the function u_0 is γ-Hölder continuous for some Hölder exponent γ because T has Hölder continuous local potentials. The above property **(H'f)** implies that $(f^n)_* u_0$ is $\gamma\kappa^{-n}$-Hölder continuous for all $n \ge n_\kappa$. More precisely, we have

$$m(d^{-kn}(f^n)_* u_0, r) \le c' r^{\gamma\kappa^{-n}} \qquad \text{and hence} \qquad m(u_n, r) \le c' d^n r^{\gamma\kappa^{-n}}$$

for some positive constant c' independent of $n \ge n_\kappa$ and r. Observe also that for $0 \le n \le n_\kappa$ all the u_n are α_κ-Hölder continuous for some $\alpha_\kappa > 0$. Indeed, as the multiplicity of f^n at a point is at most d^{kn}, we have (see again [DS10b, Corollary 4.4]):

(H"f) there is a constant $c_0 > 0$ such that for every $n \ge 0$, for all $x, y \in \mathbb{P}^k$, we can write $f^{-n}(x) = \{x_1, \ldots, x_{d^{kn}}\}$ and $f^{-n}(y) = \{y_1, \ldots, y_{d^{kn}}\}$ (counting multiplicity) with the property that

$$\mathrm{dist}(x_j, y_j) \le c_0 \, \mathrm{dist}(x,y)^{1/d^{kn}} \quad \text{for } j = 1, \ldots, d^{kn}.$$

Therefore, we have

$$m(u,r) \lesssim r^{\alpha_\kappa} + \sum_{n_\kappa \le n \le N} (\beta d)^n \, r^{\gamma\kappa^{-n}} + (A\beta)^N. \tag{2.6}$$

Choose κ close enough to 1 so that $2q \log \kappa < |\log(A\beta)|$ and take

$$N = \frac{1}{2\log \kappa} \log |\log r|$$

(recall that we only need to consider $r \le 1/2$). Then, the last term in (2.6) satisfies

$$(A\beta)^N = e^{N \log(A\beta)} < e^{-2Nq\log\kappa} = |\log r|^{-q}.$$

It remains to prove that the sum in (2.6) satisfies a similar estimate. We have

$$\sum_{n\le N} (\beta d)^n \, r^{\gamma\kappa^{-n}} \le \sum_{n \le N} \beta^n d^N r^{\gamma\kappa^{-N}} \lesssim d^N r^{\gamma\kappa^{-N}} = e^{\frac{\log d}{2\log\kappa}\log|\log r|} e^{\gamma(\log r) e^{-\frac{1}{2}\log|\log r|}} = \frac{|\log r|^{\frac{\log d}{2\log\kappa}}}{e^{\gamma\sqrt{|\log r|}}}.$$

The last expression is smaller than a constant times $|\log r|^{-q}$ because $e^t \gg t^M$ when $t \to \infty$ for every $M \ge 0$. This, together with the above estimates, gives $m(u,r) \lesssim |\log r|^{-q}$ and ends the proof of the lemma. □

2.2.3 Existence, uniqueness, and first properties

In this section I give the main ideas of the proof of Theorem 2.2.1, and in particular of the existence of a good scaling ratio λ. Recall that the Perron-Frobenius operator $\mathcal{L}$ is defined as in (2.1). A direct computation gives

$$\mathcal{L}^n(g)(y) = \sum_{f^n(x)=y} e^{\varphi(x)+\varphi(f(x))+\cdots+\varphi(f^{n-1}(x))} g(x).$$

The following is the main statement.

Theorem 2.2.13. *Let f and φ be as in Theorem 2.2.1. There exist a number $\lambda > 0$ and a continuous function $\rho > 0$ on $\mathbb{P}^k$ such that $\mathcal{L}\rho = \lambda\rho$ and for every continuous function $g\colon \mathbb{P}^k \to \mathbb{R}$ the sequence $\lambda^{-n}\mathcal{L}^n(g)$ is equicontinuous and converges uniformly to $c_g\rho$, where c_g is a constant depending linearly on g. Moreover, if g is strictly positive, then c_g is strictly positive and the sequence $\mathcal{L}^n(g)^{1/n}$ converges uniformly to λ as n tends to infinity.*

For simplicity, I will give the full proof of the convergence in Theorem 2.2.13 in the case where g is equal to $\mathbb{1}$, the function constantly equal to 1, see Sections 2.2.3.1 and Section 2.2.3.2. The general case can be deduced from this particular case, and all the main ideas are already contained in the proof of this case. I briefly describe the general case and the conclusion of the proof of Theorem 2.2.13 in Section 2.2.3.3.

Define $\mathbb{1}_n := \mathcal{L}^n(\mathbb{1})$. Denote by ρ_n^+ and ρ_n^- the maximum and the minimum of $\mathbb{1}_n$, respectively. Consider also the ratio $\theta_n := \rho_n^+/\rho_n^-$ and the function $\mathbb{1}_n^* := (\rho_n^-)^{-1}\mathbb{1}_n$. Observe that the last function satisfies $\min \mathbb{1}_n^* = 1$.

I will also first make the following simplifying assumption on φ: I will assume that φ is $\mathcal{C}^2$, and that $\|\varphi\|_{\mathcal{C}^2}$ is sufficiently small. Again, the main ideas are all already there in this case. The general case is more technical and relies on a suitable interpolation for φ (and Lemma 2.2.5). This is how the assumption on $\|\varphi\|_{\log^q} < \infty$ appears in the main statement. I will mention the main modifications, as well as give some statements that will be needed later, in Section 2.2.3.2.

2.2.3.1 The case $g \equiv \mathbb{1}$ and $\|\varphi\| \ll 1$.

As mentioned above, we assume here that $\|\varphi\|_{\mathcal{C}^2} \ll 1$. The following result is in practice the key the prove Theorem 2.2.13, and it is the only part that will be detailed here.

Proposition 2.2.14. *Under the hypotheses of Theorem 2.2.13, the sequence $\{\theta_n\}$ is bounded and the sequence of functions $\{\mathbb{1}_n^*\}$ is uniformly bounded and equicontinuous.*

Proposition 2.2.14 follows from the following crucial estimate for $dd^c\mathbb{1}_n$. We will see here the role of the estimate of the $\mathcal{C}^2$ norm of φ.

Proposition 2.2.15. *There exists a a positive constant $c = c(\|\varphi\|_{\mathcal{C}^2})$, independent of n, such that for all $n \geq 1$ we have*

$$|dd^c\mathbb{1}_n| \leq c \sum_{m=0}^{n} m^3 e^{m \max \varphi} \rho^+_{n-m} d^{(k-1)m} \omega_m.$$

Recall that the function $\mathbb{1}_n$ is given by

$$\mathbb{1}_n(y) = \sum_{f^n(x)=y} e^{\varphi(x)+\varphi(f(x))+\cdots+\varphi(f^{n-1}(x))}.$$

In order to estimate $dd^c\mathbb{1}_n$, we will use a now classical construction due to Gromov [Gro03]. Define the manifold $\Gamma_n \subset (\mathbb{P}^k)^{n+1}$ by

$$\Gamma_n := \{(z, f(z), \ldots, f^n(z)) : z \in \mathbb{P}^k\},$$

which can also be seen as the graph of the map $(f, f^2, \ldots, f^n)$ in the product space $(\mathbb{P}^k)^{n+1}$. Consider the function $\mathbb{h}$ on $(\mathbb{P}^k)^{n+1}$ given by

$$\mathbb{h}(x) = \mathbb{h}(x_0, \ldots, x_n) := e^{\varphi(x_0)+\varphi(x_1)+\cdots+\varphi(x_{n-1})}.$$

The function $\mathbb{1}_n$ on $\mathbb{P}^k$ is equal to the push-forward of the function $\mathbb{h}_{|\Gamma_n}$ to the last factor $\mathbb{P}^k$ of $(\mathbb{P}^k)^{n+1}$. Indeed, denoting by π_n the restriction of the projection $x \mapsto x_n$ to Γ_n, we have

$$(\pi_n)_* (\mathbb{h})(y) = \sum_{(x_0,\ldots,x_n)\in\Gamma_n :\, x_n=y} \mathbb{h}(x) = \sum_{z\in f^{-n}(y)} e^{\varphi(z)+\cdots+\varphi(f^{n-1}(z))} = \mathbb{1}_n(y).$$

Recall that, since $dd^c\mathbb{1}_n$ is real, estimating $|dd^c\mathbb{1}_n|$ means finding a good positive closed $(1,1)$-current S on $\mathbb{P}^k$ such that both $S \pm dd^c\mathbb{1}_n$ are positive. According to the identities above, we have

$$dd^c\mathbb{1}_n = (\pi_n)_* (dd^c\mathbb{h}).$$

Thus, we need to estimate $dd^c\mathbb{h}$ on $(\mathbb{P}^k)^{n+1}$ and Γ_n. We define $\omega^{(m)}$ as the pullback of the Fubini-Study form ω_{FS} to $(\mathbb{P}^k)^{n+1}$ by the projection $x \mapsto x_m$. Equivalently, $\omega^{(m)}$ is a $(1,1)$-form on $(\mathbb{P}^k)^{n+1}$ such that $\omega^{(m)}(x) = \omega_{\text{FS}}(x_m)$.

Lemma 2.2.16. *There exists a positive constant $c = c(\|\varphi\|_{\mathcal{C}^2})$, independent of n, such that*

$$|dd^c \mathbb{h}| \leq c\, \mathbb{h} \sum_{m=0}^{n-1} (n-m)^3 \omega^{(m)}.$$

Proof. A direct computation gives

$$i\partial\overline{\partial}\mathbb{h} = \mathbb{h}\Big(\sum_{m=0}^{n-1} i\partial\overline{\partial}\varphi(x_m) + \sum_{m,m'=0}^{n-1} i\partial\varphi(x_m)\wedge\overline{\partial}\varphi(x_{m'})\Big).$$

For the first sum, observe that

$$|i\partial\overline{\partial}\varphi(x_m)| \lesssim \|\varphi\|_{\mathcal{C}^2}\omega^{(m)}(x).$$

For the second sum, consider $m' \leq m \leq n-N-1$. By using Cauchy-Schwarz's inequality, we have

$$\begin{aligned} |i\partial\varphi(x_m)\wedge&\overline{\partial}\varphi(x_{m'})| \\ &\leq (m-m'+1)^{-2} i\partial\varphi(x_m)\wedge\overline{\partial}\varphi(x_m) + (m-m'+1)^2 i\partial\varphi(x_{m'})\wedge\overline{\partial}\varphi(x_{m'}) \\ &\lesssim (m-m'+1)^{-2}\|\varphi\|^2_{\mathcal{C}^1}\omega^{(m)}(x) + (m-m'+1)^2\|\varphi\|^2_{\mathcal{C}^1}\omega^{(m')}(x). \end{aligned} \tag{2.7}$$

This and the fact that $\sum_{j=1}^\infty j^{-2}$ is finite imply that

$$\begin{aligned} \Big|\sum_{0\leq m'\leq m\leq n-1} & i\partial\varphi(x_m)\wedge\overline{\partial}\varphi(x_{m'})\Big| \\ &\lesssim \sum_{m=0}^{n-1}\|\varphi\|^2_{\mathcal{C}^1}\omega^{(m)}(x) + \sum_{m'=0}^{n-1}(n-m'+1)^3\|\varphi\|^2_{\mathcal{C}^1}\omega^{(m')}(x) \\ &\lesssim \sum_{m=0}^{n-1}(n-m)^3\|\varphi\|^2_{\mathcal{C}^1}\omega^{(m)}(x). \end{aligned} \tag{2.8}$$

We obtain by symmetry a similar estimate for the case where $m < m' \leq n-N-1$.

Finally, combining all the above identities and estimates we get

$$|i\partial\overline{\partial}\mathbb{h}| \lesssim \mathbb{h}\sum_{m=0}^{n-1}(n-m)^3(\|\varphi\|_{\mathcal{C}^2} + \|\varphi\|^2_{\mathcal{C}^1})\omega^{(m)}.$$

The lemma follows. □

Proof of Proposition 2.2.15. We are only interested in the restriction of $\mathbb{h}$ to the graph Γ_n. We deduce from Lemma 2.2.16 that

$$|dd^c\mathbb{1}_n| = |(\pi_n)_*\, dd^c\mathbb{h}| \lesssim \sum_{m=0}^{n-1}(n-m)^3(\pi_n)_*\left(\mathbb{h}\omega^{(m)}\right), \tag{2.9}$$

where the implicit constant depends on $\|\varphi\|_{\mathcal{C}^2}$ but not on n.

Using the definition of $\mathbb{h}$, we have

$$\mathbb{h} \lesssim e^{(n-m)\max\varphi} e^{\varphi(x_0)+\varphi(x_1)+\cdots+\varphi(x_{m-1})} \lesssim e^{(n-m)\max\varphi}\mathbb{h}',$$

where

$$\mathbb{h}' := e^{\varphi(x_0)+\varphi(x_1)+\cdots+\varphi(x_{m-1})}.$$

Note that the sum in the definition of $\mathbb{h}'$ contains m terms while the one of $\mathbb{h}$ contains n terms. The specific choice of $\mathbb{h}'$ is convenient for our next computation as it is related to the function $\mathbb{1}_m$.

Consider the map $\pi' : \Gamma_n \to (\mathbb{P}^k)^{n-m+1}$ defined by $\pi'(x) := x' := (x_m, \dots, x_n)$. Denote by Γ' the image of Γ_n by π'. It is the graph of the map $(f, \dots, f^{n-m})$ from $\mathbb{P}^k$ to $(\mathbb{P}^k)^{n-m}$. We also have for $x' \in \Gamma'$

$$\pi'^{-1}(x') = \big\{(y, f(y), \dots, f^{m-1}(y), x') \quad \text{with} \quad y \in f^{-m}(x_m)\big\}.$$

So $\pi' : \Gamma_n \to \Gamma'$ is a ramified covering of degree d^{km}.

Consider the map $\pi'' : \Gamma' \to \mathbb{P}^k$ defined by $\pi''(x') := x_n$. We have, for $x_n \in \mathbb{P}^k$,

$$\pi''^{-1}(x_n) = \big\{(z, f(z), \dots, f^{n-m}(z)) \quad \text{with} \quad z \in f^{-n+m}(x_n)\big\}.$$

So $\pi'' : \Gamma' \to \mathbb{P}^k$ is a ramified covering of degree $d^{k(n-m)}$. We have $\pi_n = \pi'' \circ \pi'$. Observe that $\pi'_*(\mathbb{h}'\omega^{(m)})$ is a $(1,1)$-form on Γ' such that

$$\begin{aligned}
\pi'_*(\mathbb{h}'\omega^{(m)})(x') &= \Big(\sum_{y \in f^{-m}(x_m)} e^{\varphi(y)+\cdots+\varphi(f^{m-1}(y))}\Big)\omega_{\mathrm{FS}}(x_m) \\
&\leq \rho_m^+ \omega_{\mathrm{FS}}(x_m) =: \rho_m^+ \omega'(x'),
\end{aligned}$$

where we define ω' as the pull-back of ω_{FS} to Γ' by the map $x' \mapsto x_m$. We also have

$$\pi''_*(\omega')(x_n) = \sum_{x_m \in f^{-n+m}(x_n)} \omega_{\mathrm{FS}}(x_m) = (f^{n-m})_*(\omega_{\mathrm{FS}})(x_n) = d^{(k-1)(n-m)}\omega_{n-m}(x_n).$$

Thus,

$$(\pi_n)_*(\mathbb{h}\omega^{(m)}) \lesssim e^{(n-m)\max\varphi}\pi''_*\pi'_*(\mathbb{h}'\omega^{(m)}) \leq e^{(n-m)\max\varphi}\rho_m^+ d^{(k-1)(n-m)}\omega_{n-m}$$

and

$$\begin{aligned}
\sum_{m=0}^{n-1}(n-m)^3(\pi_n)_*\left(\mathbb{h}\omega^{(m)}\right) &\lesssim \sum_{m=0}^{n-1}(n-m)^3 e^{(n-m)\max\varphi}\rho_m^+ d^{(k-1)(n-m)}\omega_{n-m} \\
&= \sum_{m=1}^{n} m^3 e^{m\max\varphi}\rho_{n-m}^+ d^{(k-1)m}\omega_m. \qquad (2.10)
\end{aligned}$$

Finally, we deduce the proposition from (2.9) and (2.10) by multiplying c with a large enough constant. □

We are now in the position to apply the comparisons principles in Section 2.2.2. Proposition 2.2.14 follows from the following two Lemmas 2.2.17 and 2.2.18. Recall that u_m is the dynamical potential of $\omega_n := d^{-(k-1)n}(f^n)_*\omega_{\mathrm{FS}}$, see Section 2.2.2.3.

Lemma 2.2.17. *Under the hypotheses of Theorem 2.2.13, the sequence (θ_n) is bounded.*

Proof. Observe that, by the definition of $\rho_n^\pm, \theta_n$, and $\Omega(\cdot)$, for every $K \geq 1$ the two inequalities $\theta_n \leq K$ and $\Omega(\mathbb{1}_n) \leq (K-1)\rho_n^-$ are equivalent. Hence, in order to get the assertion, it is enough to show that $\Omega(\mathbb{1}_n)/\rho_n^-$ is bounded. The constants that we use below are independent of n. Fix a constant δ such that $e^{\Omega(\varphi)} < \delta < d$.

We use Proposition 2.2.15 and Corollary 2.2.7 in order to estimate $\Omega(\mathbb{1}_n)$ in terms of $\Omega(u_m)$. Recall that u_m is the dynamical potential of ω_m. We also use Lemma 2.2.11, which gives $\Omega(u_m) \lesssim d^m \delta'^{-m}$ for any δ' such that $\delta < \delta' < d$. More precisely, we obtain from those results that

$$\Omega(\mathbb{1}_n) \lesssim \sum_{m=1}^{n} m^3 e^{m \max \varphi} d^{km} \delta'^{-m} \rho_{n-m}^+.$$

Since $\delta < \delta'$, we have

$$\Omega(\mathbb{1}_n) \lesssim \sum_{m=1}^{n} e^{m \max \varphi} d^{km} \delta^{-m} \rho_{n-m}^+. \tag{2.11}$$

Working by induction on n, we can assume that $\Omega(\mathbb{1}_m)/\rho_m^- \leq 1$ (recall that we are here assuming that $\|\varphi\|_{\mathcal{C}^2} \ll 1$, in order to simplify; the induction is slightly more complicated otherwise), which implies $\rho_m^+ \leq \rho_m^-$, for all $m < n$. We need to prove the same inequality for $m = n$. I prove that $\Omega(\mathbb{1}_n)/\rho_n^-$ is bounded, with a bound independent of n the actual argument requires slightly more involved estimates.

By (2.11), the induction hypothesis, and the inequality $\rho_{n-m}^- d^{km} e^{m \min \varphi} \leq \rho_n^-$, we have

$$\Omega(\mathbb{1}_n) \lesssim \sum_{m=1}^{n} e^{m \max \varphi} d^{km} \delta^{-m} \rho_{n-m}^- \leq \sum_{m=1}^{n} e^{m\Omega(\varphi)} \delta^{-m} \rho_n^-.$$

Then, using that $\delta > e^{\Omega(\varphi)}$, we obtain

$$\frac{\Omega(\mathbb{1}_n)}{\rho_n^-} \lesssim \sum_{m=1}^{n} e^{m\Omega(\varphi)} \delta^{-m}.$$

which gives the announced bound. Observe that the sum can be assumed to be arbitrarily small if we take $\|\varphi\|_{\mathcal{C}^2} \ll 1$, as in our simplifying assumption. This ends the proof of the lemma. □

Lemma 2.2.18. *Under the hypotheses of Theorem 2.2.13, for all $p > 0$ the sequence $\|\mathbb{1}_n^*\|_{\log^p}$ is bounded. In particular, the sequence of functions $\mathbb{1}_n^*$ is equicontinuous.*

Proof. By Lemma 2.2.17 the sequence (θ_n) is bounded. This and Proposition 2.2.15 imply that

$$|dd^c \mathbb{1}_n^*| \lesssim \frac{1}{\rho_n^-} \sum_{m=1}^{n} m^3 e^{m \max \varphi} \rho_{n-m}^- d^{(k-1)m} \omega_m.$$

Then, using $\rho_{n-m}^- d^{km} e^{m \min \varphi} \leq \rho_n^-$ we obtain

$$|dd^c \mathbb{1}_n^*| \lesssim \sum_{m=1}^{n} m^3 e^{m\Omega(\varphi)} d^{-m} \omega_m.$$

Finally, since $e^{\Omega(\varphi)} < d$, Lemma 2.2.12 implies the result. □

2.2.3.2 The case $g \equiv \mathbb{1}$ and general φ

Let us now describe how the proof in the previous section should be modified to handle a general φ with $\|\varphi\|_{\log^q} < \infty$, for some $q > 2$.

By Lemma 2.2.5 applied to φ instead of g, we can find functions φ_n and ψ_n such that

$$\varphi = \varphi_n + \psi_n, \qquad \|\varphi_n\|_{\mathcal{C}^2} \leq c\,\|\varphi\|_\infty\, e^{\frac{1}{2}n^{2/q}}, \qquad \text{and} \qquad \|\psi_n\|_\infty \leq c\,\|\varphi\|_{\log^q} n^{-2}. \tag{2.12}$$

Consider two integers $J \geq 0$ and $N \geq 0$. Define for $n \geq N+1$

$$\widehat{\mathcal{L}}_n(g)(x) := \sum_{f^n(x)=y} e^{\varphi_{n+J}(x)+\varphi_{n+J-1}(f(x))+\cdots+\varphi_{J+N+1}(f^{n-N-1}(x))} g(x).$$

This operator will be used to approximate $\mathcal{L}^n$. The gain here is the fact that the involved functions φ_m have controlled $\mathcal{C}^2$ norms. As above, we define

$$\widehat{\mathbb{1}}_n := \widehat{\mathcal{L}}_n \mathbb{1}, \quad \widehat{\rho_n^+} := \max \widehat{\mathbb{1}}_n, \quad \widehat{\rho_n^-} := \min \widehat{\mathbb{1}}_n, \quad \widehat{\theta}_n := \widehat{\rho_n^+}/\widehat{\rho_n^-}, \quad \text{and} \quad \widehat{\mathbb{1}}^*_n := (\widehat{\rho_n^-})^{-1}\widehat{\mathbb{1}}_n.$$

The following lemma allows us to reduce our problem to the study of the functions $\widehat{\mathbb{1}}_n$.

Lemma 2.2.19. *There exists a positive constant $c = c(N)$ such that, for all $n > N \geq 0$ and J,*

$$c^{-1} \leq \rho_n^+/\widehat{\rho_n^+} \leq c \quad \textit{and} \quad c^{-1} \leq \rho_n^-/\widehat{\rho_n^-} \leq c.$$

In particular, the sequence $\{\widehat{\theta}_n\}$ is bounded if and only if the sequence $\{\,\theta_n\,\}$ is bounded.

The following is the counterpart of Proposition 2.2.15 for a general φ, see [BD23a, Proposition 3.4]. The details will not be given here.

Proposition 2.2.20. *There exists a sub-exponential function $\eta(t) = ct^3 e^{(t+J)^{2/q}}$ with a positive constant $c = c(\|\varphi\|_{\log^q}, \|\varphi\|_\infty)$ independent of n, J and N such that for all $n > N \geq 0$ we have*

$$\left| dd^c \widehat{\mathbb{1}}_n \right| \leq \sum_{m=N+1}^{n-N} \eta(m) e^{m \max\varphi} \widehat{\rho}^+_{n-m} d^{(k-1)m}\omega_m + \sum_{m=n_0}^{n} d^{kN}\eta(m)e^{(n-N)\max\varphi} d^{(k-1)m}\omega_m,$$

where $n_0 := \max(n-N+1, N+1)$.

Idea of the proof of Proposition 2.2.14 for a general φ. Thanks to Proposition 2.2.20, we can show that $\{\widehat{\theta}_n\}$ is bounded (with similar arguments as in Lemma 2.2.17). By Lemma 2.2.19, this shows that the sequence $\{\theta_n\}$ is bounded.

In order to show that the sequence $\{\mathbb{1}^*_n\}$ is equicontinuous, it is enough to approximate it uniformly by an equicontinuous sequence. Take $N = 0$. Fix an arbitrary constant $0 < \varepsilon < 1$. Since $\|\varphi - \varphi_m\|_\infty \lesssim m^{-2}$ by (2.12), we can choose an integer J large enough so that for every $n \geq 0$ we have

$$(1-\varepsilon)\widehat{\mathbb{1}}_n \leq \mathbb{1}_n \leq (1+\varepsilon)\widehat{\mathbb{1}}_n.$$

This implies

$$\frac{1-\varepsilon}{1+\varepsilon}\,\widehat{\mathbb{1}}^*_n \leq \mathbb{1}^*_n \leq \frac{1+\varepsilon}{1-\varepsilon}\,\widehat{\mathbb{1}}^*_n.$$

Therefore, $|\mathbb{1}^*_n - \widehat{\mathbb{1}}^*_n|$ is bounded uniformly by a constant times ε. By arguments as in Lemma 2.2.18, we can show that the sequence $(\widehat{\mathbb{1}}^*_n)$ is equicontinuous. We deduce that the sequence $(\mathbb{1}^*_n)$ is equicontinuous as well. □

2.2.3.3 Proof of Theorem 2.2.13

We first define the scaling ratio λ. By definition of ρ_n^+, we easily see that the sequence (ρ_n^+) is sub-multiplicative, that is, $\rho_{n+m}^+ \leq \rho_m^+ \rho_n^+$ for all $m, n \geq 0$. It follows that the first limit in the following line exists

$$\lambda := \lim_{n\to\infty} \left(\rho_n^+\right)^{1/n} = \lim_{n\to\infty} \left(\rho_n^-\right)^{1/n},$$

where the last identity is due to the fact that (θ_n) is bounded, see Lemma 2.2.17. We have the following lemma.

Lemma 2.2.21. *The sequences $(\lambda^{-n}\rho_n^+)$ and $(\lambda^{-n}\rho_n^-)$ are both bounded above and below by positive constants. In particular, the sequence $(\lambda^{-n}\mathbb{1}_n)$ is uniformly bounded and equicontinuous.*

Proof. It is clear that the second assertion is a consequence of the first one and Proposition 2.2.14. We prove now the first assertion. Since the sequence ρ_n^+ is sub-multiplicative, it is well-known that $\inf_n(\rho_n^+)^{1/n}$ is equal to λ. Hence, we have $\lambda^{-n}\rho_n^+ \geq 1$. Since θ_n is bounded, we have $\rho_n^+ \lesssim \rho_n^-$. It follows that both $\lambda^{-n}\rho_n^\pm$ are bounded from below by positive constants. Similarly, the sequence ρ_n^- is super-multiplicative, i.e., $\rho_{n+m}^- \geq \rho_m^-\rho_n^-$ for all $m, n \geq 0$, and we deduce that that both $\lambda^{-n}\rho_n^\pm$ are bounded from above by positive constants. The lemma follows. □

The above completes the proof of the convergence in Theroem 2.2.13 in the case where $g \equiv \mathbb{1}$. We then extend the above result to all continuous test functions.

Lemma 2.2.22. *Let $\mathcal{F}$ be a uniformly bounded and equicontinuous family of real-valued functions on $\mathbb{P}^k$. Then the family*

$$\mathcal{F}_{\mathbb{N}} := \{\lambda^{-n}\mathcal{L}^n(g) \ : \ g \in \mathcal{F}, n \geq 0\}$$

is also uniformly bounded and equicontinuous.

I only give a sketch of the proof of the above lemma. For simplicity, I will assume that all the functions in $\mathcal{F}$ are $\mathcal{C}^2$, and that the same is true for φ. The general case (see [BD23a, Lemma 3.9]) can be handled by approximating elements of $\mathcal{F}$ with $\mathcal{C}^2$ functions, and by an interpolation on φ as in the previous section. In particular, I assume that $\|g\|_{\mathcal{C}^2}$ is bounded by a constant for $g \in \mathcal{F}$. The constants involved in the proof below do not depend on $g \in \mathcal{F}$.

Sketch of the proof of Lemma 2.2.22 in the $\mathcal{C}^2$ case. By Lemma 2.2.21, the family $\mathcal{F}_{\mathbb{N}}$ is uniformly bounded. We prove that it is equicontinuous.

We will use the same idea as in Proposition 2.2.15 and Lemma 2.2.16. Instead of the function $\mathbb{h}$, we need to consider the following slightly different function

$$\mathbb{H}(x_0, \dots, x_n) := e^{\varphi(x_0)+\varphi(x_1)+\cdots+\varphi(x_{n-1})} g(x_0) = \mathbb{h}(x_0, \dots, x_n) g(x_0).$$

We have

$$i\partial\overline{\partial}\mathbb{H} = (i\partial\overline{\partial}\mathbb{h})g(x_0) + \mathbb{h}(i\partial\overline{\partial}g(x_0)) + i\partial\mathbb{h} \wedge \overline{\partial}g(x_0) - i\overline{\partial}\mathbb{h} \wedge \partial g(x_0).$$

Applying Cauchy-Schwarz's inequality to the last two terms, and since g has a bounded $\mathcal{C}^2$ norm, we obtain

$$\begin{aligned}
|i\partial\overline{\partial}\mathbb{H}| &\leq |(i\partial\overline{\partial}\mathbb{h})g(x_0)| + |\mathbb{h}(i\partial\overline{\partial}g(x_0))| + i\mathbb{h}^{-1}\partial\mathbb{h} \wedge \overline{\partial}\mathbb{h} + i\mathbb{h}\partial g(x_0) \wedge \overline{\partial}g(x_0) \\
&\lesssim |i\partial\overline{\partial}\mathbb{h}| + \mathbb{h}\omega_{\mathrm{FS}}(x_0) + i\mathbb{h}^{-1}\partial\mathbb{h} \wedge \overline{\partial}\mathbb{h} + \mathbb{h}\omega_{\mathrm{FS}}(x_0) \\
&\lesssim |i\partial\overline{\partial}\mathbb{h}| + \mathbb{h}\omega_{\mathrm{FS}}(x_0) + i\mathbb{h}^{-1}\partial\mathbb{h} \wedge \overline{\partial}\mathbb{h}.
\end{aligned}$$

We claim that the last sum satisfies

$$|i\partial\overline{\partial}\mathbb{h}| + \mathbb{h}\omega_{\mathrm{FS}}(x_0) + i\mathbb{h}^{-1}\partial\mathbb{h}\wedge\overline{\partial}\mathbb{h} \lesssim \mathbb{h}\sum_{m=0}^{n-1}(n-m)^3\omega^{(m)}.$$

Lemma 2.2.16 shows that the first term $|i\partial\overline{\partial}\mathbb{h}|$ of the LHS is bounded by the RHS. The second term clearly satisfies the same property (consider $m=0$ in the above sum). For the last term, by Cauchy-Schwarz's inequality and using a computation as in the proof of Lemma 2.2.16, we have

$$i\mathbb{h}^{-1}\partial\mathbb{h}\wedge\overline{\partial}\mathbb{h} = \mathbb{h}\sum_{m,m'=0}^{n-1} i\partial\varphi(x_m)\wedge\overline{\partial}\varphi(x_{m'}) \lesssim \mathbb{h}\sum_{m=0}^{n-1}(n-m)^3\omega^{(m)}.$$

This implies the claim and gives a bound for $|i\partial\overline{\partial}\mathbb{H}|$.

Since $\mathcal{L}^n(g) = (\pi_n)_*(\mathbb{H})$, we obtain as in the proof of Proposition 2.2.15 that

$$|dd^c\lambda^{-n}\mathcal{L}^n(g)| \lesssim \lambda^{-n}\sum_{m=1}^{n} m^3 e^{m\max\varphi}\rho^+_{n-m}d^{(k-1)m}\omega_m.$$

By Lemmas 2.2.19 and 2.2.21 we have $\rho^{\pm}_{n-m}\lesssim\lambda^{n-m}$. Therefore, we obtain

$$|dd^c\lambda^{-n}\mathcal{L}^n(g)| \lesssim \sum_{m=1}^{n} m^3 e^{m\max\varphi}\lambda^{-m}d^{(k-1)m}\omega_m.$$

Finally, since $\lambda\geq d^k e^{\min\varphi}$ by definition of λ, the last estimate implies that

$$|dd^c\lambda^{-n}\mathcal{L}^n(g)| \lesssim \sum_{m=1}^{n} m^3 e^{m\Omega(\varphi)}d^{-m}\omega_m.$$

Lemma 2.2.12 and the fact that $d > e^{\Omega(\varphi)}$ imply the result. □

We now construct a density function ρ on $\mathbb{P}^k$. Recall that the sequence $\lambda^{-n}\mathbb{1}_n$ is uniformly bounded and equicontinuous. Therefore, the Cesaro sums

$$\widetilde{\mathbb{1}}_n := \frac{1}{n}\sum_{j=0}^{n-1}\lambda^{-j}\mathbb{1}_j$$

also form a uniformly bounded and equicontinuous sequence of functions. It follows that there is a subsequence of $\widetilde{\mathbb{1}}_n$ which converges uniformly to a continuous function ρ. Observe that $\rho\geq\inf_n\lambda^{-n}\rho^-_n$. Hence, by Lemma 2.2.21, the function ρ is strictly positive. A direct computation gives

$$\lambda^{-1}\mathcal{L}(\widetilde{\mathbb{1}}_n) - \widetilde{\mathbb{1}}_n = \frac{1}{n}(\lambda^{-n}\mathbb{1}_n - \mathbb{1}_0).$$

Since $\lambda^{-n}\mathbb{1}_n$ is bounded uniformly in n, the last expression tends uniformly to 0 when n tends to infinity. We then deduce from the definition of ρ that $\lambda^{-1}\mathcal{L}(\rho)=\rho$.

End of the proof of Theorem 2.2.13. Observe that we only need to show that $\lambda^{-n}\mathcal{L}^n(g)$ converges to $c_g\rho$ for some constant c_g, where ρ is a unique function satisfying $\mathcal{L}\rho = \lambda\rho$ (up to a multiplicative constant). The remaining part of the theorem is then clear. Let $\mathcal{G}$ denote the family of all limit functions of subsequences of $\lambda^{-n}\mathcal{L}^n(g)$. By Lemma 2.2.22, the sequence $\lambda^{-n}\mathcal{L}^n(g)$ is uniformly bounded and equicontinuous. Therefore, by Arzelà-Ascoli theorem, $\mathcal{G}$ is a uniformly bounded and equicontinuous family of functions which is compact for the uniform topology. Observe also that $\mathcal{G}$ is invariant under the action of $\lambda^{-1}\mathcal{L}$. Using the equidistribution of preimages of generic points towards the measure of maximal entropy, whose support is the Julia set, it is not difficult to deduce that $\mathcal{G}$ consists of the single function ρ constructed above. This completes the proof of the theorem. □

Once Theorem 2.2.13 is established, Theorem 2.2.1 follows from more standard arguments, that I do not recall here, see for instance [PU10; UZ13] and [BD23a, Section 4.2]. We just state below two specific results that will be needed in the proof of the spectral gap in the next sections. The proofs are omitted as they follow from the above and more standard arguments.

For positive real numbers q, M, and Ω with $q > 2$ and $\Omega < \log d$, consider the following set of weights

$$\mathcal{P}(q, M, \Omega) := \{\varphi\colon \mathbb{P}^k \to \mathbb{R} \ : \ \|\varphi\|_{\log^q} \leq M, \ \Omega(\varphi) \leq \Omega\}$$

and the uniform topology induced by the sup norm. Observe that this family is equicontinuous. The two lemmas below give the dependence on $\varphi \in \mathcal{P}(q, M, \Omega)$ of the objects introduced in this section. Therefore, we will use the index φ or parameter φ for objects which depend on φ, e.g., we will write $\lambda_\varphi, \mathcal{L}_\varphi, \rho_\varphi, \mathbb{1}_n(\varphi)$ instead of $\lambda, \mathcal{L}, \rho$ and $\mathbb{1}_n$.

Lemma 2.2.23. *Let q, M, and Ω be positive real numbers such that $q > 2$ and $\Omega < \log d$. The maps $\varphi \mapsto \lambda_\varphi$, $\varphi \mapsto m_\varphi$, $\varphi \mapsto \mu_\varphi$, and $\varphi \mapsto \rho_\varphi$ are continuous on $\varphi \in \mathcal{P}(q, M, \Omega)$ with respect to the standard topology on $\mathbb{R}$, the weak topology on measures, and the uniform topology on functions. In particular, ρ_φ is bounded from above and below by positive constants which are independent of $\varphi \in \mathcal{P}(q, M, \Omega)$. Moreover, $\|\lambda_\varphi^{-n}\mathcal{L}_\varphi^n\|_\infty$ is bounded by a constant which is independent of n and of $\varphi \in \mathcal{P}(q, M, \Omega)$.*

Lemma 2.2.24. *Let q, M, and Ω be positive real numbers such that $q > 2$ and $\Omega < \log d$. Let $\mathcal{F}$ be a uniformly bounded and equicontinuous family of real-valued functions on $\mathbb{P}^k$. Then the family*

$$\{\lambda_\varphi^{-n}\mathcal{L}_\varphi^n(g) \ : \ n \geq 0, \ \varphi \in \mathcal{P}(p, M, \Omega), \ g \in \mathcal{F}\}$$

is equicontinuous. Moreover, $\|\lambda_\varphi^{-n}\mathcal{L}_\varphi^n(g) - \langle m_\varphi, g\rangle\|_\infty$ tends to 0 uniformly on $\varphi \in \mathcal{P}(p, M, \Omega)$ and $g \in \mathcal{F}$ when n goes to infinity.

2.2.4 A game of norms

Once Theorem 2.2.1 is established, we move towards the establishment of the main Theorem 2.2.2. This requires the introduction of several intermediates (semi-)norms, and a precise study of the action of the operator $(f^n)_*$ on functions and currents with respect to such norms. Recall that we always assume that f satisfies the Assumption **(Hf)** in Section 2.2.1.

Before introducing the final norm, and the several intermediate norms that we will need, in the next section, I give here a rough description of the properties that a norm $\|\cdot\|_\diamond$ should have in order to satisfy Theorem 2.2.2. They are essentially as follows. We fix $0 < \gamma \leq 2$ as in that statement. Recall that we are trying to find a norm with respect to which the operator $\lambda^{-1}\mathcal{L}$ becomes a contraction (all the norms below are actually semi-norms, and give norms on a complement of the span on ρ. We will just refer to them as norms for simplicity, by only considering functions g such that $\langle m_\varphi, g\rangle = 0$).

(N1) $\| \cdot \|_\diamond \lesssim \| \cdot \|_{\mathcal{C}^\gamma}$;

(N2) $\{g \colon \|g\|_\diamond \leq 1\}$ is an equicontinuous family of continuous functions;

(N3) $\|f_*\|_\diamond < 1$, i.e., there exists $\beta < 1$ such that $\|f_* g\|_\diamond < \beta \|g\|_\diamond$ for every g;

(N4) $\|gh\|_\diamond \lesssim \|g\|_\infty \|h\|_\diamond + \|g\|_\diamond \|h\|_\infty$.

The first property is clear. In practice, this will be addressed with the final norm $\|\cdot\|_{\langle p,\alpha\rangle,\gamma}$ described in Section 2.2.5.5. We will first construct a norm $\|\cdot\|_{\langle p,\alpha\rangle}$ satisfying **(N2)**, **(N3)**, **(N4)**, and

(N1') $\| \cdot \|_\diamond \lesssim \| \cdot \|_{\mathcal{C}^2}$,

see Section 2.2.5.4. The norm $\|\cdot\|_{\langle p,\alpha\rangle,\gamma}$ will then be built from $\|\cdot\|_{\langle p,\alpha\rangle}$ by means of an interpolation (in the same way as Hölder continuous functions can be constructed by an interpolation between continuous and smooth ones, see Lemma 2.2.39).

The first idea to have a norm bounded by $\| \cdot \|_{\mathcal{C}^2}$ is clearly to have a term of the form $\|dd^c g\|_*$ in the definition of the norm (where we recall that this is, roughly speaking, the sum of the masses of the positive and negative parts of $dd^c g$). This is the DSH norm $\|g\|_{\mathrm{DSH}}$ introduced by Dinh-Sibony [DS06b; DS06a]. We also recall that such norm satisfies **(N3)**. It is then a good starting building block. Recalling that $\mathcal{L}(g) = f_*(e^\varphi g)$, an estimate for $dd^c \lambda^{-1}\mathcal{L}(g)$ is given by

$$\left|\frac{dd^c \mathcal{L}(g)}{\lambda}\right| \lesssim \left(\frac{e^{\Omega(\varphi)}}{d}\right) \Big(|f_*(dd^c g)| + \|g\|_\infty |f_*(dd^c\varphi)| + f_*(|i\partial g \wedge \overline{\partial}\varphi + i\partial\varphi \wedge \overline{\partial} g|)\Big).$$

In practice, since we would like to use the information on the convergence of $\lambda^{-n}\mathcal{L}^n(g)$ towards $\langle m_\varphi, g\rangle\rho$, we will be lead to bound $dd^c\lambda^{-n}\mathcal{L}^n(g)$ for some large n. In this case, by similar computations as in the previous section, we have

$$\begin{aligned}\left|\frac{dd^c \mathcal{L}^n g}{\lambda^n}\right| &\lesssim \left(\frac{e^{\Omega(\varphi)}}{d}\right)^n |f_*^n(dd^c g)| + \sum_{j=1}^n \left(\frac{e^{\Omega(\varphi)}}{d}\right)^j \left\|\frac{\mathcal{L}^{n-j}(g)}{\lambda^{n-j}}\right\|_\infty |f_*^j(dd^c\varphi)| + M_n(g,\varphi) \\ &= S_n(g,\varphi) + M_n(g,\varphi),\end{aligned}$$

where $M_n(g,\varphi)$ is a sum involving, for instance, mixed products of the form $i\partial g \wedge \overline{\partial}\varphi + i\partial\varphi \wedge \overline{\partial} g$.

Let us forget for now the term $M_n(g,\varphi)$. We see that the property **(N2)** is needed to be able to apply Lemma 2.2.24 uniformly on all the space $\{g \colon \|g\|_\diamond \leq 1\}$, and in particular to bound the terms $\|\lambda^{-n+j}\mathcal{L}^{n-j}(g)\|_\infty$ in the above expression. It is enough for this to add any norm (stronger than $\| \cdot \|_\infty$!) in the definition of $\|\cdot\|_\diamond$. A Hölder norm would be a natural guess, but we already saw that this norm is very badly suited for the problem, as we lose regularity at every step of the iteration. Moreover, if we choose a Hölder norm, this and condition **(N1)** would imply that we are in practice using a Hölder norm for $\|\cdot\|_\diamond$, which we know that cannot work in general. As we saw in the previous sections that the norm $\|\cdot\|_{\log^p}$ is better suited to the problem, the next natural norm is given by $\|g\|_p := \|dd^c g\|_* + \|g\|_{\log^p}$. This is the norm studied in Section 2.2.5.2, and the cornerstone of all the subsequent norms. Observe that we can also define a norm $\|R\|_p$ on (positive closed) $(1,1)$-currents, by setting $\|R\|_p = \|R\|_* + \|u_R\|_{\log^p}$, where u_R is a dynamical potential for R, see Section 2.2.2.3.

Let us now move to **(N3)**. As mentioned above, the DSH norm would satisfy this request, but not **(N2)**. The first key point in the construction of the norm is to address **(N2)** and **(N3)** together, i.e., to find a norm which has a spectral gap at least in the case $\varphi = 0$, and still gives a compact space of continuous functions (observe, in particular, that the term $M(g,\varphi)$ does not appear when $\varphi = 0$). The main idea of the construction is as follows. Because of the estimates that regularly appeared in the constructions, it is natural to define a norm (depending on some $\alpha \in (0,1)$) on $(1,1)$-currents S as

$$\|S\|_\alpha := \min\Big\{c\colon |S| \le c\sum_{j=0}^{\infty} \alpha^j \frac{f_*^j(\omega_{\mathrm{FS}})}{d^{(k-1)j}}\Big\}.$$

and to then set $\|g\|_\alpha := \|dd^c g\|_\alpha$. By Lemma 2.2.12, we have $\|g\|_{\log^q} \lesssim \|g\|_\alpha$ for every q. The norm $\|\cdot\|_\alpha$ is also tailor-suited to work well under the iteration of f_*. This is in practice the idea of the norm $\|\cdot\|_{p,\alpha}$ introduced in Section 2.2.5.3. With respect to the sum above, in the definition of $\|\cdot\|_{p,\alpha}$, we replace ω_{FS} by any $(1,1)$-current with $\|R\|_p \le 1$, to gain more flexibility in the construction and estimates (since, for instance, the functions φ that we will use will not always be smooth). Lowering the regularity to $\|R\|_p \le 1$ may a priori create problems with **(N2)**, but we show that this is not the case. Indeed, we can show that $\|\cdot\|_q \lesssim \|\cdot\|_{p,\alpha}$ for some explicit q depending on p, α, and the degree of f, see Lemma 2.2.30. The compactness of the space defined by $\|\cdot\|_{p,\alpha} \le 1$ is one of the key points in all the construction. An important point in the proof of this fact is that we can prove preliminary estimates on $\|\cdot\|_{\log^p}$ showing that, at least, the operator $d^{-k} f_*$ is Lipschitz with respect to this norm, with a constant that can be taken arbitrarily close to 1 when taking iterates (thanks to the genericity assumption **(Hf)**), see Lemma 2.2.25 and Theorem 2.2.26.

Until now, the above would give a norm which has a spectral gap for the operator f_*. This is already good enough for some applications, but not for those which require to control the perturbation of the transfer operator. Recalling that $\mathcal{L}g = f_*(e^\varphi g)$, up to developing the exponential we see why the last condition **(N4)** should be addressed to completely solve the problem. Another, analytical, way to see this is that a norm on g just based on some control on $dd^c g$ will not be able to control the terms of the form $i\partial g \wedge \overline{\partial}\varphi$ appearing in the development of $\mathcal{L}^n g$, and in particular the term $M_n(g,\varphi)$. The fact that this is the main difficulty in all the construction is essentially related to the fact that the operator $f_*(e^\varphi \cdot)$ is a *real* perturbation of a complex operator. In particular, it behaves badly with respect to dd^c. We explicitly notice that it is not enough to get an estimate of the form $\|gh\|_\diamond \lesssim \|g\|_\diamond \cdot \|h\|_\diamond$ instead of **(N4)**.

The solution to this final problem leads to the norm $\|\cdot\|_{\langle p,\alpha\rangle}$, described in Section 2.2.5.4, and essentially defined as

$$\|g\|_{\langle p,\alpha\rangle} := \|\partial g \wedge \overline{\partial} g\|_{p,\alpha}^{1/2}.$$

The idea of this norm is that, instead of trying to bound $dd^c\mathcal{L}^n g$ by means of some bound on $dd^c g$ and $dd^c\varphi$, we try to bound $\partial\mathcal{L}^n g \wedge \overline{\partial}\mathcal{L}^n g$ by means on some bound on $\partial g \wedge \overline{\partial} g$ and $\partial\varphi \wedge \overline{\partial}\varphi$. A development of $\partial\mathcal{L}^n g \wedge \overline{\partial}\mathcal{L}^n g$ shows that we have

$$\frac{i\partial\mathcal{L}^n g \wedge \overline{\partial}\mathcal{L}^n g}{\lambda^{2n}} \lesssim \left(\frac{e^{\Omega(\varphi)}}{d}\right)^{2n} f_*^n(i\partial g \wedge \overline{\partial} g) + \sum_{j=1}^{n} \left(\frac{e^{\Omega(\varphi)}}{d}\right)^{2j} \left\|\frac{\mathcal{L}^{n-j}(g)}{\lambda^{n-j}}\right\|_\infty^2 f_*^j(i\partial\varphi \wedge \overline{\partial}\varphi).$$

The details for the bounds of the operator $\mathcal{L}$ in the norm $\|\cdot\|_{\langle p,\alpha\rangle}$ (with both g and φ bounded in this norm) are given in Section 2.2.6, which essentially consists of the proof of Theorem 2.2.2 up to admitting that the norm $\|\cdot\|_{\langle p,\alpha\rangle}$ satisfies the properties **(N1')**, **(N2)**, **(N3)**, and **(N4)**.

Proving **(N4)** for the norm $\|\cdot\|_{\langle p,\alpha\rangle}$ requires to prove related intermediate estimates for the norms $\|\cdot\|_p$ and $\|\cdot\|_{p,\alpha}$. And while it is not difficult to see that the norm $\|\cdot\|_{\langle p,\alpha\rangle}$ also satisfies **(N1')** and **(N3)**, we should now make sure that $\|\cdot\|_{\langle p,\alpha\rangle}$ still satisfies **(N2)**. Indeed, all the verifications of such properties for the previous norms were based on the comparison principles for the operator dd^c introduced in Section 2.2.2. In this case, we need a more refined comparison principle, involving complex Sobolev function. Such comparison is still valid (although more technical), and given in Proposition 2.2.33. Together with the previous verifications, this completes the proof of the fact that the norm $\|\cdot\|_{\langle p,\alpha\rangle}$ satisfies all the properties above.

As mentioned above, the norm $\|\cdot\|_{\langle p,\alpha\rangle}$ satisfies **(N1')** instead of **(N1)**. The final norm $\|\cdot\|_{\langle p,\alpha\rangle,\gamma}$, introduced by interpolation in Section 2.2.5.5, satisfies this last property. Up to a couple of further technical arguments (essentially involving the fact that we need to iterate the system a finite number of times to get a spectral gap for the norms $\|\cdot\|_{\langle p,\alpha\rangle}$ and $\|\cdot\|_{\langle p,\alpha\rangle,\gamma}$), this is in practice the final norm in Theorem 2.2.2.

2.2.5 The norms and corresponding estimates

In this section I give the precise definitions of all the (semi-)norms that we introduced in our study, as well as the property of each of them that we will need. All the details which are not given here are in [BD22, Section 3].

2.2.5.1 Bounds with respect to the semi-norm $\|\cdot\|_{\log^p}$

In this section, we study the action of the operator f_* on functions with bounded semi-norm $\|\cdot\|_{\log^p}$. We first prove that, with respect to this semi-norm, the operator f_* is Lipschitz. We omit the proof as it is elementary, but, as Lemma 2.2.11, observe that it relies on the assumption **(Hf)**.

Lemma 2.2.25. *For every constant $A > 1$, there exists a positive constant $c = c(A)$ such that for every $n \geq 0$, $p > 0$, and continuous function $g\colon \mathbb{P}^k \to \mathbb{R}$, we have*

$$\left\|d^{-kn}(f^n)_* g\right\|_{\log^p} \leq c^p A^{pn} \|g\|_{\log^p}.$$

We will need the following result which is an improvement of [DS10b, Theorem 1.1] in the case where f satisfies the Assumption **(Hf)**. By duality, this result implies an exponential equidistribution of $d^{-kn}(f^n)^*\nu$ towards μ for every probability measure ν. The assumption **(Hf)** is necessary here to get the estimate in the norm $\|\cdot\|_\infty$. Since this is an interesting result in itself, we give the complete proof (assuming Lemma 2.2.25).

Theorem 2.2.26. *Let f be an endomorphism of $\mathbb{P}^k$ of algebraic degree $d \geq 2$ and satisfying the Assumption **(Hf)**. Consider a real number $p > 0$. Let $g\colon \mathbb{P}^k \to \mathbb{R}$ be such that $\|dd^c g\|_* \leq 1$, $\langle \mu, g\rangle = 0$ and $\|g\|_{\log^p} \leq 1$. Then, for every constant $d^{-p/(p+1)} < \eta < 1$, there is a positive constant c independent of g such that for every $n \geq 0$*

$$\|d^{-kn}(f^n)_* g\|_\infty \leq c\eta^n.$$

Proof. Set $g_n := d^{-kn}(f^n)_* g$. Recall (see, e.g., [Sko72; DNS10]) that there exists a positive constant c_0 independent of g and n such that

$$\int_{\mathbb{P}^k} e^{d^n |g_n|} \leq c_0,$$

where the integral above is taken with respect to the Lebesgue measure associated to the volume form ω_{FS}^k on $\mathbb{P}^k$.

Fix a constant $A > 1$ such that $\eta > (A/d)^{p/(p+1)}$. Suppose by contradiction that for infinitely many n there exists a point $a_n \in \mathbb{P}^k$ such that $|g_n(a_n)| \geq 3\eta^n$ for some g as above. Choose $r := e^{-c_A A^n \eta^{-n/p}}$ with c_A the constant given by Lemma 2.2.25 (we write c_A instead of c in order to avoid confusion). By that lemma, when $\mathrm{dist}(z, a_n) < r$, we have

$$|g_n(z)| \geq |g_n(a_n)| - |g_n(z) - g_n(a_n)| \geq 3\eta^n - c_A^p A^{pn}(1 + |\log r|)^{-p} \geq \eta^n.$$

This implies that

$$c_0 \geq \int_{\mathbb{P}^k} e^{d^n|g_n|} \geq \int_{\mathrm{dist}(z,a_n)<r} e^{d^n|g_n(z)|} \gtrsim r^{2k} e^{d^n\eta^n} \gtrsim e^{-2kc_A A^n \eta^{-n/p} + d^n \eta^n}.$$

By the choice of A, the last expression diverges when n tends to infinity. This is a contradiction. The theorem follows. □

2.2.5.2 The semi-norm $\|\cdot\|_p$

We now combine the semi-norms $\|\cdot\|_{\log^p}$ and $\|\cdot\|_*$ to build a new semi-norm $\|\cdot\|_p$. For every positive closed $(1,1)$-current S on $\mathbb{P}^k$ we first define

$$\|S\|'_p := \|S\| + \|u_S\|_{\log^p},$$

where u_S is the dynamical potential of S, see Section 2.2.2.3. When R is any $(1,1)$-current we define

$$\|R\|_p = \inf \|S\|'_p\,,$$

where the infimum is taken over all positive closed $(1,1)$-currents S such that $|R| \leq S$, and we set $\|R\|_p := \infty$ when no such S exists. By the compactness of positive closed currents with bounded $\|\cdot\|'_p$-norm, this infimum is actually a minimum when it is finite. Finally, for all $g\colon \mathbb{P}^k \to \mathbb{R}$, define

$$\|g\|_p := \|dd^c g\|_p.$$

The following lemma, whose proof is based of the comparison principles and estimates of Section 2.2.2.2, shows in particular that the norm $\|\cdot\|_p$ is equivalent to the norm $\|\cdot\|'_p$ when both are defined. We will thus just consider the norm $\|\cdot\|_p$ in the sequel.

Lemma 2.2.27. *Let S be a positive closed $(1,1)$-current on $\mathbb{P}^k$ and let $g\colon \mathbb{P}^k \to \mathbb{R}$ be a continuous function. Then*

$$\|u_S\|_p \leq 2\|S\|'_p, \qquad \|S\|_p \leq \|S\|'_p \leq c\,\|S\|_p\,, \qquad \textit{and} \qquad \|g\|_{\log^p} \leq c\,\|g\|_p$$

for some positive constant $c = c(p)$ independent of S and g.

The following lemma is the first step towards the property **(N4)**.

Lemma 2.2.28. *There is a positive constant $c = c(p)$ such that for all continuous functions $g, h\colon \mathbb{P}^k \to \mathbb{R}$ with finite $\|\cdot\|_p$ semi-norms and $\mathcal{C}^2$ or convex and Lipschitz function $\chi : \mathbb{R} \to \mathbb{R}$ we have*

$$\left\|\partial g \wedge \overline{\partial} h\right\|_p \leq c(\Omega(g)\|h\|_p + \|g\|_p\Omega(h)) \quad \textit{and} \quad \|\chi(g)\|_p \leq c\|\chi'(g)\|_\infty \|g\|_p.$$

2.2.5.3 The dynamical norm $\|\cdot\|_{p,\alpha}$

We now define the main norms $\|\cdot\|_{p,\alpha}$ for $(1,1)$-currents that we will use to quantify the convergence (2.2). We will see that these norms satisfy the inequalities

$$\|\cdot\|_q \lesssim \|\cdot\|_{p,\alpha} \lesssim \|\cdot\|_p$$

for some explicit q depending on p, α, and d. In particular, the new norms are at the same time weaker than the previous norm $\|\cdot\|_p$, but still inherit the main properties of a similar norm $\|\cdot\|_q$.

Definition 2.2.29. *Given a positive closed $(1,1)$-current S on $\mathbb{P}^k$ and a real number α such that $d^{-1} \leq \alpha < 1$, we define the current S_α by*

$$S_\alpha := \sum_{n=0}^{\infty} \alpha^n \frac{(f^n)_*(S)}{d^{(k-1)n}}.$$

For any $(1,1)$-current R on $\mathbb{P}^k$ and real number $p > 0$, we define

$$\|R\|_{p,\alpha} := \inf\{\, c \in \mathbb{R}\colon \exists S \textit{ positive closed}\colon\ \|S\|_p \leq 1,\ |\,R| \leq cS_\alpha \,\} \tag{2.13}$$

and we set $\|R\|_{p,\alpha} := \infty$ if such a number c does not exist.

Recall that the mass of $d^{-(k-1)n}(f^n)_*(S)$ is independent of n. Hence, we have $\|S_\alpha\| = \sum_{n\geq 0} \alpha^n$. Note also that when $\|R\|_{p,\alpha}$ is finite, by compactness, the infimum in (2.13) is actually a minimum and that, by definition we have $\|\xi R\|_{p,\alpha} \leq \|R\|_{p,\alpha}$ for every $(1,1)$-current R and every $\xi\colon \mathbb{P}^k \to \mathbb{C}$ with $|\xi| \leq 1$. We have the following lemma where the assumption $d^{-1} \leq \alpha < d^{-1/(p+1)}$ is equivalent to $0 < q_0 \leq p$.

Lemma 2.2.30. *Let α and p be positive and such that $d^{-1} \leq \alpha < d^{-1/(p+1)}$. Then, for every $0 < q < q_0 := \frac{|\log \alpha|}{\log d}(p+1) - 1$, there are positive constants $c_1 = c_1(p,\alpha)$ and $c_2 = c_2(p,\alpha,q)$ such that, for every $(1,1)$-current R,*

$$\|R\|_{p,\alpha} \leq c_1\,\|R\|_p \qquad \textit{and} \qquad \|R\|_q \leq c_2\,\|R\|_{p,\alpha}\,.$$

Proof. The first inequality holds by the definition of $\|\cdot\|_{p,\alpha}$ and Lemma 2.2.27. We prove the second inequality. Consider a current R such that $\|R\|_{p,\alpha} = 1$. We have to show that $\|R\|_q$ is bounded by a constant.

From the definition of $\|\cdot\|_{p,\alpha}$, we can find a positive closed current S such that $\|S\|_p = 1$ and $|R| \leq S_\alpha$. By the definition of the norm $\|\cdot\|_q$ and Lemma 2.2.27 applied to S_α, it is enough to show that $\|S_\alpha\|'_q$ is bounded. Denote by u_α the dynamical potential of S_α. Since the mass of S_α is bounded, we only need to show that $\|u_\alpha\|_{\log^q}$ is bounded. By definition of S_α, we have

$$u_\alpha = \sum_{n=0}^{\infty} \alpha^n \frac{(f^n)_* u_S}{d^{(k-1)n}}.$$

It follows that, for every positive number N,

$$m(u_\alpha, r) \leq \sum_{n \leq N} \alpha^n d^{-(k-1)n} \|(f^n)_* u_S\|_{\log^p} (\log^\star r)^{-p} + 2 \sum_{n>N} \alpha^n d^{-(k-1)n} \|(f^n)_* u_S\|_\infty .$$

Fix constants $A > 1$ close enough to 1, $\eta > d^{-p/(p+1)}$ close enough to $d^{-p/(p+1)}$ and $\alpha' > \alpha A^p$ close enough to α. In particular, by the assumption on α and the choice of η, we have that $\alpha d\eta$ is close to $\alpha d^{1/(p+1)}$ and smaller than 1. By Lemma 2.2.25 and Theorem 2.2.26 we know that $\|(f^n)_* u_S\|_{\log^p} \lesssim d^{kn} A^{pn}$ and $\|(f^n)_* u_S\|_\infty \lesssim d^{kn}\eta^n$. This, the above estimate on $m(u_\alpha, r)$, and the fact that $\alpha' d > \alpha d \geq 1$ imply that

$$m(u_\alpha, r) \lesssim \sum_{n \leq N} \alpha^n d^n A^{pn} (\log^\star r)^{-p} + \sum_{n>N} \alpha^n d^n \eta^n \lesssim (\alpha' d)^N (\log^\star r)^{-p} + (\alpha d\eta)^N .$$

Finally, choose $N = \frac{p+1}{\log d} \log\log^\star r$. Observe that if we replace α' by α and $\alpha d\eta$ by $\alpha d^{1/(p+1)}$, the last sum is equal to $2(\log^\star r)^{-q_0}$. So, this sum is bounded by a constant times $(\log^\star r)^{-q}$ for $q < q_0$ because α' is chosen close to α and $\alpha d\eta$ is close to $\alpha d^{1/(p+1)}$. This concludes the proof of the lemma. $\square$

The following shifting property of the norm $\|\cdot\|_{p,\alpha}$ will be very useful when we work with the action of f, and is the key property that we need of this norm. The proof follows from the definition of the norm $\|\cdot\|_{p,\alpha}$.

Lemma 2.2.31. *For every $n \geq 0$ and every $(1,1)$-current R on $\mathbb{P}^k$, we have*

$$\left\| d^{-kn} (f^n)_* R \right\|_{p,\alpha} \leq \frac{1}{d^n \alpha^n} \|R\|_{p,\alpha} .$$

2.2.5.4 The dynamical Sobolev semi-norm $\|\cdot\|_{\langle p,\alpha\rangle}$

We can now define the first semi-norm for functions $g\colon \mathbb{P}^k \to \mathbb{R}$ with respect to which we will be able to prove the existence of a spectral gap for the transfer operator. We can also define this norm for 1-forms. Recall that this will not be the final norm, as it is only bounded by the $\|\cdot\|_{C^1}$ norm, and not by the Hölder norms as in Theorem 2.2.2.

Definition 2.2.32. *Let p and α be real numbers such that $p > 0$ and $d^{-1} \leq \alpha < 1$. For any function $g\colon \mathbb{P}^k \to \mathbb{R}$ we set*

$$\|g\|_{\langle p,\alpha\rangle} := \left\| i\partial g \wedge \overline{\partial} g \right\|_{p,\alpha}^{1/2} .$$

We following proposition is the key to extend the results that we proved for the norm $\|\cdot\|_{p,\alpha}$ to the norm $\|\cdot\|_{\langle p,\alpha\rangle}$, see [BD22, Proposition 2.7]

Proposition 2.2.33. *Let $u\colon \mathbb{B}_5^k \to \mathbb{R}$ be continuous and such that $\|\partial u\|_{L^2(\mathbb{B}_5^k)} < \infty$. Assume that $i\partial u \wedge \overline{\partial} u \leq dd^c v$ where $v\colon \mathbb{B}_5^k \to \mathbb{R}$ is continuous, p.s.h., and such that*

$$\int_0^1 m_{\mathbb{B}_4^k}(v,t)(\log\log^\star t)^4 t^{-1} dt < +\infty.$$

Then there is a positive constant c, independent of u and v, such that, for all $0 < r \leq 1/2$, we have

$$m_{\mathbb{B}_1^k}(u,r) \leq c\Big(\int_0^{r^{1/2}} m_{\mathbb{B}_4^k}(v,t)(\log\log^\star t)^2 t^{-1} dt\Big)^{1/2} + c m_{\mathbb{B}_4^k}(v,r)^{1/3}\Omega_{\mathbb{B}_4^k}(v)^{1/6}(\log^\star r)^{1/2} + c\Omega_{\mathbb{B}_4^k}(v)^{1/2} r^{1/2}(\log^\star r)^{1/2}.$$

Corollary 2.2.34. *Let S_0 be a positive closed $(1,1)$-current on $\mathbb{P}^k$ of unit mass, whose dynamical potential u_S satisfies $\|u_S\|_{\log^p} \leq 1$ for some $p > 3/2$. Let $\mathcal{F}(S_0)$ denote the set of all continuous functions $g\colon \mathbb{P}^k \to \mathbb{R}$ such that $i\partial g \wedge \overline{\partial} g \leq S_0$. Then for any positive number $q < \frac{p}{3} - \frac{1}{2}$ we have $\|g\|_{\log^q} \leq c$ for some positive constant $c = c(p,q)$ independent of S_0. In particular, the family $\mathcal{F}(S_0)$ is equicontinuous.*

Remark 2.2.35. *The conclusion of Lemma 2.2.12 stays true if the assumption* (2.5) *is replaced by*

$$i\partial g \wedge \overline{\partial} g \leq \sum_{n=0}^{\infty} \beta^n \omega_n.$$

It suffices to use Corollary 2.2.34 instead of Corollary 2.2.9.

The following two lemmas give the main properties of the semi-norm $\|\cdot\|_{\langle p,\alpha\rangle}$ that we will need in Section 5, together with Lemma 2.2.31. Recall that q_0 is defined in Lemma 2.2.30. Note that the hypothesis $p > 3/2$ ensures that $d^{-1} < d^{-5/(2p+2)}$ and the hypothesis on α ensures that $q_0 > 3/2$, and hence that q_1 as in the statement below is positive.

Lemma 2.2.36. *Let α and p be positive numbers such that $p > 3/2$ and $d^{-1} \leq \alpha < d^{-5/(2p+2)}$. Then, for every $0 < q < q_1 := \frac{q_0}{3} - \frac{1}{2}$, there are positive constants $c_1 = c_1(p,\alpha,q)$ and $c_2 = c_2(p,\alpha)$ such that for every $g\colon \mathbb{P}^k \to \mathbb{R}$ we have*

$$\|g\|_{\log^q} \leq c_1 \|g\|_{\langle p,\alpha\rangle}, \quad \|g\|_{\langle p,\alpha\rangle} \leq c_2 \|g\|_p, \quad \text{and} \quad \|g\|_{\langle p,\alpha\rangle} \leq c_2 \|g\|_{\mathcal{C}^1}.$$

Proof. We can assume that $\|g\|_{\langle p,\alpha\rangle} \leq 1$. By the definition of the norm $\|\cdot\|_{\langle p,\alpha\rangle}$ and Lemma 2.2.30, $\big\|i\partial g \wedge \overline{\partial} g\big\|_{q'}$ is bounded by a constant for any $q' < q_0$. Therefore, we have $i\partial g \wedge \overline{\partial} g \leq R$ for some positive closed current R such that $\|R\|$ and $\|u_R\|_{\log^{q'}}$ are bounded by a constant. The first inequality follows from Corollary 2.2.34. The second assertion follows from Lemmas 2.2.30 and 2.2.28. The last assertion follows from Definition 2.2.32. □

The following lemma, related to **(N4)**, follows from a direct development of $i\partial(gh) \wedge \overline{\partial}(gh)$.

Lemma 2.2.37. *Let α and p be positive numbers such that $d^{-1} \leq \alpha < 1$. Then for all functions $g,h\colon \mathbb{P}^k \to \mathbb{R}$ we have*

$$\|gh\|_{\langle p,\alpha\rangle} \leq \sqrt{2}\big(\|g\|_{\langle p,\alpha\rangle} \|h\|_\infty + \|g\|_\infty \|h\|_{\langle p,\alpha\rangle}\big).$$

2.2.5.5 The semi-norm $\|\cdot\|_{\langle p,\alpha\rangle,\gamma}$

The following semi-norm defines the final space of functions that we will use in our study of the transfer operator. We use here some ideas from the theory of interpolation between Banach spaces, see also [Tri95].

Definition 2.2.38. *For all real numbers $d^{-1} \leq \alpha < 1$, $\gamma > 0$ and $p > 0$, we define for a continuous function $g\colon \mathbb{P}^k \to \mathbb{R}$*

$$\|g\|_{\langle p,\alpha\rangle,\gamma} := \inf\Big\{c \geq 0\colon \forall\, 0 < \varepsilon \leq 1\ \exists\, g_\varepsilon^{(1)}, g_\varepsilon^{(2)}\colon \\ g = g_\varepsilon^{(1)} + g_\varepsilon^{(2)}, \|g_\varepsilon^{(1)}\|_{\langle p,\alpha\rangle} \leq c(1/\varepsilon)^{1/\gamma}, \|g_\varepsilon^{(2)}\|_\infty \leq c\varepsilon\Big\}.$$

When such a number c does not exist, we set $\|g\|_{\langle p,\alpha\rangle,\gamma} := \infty$.

The following lemma is the counterpart of Lemma 2.2.5 for Hölder-continuous function. Applied with $s = 1$, it implies that $\|\cdot\|_{\langle p,\alpha\rangle,\gamma} \lesssim \|\cdot\|_{\mathcal{C}^\gamma}$ because $\|\cdot\|_{\langle p,\alpha\rangle} \lesssim \|\cdot\|_{\mathcal{C}^1}$, see Lemma 2.2.36.

Lemma 2.2.39. *Let $0 < \gamma \leq 1$ be a constant. Then, for every $\mathcal{C}^\gamma$ function $g\colon \mathbb{P}^k \to \mathbb{R}$, $s \geq 1$, and $0 < \varepsilon \leq 1$, there exist a $\mathcal{C}^s$ function $g_\varepsilon^{(1)}$ and a continuous function $g_\varepsilon^{(2)}$ such that*

$$g = g_\varepsilon^{(1)} + g_\varepsilon^{(2)}, \qquad \|g_\varepsilon^{(1)}\|_{\mathcal{C}^s} \leq c\,\|g\|_\infty\,(1/\varepsilon)^{s/\gamma} \qquad \text{and} \qquad \|g_\varepsilon^{(2)}\|_\infty \leq c\,\|g\|_{\mathcal{C}^\gamma}\,\varepsilon,$$

where $c = c(\gamma, s)$ is a positive constant independent of g and ε.

The following two lemmas are the counterparts of Lemmas 2.2.36 and 2.2.37 for the semi-norm $\|\cdot\|_{\langle p,\alpha\rangle,\gamma}$. Recall that q_1 is defined in Lemma 2.2.36.

Lemma 2.2.40. *For all positive numbers p, α, γ, q satisfying $p > 3/2$, $d^{-1} \leq \alpha < d^{-5/(2p+2)}$ and $q < q_2 := \frac{\gamma}{\gamma+1} q_1$, there is a positive constant $c = c(p, \alpha, \gamma, q)$ such that*

$$\|g\|_{\log^q} \leq c\,\|g\|_{\langle p,\alpha\rangle,\gamma} \quad \text{and} \quad \|g\|_{\langle p,\alpha\rangle,\gamma} \leq \|g\|_{\langle p,\alpha\rangle}$$

for every continuous function $g\colon \mathbb{P}^k \to \mathbb{R}$. Moreover, if $\chi : I \to \mathbb{R}$ is a Lipschitz function with Lipschitz constant κ on an interval $I \subset \mathbb{R}$ containing the image of g, then we have

$$\|\chi(g)\|_{\langle p,\alpha\rangle,\gamma} \leq \kappa \|g\|_{\langle p,\alpha\rangle,\gamma}.$$

The key point of this lemma is the first inequality. We give a sketch of its proof.

Proof of the first inequality. We can assume that $\|g\|_{\langle p,\alpha\rangle,\gamma} \leq 1$. Lemma 2.2.36 implies that $g_\varepsilon^{(1)}$ has $\|\cdot\|_{\log^{q'}}$ semi-norm bounded by a constant times $(1/\varepsilon)^{1/\gamma}$ when $q' < q_1$. Therefore, we have for $r > 0$

$$m(g, r) \leq m(g_\varepsilon^{(1)}, r) + m(g_\varepsilon^{(2)}, r) \lesssim \frac{(1/\varepsilon)^{1/\gamma}}{(\log^\star r)^{q'}} + \varepsilon.$$

Choosing $\varepsilon = (\log^\star r)^{-K}$ with $K = q'/(1 + 1/\gamma)$ gives

$$m(g, r) \lesssim (\log^\star r)^{-q'+K/\gamma} + (\log^\star r)^{-K} = 2(\log^\star r)^{-q'/(1+1/\gamma)}.$$

The assertion follows by choosing q' close enough to q_1. □

Lemma 2.2.41. *For all positive numbers p, α, γ such that $d^{-1} \leq \alpha < 1$ we have*

$$\|gh\|_{\langle p,\alpha\rangle,\gamma} \leq 3\big(\|g\|_{\langle p,\alpha\rangle,\gamma} \|h\|_\infty + \|g\|_\infty \|h\|_{\langle p,\alpha\rangle,\gamma} \big)$$

for every continuous functions $g, h\colon \mathbb{P}^k \to \mathbb{R}$.

2.2.6 Spectral gap for the transfer operator

Assuming the controls on the norms in the previous section, in this section I give the main ideas in the proof our main Theorem 2.2.2. Theorem 2.2.1 and (2.2) give the scaling ratio λ, the density function ρ as an eigenfunction for the operator $\mathcal{L} = \mathcal{L}_\varphi$, and the probability measures m_φ and μ_φ, all under the hypothesis that $\|\varphi\|_{\log^q} < \infty$ for some $q > 2$. The semi-norms $\|\cdot\|_{\langle p,\alpha\rangle}$ and $\|\cdot\|_{\langle p,\alpha\rangle,\gamma}$ were introduced in Sections 2.2.5.4 and 2.2.5.5, respectively.

The following is the main result of [BD22]. Theorem 2.2.2 follows from it from more standard arguments and other similar estimates, that I will not give here, recalling that $\|\cdot\|_{\langle p,\alpha\rangle,\gamma} \lesssim \|\cdot\|_{\mathcal{C}^\gamma}$.

Theorem 2.2.42. *Let $f, \varphi, m_\varphi, \rho$ be as in Theorem 2.2.1, $\mathcal{L}$ the Perron-Frobenius operator associated to φ as in* (2.1)*, and λ the scaling ration as in* (2.2)*. Let $p, \alpha, \gamma, A, \Omega$ be positive constants and q_2 as in Lemma 2.2.40 such that $p > 3/2$, $d^{-1} \leq \alpha < d^{-5/(2p+2)}$, $\Omega < \log(d\alpha)$, and $q_2 > 2$. Assume that $\|\varphi\|_{\langle p,\alpha\rangle,\gamma} \leq A$ and $\Omega(\varphi) \leq \Omega$. Then we have*

$$\|\lambda^{-n}\mathcal{L}^n\|_{\langle p,\alpha\rangle,\gamma} \leq c$$

for some positive constant $c = c(p, \alpha, \gamma, A, \Omega)$ independent of φ and n. Moreover, for every constant $0 < \beta < 1$ there is a positive integer $N = N(p, \alpha, \gamma, A, \Omega, \beta)$ independent of φ such that

$$\|\lambda^{-N}\mathcal{L}^N g\|_{\langle p,\alpha\rangle,\gamma} \leq \beta \|g\|_{\langle p,\alpha\rangle,\gamma} \tag{2.14}$$

for every function $g\colon \mathbb{P}^k \to \mathbb{R}$ with $\langle m_\varphi, g\rangle = 0$. Furthermore, there exists $N' = N'(p, \alpha, \gamma, \Omega)$ such that for any given constant $1 < \delta < (d\alpha)^{\gamma/(2\gamma+2)}$, when A is small enough (depending on the choice of N') (2.14) *holds with $N = N'$ and $\beta = \delta^{-N'}$.*

Notice that Lemma 2.2.40 and the assumption $q_2 > 2$ imply that $\|\varphi\|_{\log^q} < \infty$ for some $q > 2$. Hence, the scaling ratio λ, the density function ρ, and the measures m_φ and μ_φ are well defined by Theorem 2.2.1. Notice also that $q_2 > 2$ implies that the condition $\alpha < d^{-5/(2p+2)}$ is automatically satisfied.

As it was the case for Theorem 2.2.13, the proof of Theorem 2.2.42 will be reduced to a comparison between suitable currents and their norms. Theorem 2.2.42 then follows from some interpolation techniques. As in Section 2.2.3.1, I present the following simpler version of Theorem 2.2.42. Theorem 2.2.42 in practice follows from it by means of interpolation techniques, which I will briefly describe at the end of this section.

Theorem 2.2.43. *Let $f, \varphi, \lambda, m_\varphi, \rho$ be as in Theorem 2.2.1 and $\mathcal{L}$ the Perron-Frobenius operator associated to φ. Let p, α, A, Ω be positive constants and q_1 as in Lemma 2.2.36 such that $p > 3/2$, $d^{-1} \leq \alpha < d^{-5/(2p+2)}$, $\Omega < \log(d\alpha)$, and $q_1 > 2$. Assume that $\|\varphi\|_{\langle p,\alpha\rangle} \leq A$ and $\Omega(\varphi) \leq \Omega$. Then we have*

$$\|\lambda^{-n}\mathcal{L}^n\|_{\langle p,\alpha\rangle} \leq c$$

for some positive constant $c = c(p, \alpha, A, \Omega)$ independent of φ and n. Moreover, for every constant $0 < \beta < 1$ there is a positive integer $N = N(p, \alpha, A, \Omega, \beta)$ independent of φ such that

$$\|\lambda^{-N}\mathcal{L}^N g\|_{\langle p,\alpha\rangle} \leq \beta \|g\|_{\langle p,\alpha\rangle}$$

for every function $g\colon \mathbb{P}^k \to \mathbb{R}$ with $\langle m_\varphi, g\rangle = 0$. Furthermore, for every given constant $1 < \delta < (d\alpha)^{1/2}$, when A is small enough we can take $\beta = \delta^{-N}$.

Note that Lipschitz functions have finite $\|\cdot\|_{\langle p,\alpha\rangle}$ semi-norm (this follows from Lemma 2.2.36, since Lipschitz functions can be uniformly approximated by $\mathcal{C}^1$ ones whose norm is dominated by the Lipschitz constant, see also the proof of Lemma 2.2.40). So the last theorem can be applied to Lipschitz functions. For such functions we can take any p large enough and α close to 1. The rate of contraction is then almost equal to $d^{-1/2}$ when A is small enough (i.e., when φ is close to a constant function). This rate is likely optimal as it corresponds to known results obtained in the setting of zero weight, see [DS10a].

The crucial estimate that we will need in order to proof Theorem 2.2.43 is the following.

Proposition 2.2.44. *Let f be as in Theorem 2.2.1. Take $0 < \alpha < 1$ and $p > 0$. Given n functions $\varphi^{(j)}\colon \mathbb{P}^k \to \mathbb{R}$ for $j = 1, \dots, n$, set*

$$\Phi_m := \alpha^{-m} d^{(k-1)m} e^{m \max\varphi}.$$

Then there exists a positive constant $c = c(p, \alpha)$, independent of φ, such that

$$\|\mathcal{L}^n g\|_{\langle p,\alpha\rangle} \leq c\, \|\mathcal{L}^n \mathbb{1}\|_\infty^{1/2}\, \Phi_n^{1/2}\, \|g\|_{\langle p,\alpha\rangle} + c \sum_{m=1}^{n} \|\varphi\|_{\langle p,\alpha\rangle} m^{3/2} \Phi_m^{1/2}\, \|\mathcal{L}^m \mathbb{1}\|_\infty^{1/2}\, \|\mathcal{L}^{n-m} g\|_\infty$$

for every function $g\colon \mathbb{P}^k \to \mathbb{R}$.

Recalling that $\|\mathbb{1}_n\|_\infty \lesssim \lambda^n$ we obtain from Proposition 2.2.44 the inequality

$$\|\lambda^{-n}\mathcal{L}^n g\|_{\langle p,\alpha\rangle} \lesssim \|g\|_{\langle p,\alpha\rangle} \Big(\frac{e^\Omega}{d\alpha}\Big)^{n/2} + \|\varphi\|_{\langle p,\alpha\rangle} \sum_{m=1}^{n} m^{3/2} \Big(\frac{e^\Omega}{d\alpha}\Big)^{m/2} \|\lambda^{-n+m}\mathcal{L}^{n-m} g\|_\infty.$$

With this estimate, it is easy to deduce Theorem 2.2.43.

Proof of Proposition 2.2.44. By Definition 2.2.32 of the semi-norm $\|\cdot\|_{\langle p,\alpha\rangle}$, we need to bound the current $i\partial\mathcal{L}^n g \wedge \overline{\partial}\mathcal{L}^n g$. We will use here the map π_n introduced in Section 2.2.3.1. We have, using a direct computation,

$$\partial\big(e^{\varphi(x_0)+\cdots+\varphi(x_{n-1})} g(x_0)\big) = \Theta_1 + \Theta_2$$

with

$$\Theta_1 := \mathbb{h}'' \partial g(x_0), \quad \Theta_2 := \mathbb{h}'' g(x_0) \sum_{m=0}^{n-1} \partial\varphi(x_m), \quad \text{and} \quad \mathbb{h}'' := e^{\varphi(x_0)+\cdots+\varphi(x_{n-1})}.$$

Using Cauchy-Schwarz's inequality, we obtain

$$\begin{aligned} i\partial\mathcal{L}^n g \wedge \overline{\partial}\mathcal{L}^n g &= i(\pi_n)_*(\Theta_1 + \Theta_2) \wedge (\pi_n)_*(\overline{\Theta}_1 + \overline{\Theta}_2) \\ &\leq 2i(\pi_n)_*(\Theta_1) \wedge (\pi_n)_*(\overline{\Theta}_1) + 2i(\pi_n)_*(\Theta_2) \wedge (\pi_n)_*(\overline{\Theta}_2). \end{aligned}$$

We need to bound the norm $\|\cdot\|_{p,\alpha}$ of the two terms in the last sum by the square of the RHS of the inequality in the proposition.

For the first term, using again Cauchy-Schwarz's inequality, the definition of Φ_m as in the statement, and Lemma 2.2.31, we get (notice that $(\pi_n)_*(\mathbb{h}'') = \mathcal{L}_{1,n}\mathbb{1}$)

$$\begin{aligned}\left\|i(\pi_n)_*(\Theta_1)\wedge(\pi_n)_*(\overline{\Theta}_1)\right\|_{p,\alpha} &\leq \left\|(\pi_n)_*(\mathbb{h}'')(\pi_n)_*(\mathbb{h}'' i\partial g(x_0)\wedge\overline{\partial}g(x_0))\right\|_{p,\alpha}\\ &\leq \|\mathcal{L}^n\mathbb{1}\|_\infty e^{n\max\varphi}\left\|(f^n)_*(i\partial g\wedge\overline{\partial}g)\right\|_{p,\alpha}\\ &\leq \|\mathcal{L}^n\mathbb{1}\|_\infty \Phi_n\left\|i\partial g\wedge\overline{\partial}g\right\|_{p,\alpha}.\end{aligned}$$

This gives the desired estimate for the first term.

For the second term, observe that $i(\pi_n)_*(\Theta_2)\wedge(\pi_n)_*(\overline{\Theta}_2)$ is equal to

$$\begin{aligned}&\sum_{0\leq m,m'<n} i(\pi_n)_*(\mathbb{h}'' g(x_0)\partial\varphi(x_m))\wedge(\pi_n)_*(\mathbb{h}'' g(x_0)\overline{\partial}\varphi(x_{m'}))\\ &\leq\ 2\sum_{0\leq m'\leq m<n}\left|i(\pi_n)_*(\mathbb{h}'' g(x_0)\partial\varphi(x_m))\wedge(\pi_n)_*(\mathbb{h}'' g(x_0)\overline{\partial}\varphi(x_{m'}))\right|.\end{aligned}$$

Using Cauchy-Schwarz's inequality as in (2.7) and (2.8) we can bound the current in the absolute value signs by

$$\begin{aligned}&(m-m'+1)^{-2} i(\pi_n)_*(\mathbb{h}'' g(x_0)\partial\varphi(x_m))\wedge(\pi_n)_*(\mathbb{h}'' g(x_0)\overline{\partial}\varphi(x_m))\\ &\quad +(m-m'+1)^2 i(\pi_n)_*(\mathbb{h}'' g(x_0)\partial\varphi(x'_m))\wedge(\pi_n)_*(\mathbb{h}'' g(x_0)\overline{\partial}\varphi(x'_m))\end{aligned}$$

and deduce that $i(\pi_n)_*(\Theta_2)\wedge(\pi_n)_*(\overline{\Theta}_2)$ is bounded by a constant times

$$\sum_{0\leq m<n}(n-m)^3 i(\pi_n)_*(\mathbb{h}'' g(x_0)\partial\varphi(x_m))\wedge(\pi_n)_*(\mathbb{h}'' g(x_0)\overline{\partial}\varphi(x_m)).$$

Therefore, in order to get the proposition, setting $\eta := (\pi_n)_*(\mathbb{h}'' g(x_0)\partial\varphi(x_m))$ we only need to show that

$$\|i\eta\wedge\overline{\eta}\|_{p,\alpha}\leq\|\varphi\|^2_{\langle p,\alpha\rangle}\Phi_{n-m}\left\|\mathcal{L}^{n-m}\mathbb{1}\right\|_\infty\|\mathcal{L}_{n-m+1,n}g\|^2_\infty.$$

Now, using π'' as in the proof of Proposition 2.2.15, we see that

$$\eta=\pi''_*(\mathcal{L}^m g(x_m)\mathbb{h}_m\partial\varphi(x_m))\quad\text{with}\quad \mathbb{h}_m := e^{\varphi(x_m)+\cdots+\varphi(x_{n-1})}.$$

It follows from Cauchy-Schwarz's inequality that

$$\begin{aligned}i\eta\wedge\overline{\eta} &\leq \|\mathcal{L}^m g\|^2_\infty\pi''_*(\mathbb{h}_m)\pi''_*(\mathbb{h}_m i\partial\varphi(x_m)\wedge\overline{\partial}\varphi(x_m))\\ &\leq \|\mathcal{L}^m g\|^2_\infty\|\pi''_*(\mathbb{h}_m)\|_\infty\|\mathbb{h}_m\|_\infty(f^{n-m})_*(i\partial\varphi\wedge\overline{\partial}\varphi).\end{aligned}$$

Thus, by Lemma 2.2.31 and the definition of π'', we get

$$\|i\eta\wedge\overline{\eta}\|_{p,\alpha}\leq\|\mathcal{L}^m g\|^2_\infty\|\mathcal{L}^{n-m}\mathbb{1}\|_\infty\Phi_{n-m}\|i\partial\varphi\wedge\overline{\partial}\varphi\|_{p,\alpha}.$$

This ends the proof of the proposition. □

We conclude this section with an idea of how to get the actual Theorem 2.2.42 by means of more refined estimates and interpolation techniques. I only show how to prove (2.14), since the estimates required for the other assertions are similar. First of all, the following proposition is a version of Proposition 2.2.44 for the norm $\|\cdot\|_{\langle p,\alpha\rangle,\gamma}$, see [BD22, Proposition 4.4].

Proposition 2.2.45. *Let f be as in Theorem 2.2.1. Take $0<\alpha<1$ and $p>0$. Given n functions $\varphi^{(j)}\colon \mathbb{P}^k\to\mathbb{R}$ for $j=1,\dots,n$, set*

$$\Phi_m := \alpha^{-m} d^{(k-1)m} e^{\sum_{j=1}^m \max(\varphi^{(j)})} \quad \textit{and} \quad \mathcal{L}_{m,n} := \mathcal{L}_{\varphi^{(m)}}\circ\cdots\circ\mathcal{L}_{\varphi^{(n)}}.$$

Then there exists a positive constant $c=c(p,\alpha)$, independent of $\varphi^{(j)}$, such that

$$\|\mathcal{L}_{1,n} g\|_{\langle p,\alpha\rangle} \le c\,\|\mathcal{L}_{1,n}\mathbb{1}\|_\infty^{1/2}\,\Phi_n^{1/2}\,\|g\|_{\langle p,\alpha\rangle} + c\sum_{m=1}^n \|\varphi^{(m)}\|_{\langle p,\alpha\rangle} m^{3/2}\Phi_m^{1/2}\,\|\mathcal{L}_{1,m}\mathbb{1}\|_\infty^{1/2}\,\|\mathcal{L}_{m+1,n} g\|_\infty$$

for every function $g\colon\mathbb{P}^k\to\mathbb{R}$.

Once Proposition 2.2.45 is established, Theorem 2.2.42 can be deduced as follows. By subtracting from φ a constant, we can assume that φ belongs to the family of weights

$$\mathcal{Q}_0 := \{\varphi\colon \mathbb{P}^k\to\mathbb{R} \ :\ \min\varphi = 0,\ \|\varphi\|_{\langle p,\alpha\rangle,\gamma}\le A,\ \Omega(\varphi)\le\Omega\}.$$

Observe that we can apply Lemmas 2.2.23 and 2.2.24 because, by Lemma 2.2.40 and the assumptions on α and p, the family $\mathcal{Q}_0$ is contained in $\mathcal{P}_0(q,M,\Omega):=\{\varphi\in\mathcal{P}(q,M,\Omega)\colon \min\varphi=0\}$ for suitable $q>2$ and M. Observe also that $\|\varphi\|_\infty=\Omega(\varphi)\le\Omega$ and $\|\varphi\|_\infty\lesssim\|\varphi\|_{\langle p,\alpha\rangle,\gamma}\le A$.

Consider two constants $K\ge 1$ and $K'\ge 1$ whose values will depend on β. Note that the constants hidden in the signs $\lesssim$ below are independent of the parameters A,β,K,K',n and also of the constant $0<\varepsilon\le 1$ and the integer j that we consider now.

Since $\|\varphi\|_{\langle p,\alpha\rangle,\gamma}\le A$, for every $j\ge 1$ there are functions $\varphi^{(j)}$ and $\psi^{(j)}$ such that

$$\varphi=\varphi^{(j)}+\psi^{(j)},\quad \|\varphi^{(j)}\|_{\langle p,\alpha\rangle}\le A(Kj^2)^{1/\gamma}(1/\varepsilon)^{1/\gamma},\quad \text{and}\quad \|\psi^{(j)}\|_\infty\le AK^{-1}j^{-2}\varepsilon. \tag{2.15}$$

Observe that $\|\varphi^{(j)}\|_\infty$ is bounded by a constant since $\|\varphi\|_\infty$ is bounded by a constant.

We can assume for simplicity that $\|g\|_{\langle p,\alpha\rangle,\gamma}\le 1$, which implies that $\Omega(g)$ is bounded by a constant. Since $\langle m_\varphi,g\rangle=0$ by hypothesis, we deduce that $\|g\|_\infty$ is bounded by a constant. By the definition of the semi-norm $\|\cdot\|_{\langle p,\alpha\rangle,\gamma}$, we can find two functions $g_\varepsilon^{(1)}$ and $g_\varepsilon^{(2)}$ satisfying

$$g=g_\varepsilon^{(1)}+g_\varepsilon^{(2)},\ \ \|g_\varepsilon^{(1)}\|_{\langle p,\alpha\rangle}\le K'^{1/\gamma}(1/\varepsilon)^{1/\gamma},\ \ \|g_\varepsilon^{(2)}\|_\infty\le 2K'^{-1}\varepsilon,\ \ \langle m_\varphi,g_\varepsilon^{(1)}\rangle=\langle m_\varphi,g_\varepsilon^{(2)}\rangle=0.$$

Notice that without the condition $\langle m_\varphi,g_\varepsilon^{(2)}\rangle=0$ we would not need the coefficient 2 in the above estimate of $\|g_\varepsilon^{(2)}\|_\infty$. We obtain this condition by adding to $g_\varepsilon^{(2)}$ a suitable constant and subtracting the same constant from $g_\varepsilon^{(1)}$. The condition $\langle m_\varphi,g_\varepsilon^{(1)}\rangle=0$ is deduced from the hypothesis $\langle m_\varphi,g\rangle=0$ when we have $\langle m_\varphi,g_\varepsilon^{(2)}\rangle=0$. Since $\|g\|_\infty$ is bounded by a constant, $\|g_\varepsilon^{(1)}\|_\infty$ is also bounded by a constant.

Define as above $\mathcal{L}_{m,n} := \mathcal{L}_{\varphi^{(m)}} \circ \cdots \circ \mathcal{L}_{\varphi^{(n)}}$, where the $\varphi^{(j)}$'s are as in (2.15), and write

$$\lambda^{-n}\mathcal{L}^n g = \lambda^{-n}\mathcal{L}_{1,n} g_\varepsilon^{(1)} + \lambda^{-n}\big(\mathcal{L}^n g_\varepsilon^{(1)} - \mathcal{L}_{1,n} g_\varepsilon^{(1)}\big) + \lambda^{-n}\mathcal{L}^n g_\varepsilon^{(2)} =: G_{n,\varepsilon}^{(a)} + G_{n,\varepsilon}^{(b)} + G_{n,\varepsilon}^{(c)}.$$

We can prove the following estimates for the three terms in the RHS of the above expression. Lemma 2.2.47 is the key point, and is where Proposition 2.2.45 is used.

Lemma 2.2.46. *When K and K' are large enough, we have for every $n \geq 1$*

$$\|G_{n,\varepsilon}^{(b)}\|_\infty \leq \frac{1}{2}\beta\varepsilon \quad \textit{and} \quad \|G_{n,\varepsilon}^{(c)}\|_\infty \leq \frac{1}{2}\beta\varepsilon.$$

Lemma 2.2.47. *When $K \geq 1$ and K' are fixed, there is a constant $0 < \varepsilon_0 \leq 1$ independent of φ and g such that, for all $0 < \varepsilon \leq \varepsilon_0$ and all n large enough, also independent of φ and g,*

$$\|G_{n,\varepsilon}^{(a)}\|_{\langle p,\alpha\rangle} \leq \beta(1/\varepsilon)^{1/\gamma}.$$

Take now N large enough, independent of φ. It suffices to show that we can write

$$\lambda^{-N}\mathcal{L}^N g = G_{N,\varepsilon}^{(1)} + G_{N,\varepsilon}^{(2)} \quad \text{with} \quad \|G_{N,\varepsilon}^{(1)}\|_{\langle p,\alpha\rangle} \leq \beta(1/\varepsilon)^{1/\gamma} \quad \text{and} \quad \|G_{N,\varepsilon}^{(2)}\|_\infty \leq \beta\varepsilon.$$

We apply Lemmas 2.2.46 and 2.2.47 to $n := N$. When $\varepsilon \leq \varepsilon_0$, it is enough to choose $G_{N,\varepsilon}^{(1)} := G_{N,\varepsilon}^{(a)}$ and $G_{N,\varepsilon}^{(2)} := G_{N,\varepsilon}^{(b)} + G_{N,\varepsilon}^{(c)}$. Assume now that $\varepsilon_0 \leq \varepsilon \leq 1$ and choose $G_{N,\varepsilon}^{(1)} := 0$ and $G_{N,\varepsilon}^{(2)} := \lambda^{-N}\mathcal{L}^N g$. With N large enough, we have $\|G_{N,\varepsilon}^{(2)}\|_\infty \leq \beta\varepsilon_0 \leq \beta\varepsilon$ because $\|\lambda^{-n}\mathcal{L}^n g\|_\infty$ tends to 0 uniformly on φ and g when n goes to infinity, see Lemma 2.2.24. Thus, we have the desired decomposition of $\lambda^{-N}\mathcal{L}^N g$ and hence the property (2.14) for all N large enough.

2.3 Exponential mixing of all orders and Central Limit Theorems

In this section I describe the two works [BD24] and [BD23b], obtained in collaboration with Tien-Cuong Dinh. The main goal is to prove that the measures of maximal entropy of some dynamical systems of saddle type satisfy the Central Limit Theorem for Hölder observables. Observe that, for these systems, we do not know yet if a spectral gap exists on a suitable functional space, since the method of the previous section cannot be applied, due to the presence of the attracting directions. The Central Limit Theorem could be for instance deduced from the spectral gap since this would imply that the assumptions in the classical Gordin theorem [Gor69] are satisfied. More generally, the fact that the dynamics is expanding in some directions, but contracting in others, makes it not possible to apply directly the Gordin criterion, either, since this would require a control on the dynamics in the stable directions, which is not clear how to obtain in general. We will instead deduce the Central Limit Theorem from a strong version of mixing, the exponential mixing of all orders, that we are able to establish for these systems.

2.3.1 Hénon maps

Hénon maps are among the most studied dynamical systems that exhibit interesting chaotic behaviour. The main goal of the work [BD24], joint with Tien-Cuong Dinh, is to prove that the measure of maximal entropy of any complex Hénon map is *exponentially mixing of all orders* with respect to Hölder observables. As a consequence, we also solved a long-standing question proving the Central Limit Theorem for all Hölder observables with respect to the maximal entropy measures of complex Hénon maps. A similar result holds for automorphisms of compact Kähler surfaces with positive entropy [BD23b], and related versions are also true in higher dimension. Since the general strategy is similar, I mainly focus on the case of Hénon maps in this section, and briefly describe the differences in the second case in the next one.

The reader can find in the work of Bedford, Dinh, Fornaess, Lyubich, Sibony, and Smillie fundamental dynamical properties of these systems, see [BLS93; BS91; BS92; DS14; For96; FS92; Sib99] and the references therein. It is shown in [BLS93] that the measure of maximal entropy μ is Bernoulli. In particular, it is mixing of all orders [CFS12]. Namely, for every $\kappa \in \mathbb{N}$, observables $g_0, \dots, g_\kappa \in L^{\kappa+1}(\mu)$, and integers $0 =: n_0 \leq n_1 \leq \cdots \leq n_\kappa$, we have

$$\langle \mu, g_0(g_1 \circ f^{n_1}) \dots (g_\kappa \circ f^{n_\kappa}) \rangle - \prod_{j=0}^{\kappa} \langle \mu, g_j \rangle \to 0.$$

On the other hand, the control of the speed of mixing (i.e., the rate of the above convergence) for general dynamical systems and for regular enough observables is a challenging problem, and usually one can obtain it only under strong hyperbolicity assumptions on the system.

Let us recall the following general definition.

Definition 2.3.1. *Let (X, f) be a dynamical system and ν an f-invariant measure. Let $(E, \|\cdot\|_E)$ be a normed space of real functions on X with $\|\cdot\|_{L^p(\nu)} \lesssim \|\cdot\|_E$ for all $1 \leq p < \infty$. We say that ν is* exponentially mixing of order $\kappa \in \mathbb{N}^*$ *for observables in E if there exist constants $C_\kappa > 0$ and $0 < \theta_\kappa < 1$ such that, for all $g_0, \dots, g_\kappa$ in E and integers $0 =: n_0 \leq n_1 \leq \cdots \leq n_\kappa$, we have*

$$\Big|\langle \nu, g_0(g_1 \circ f^{n_1}) \dots (g_\kappa \circ f^{n_\kappa}) \rangle - \prod_{j=0}^{\kappa} \langle \nu, g_j \rangle\Big| \leq C_\kappa \cdot \Big(\prod_{j=0}^{\kappa} \|g_j\|_E \Big) \cdot \theta_\kappa^{\min_{0\leq j\leq \kappa-1}(n_{j+1}-n_j)}.$$

We say that ν is exponentially mixing of all orders *for observables in E if it is exponentially mixing of order κ for every $\kappa \in \mathbb{N}$.*

A recent major result by Dolgopyat, Kanigowski, and Rodriguez-Hertz [DKR21] ensures that, under suitable assumptions on the system, the exponential mixing of order 1 implies that the system is Bernoulli. In particular, it implies the mixing of all orders (with no control on the rate of decay of correlation). It is a main open question whether the exponential mixing of order 1 implies the exponential mixing of all orders, see for instance [DKR21, Question 1.5].

Let now f be a complex Hénon map on $\mathbb{C}^2$. It is a polynomial diffeomorphism of $\mathbb{C}^2$. We can associate to f its unique measure of maximal entropy μ [BLS93; BS91; BS92; Sib99]. It was established by Dinh in [Din05a] that such measure is exponential mixing of order 1 for Hölder observables, see also Vigny [Vig15] and Wu [Wu22]. Similar results were obtained by Liverani [Liv95] in the case of uniformly hyperbolic diffeomorphisms and Dolgopyat [Dol98] for Anosov flows.

Theorem 2.3.2 (Bianchi-Dinh [BD24]). *Let f be a complex Hénon map and μ its measure of maximal entropy. Then, for every $\kappa \in \mathbb{N}^*$, μ is exponential mixing of order κ as in Definition 2.3.1 for $\mathcal{C}^\gamma$ observables ($0 < \gamma \leq 2$), with $\theta_\kappa = d^{-(\gamma/2)^{\kappa+1}/2}$.*

For endomorphisms of $\mathbb{P}^k(\mathbb{C})$, the exponential mixing for all orders for the measure of maximal entropy and Hölder observables was established in [DNS10]. As we saw in Section 2.2, such property holds for a large class of invariant measures with strictly positive Lyapunov exponents [BD22]. In that case, this was proved by constructing a suitable (semi-)norm on functions that turns the so-called Ruelle-Perron-Frobenius operator (suitably normalized) into a contraction. As far as we know, the paper [BD24] gives the first instance where the exponential mixing of all orders is established for holomorphic dynamical systems with both positive and negative Lyapunov exponents.

The exponential mixing of all orders is one of the strongest properties in dynamics. It was recently shown to imply a number of statistical properties, see for instance [BG20; DFL21]. As an example, a consequence of Theorem 2.3.2 is the following result. Take $u \in L^1(\mu)$. As μ is ergodic, Birkhoff's ergodic theorem states that

$$n^{-1}S_n(u) := n^{-1}\big(u(x) + u \circ f(x) + \cdots + u \circ f^{n-1}(x)\big) \to \langle \mu, u\rangle \quad \text{for } \mu - \text{a.e. } x \in X.$$

We say that u *satisfies the Central Limit Theorem* (CLT) with variance $\sigma^2 \geq 0$ with respect to μ if $n^{-1/2}(S_n(u) - n\langle \mu, u\rangle) \to \mathcal{N}(0, \sigma^2)$ in law, where $\mathcal{N}(0, \sigma^2)$ denotes the (possibly degenerate, for $\sigma = 0$) Gaussian distribution with mean 0 and variance σ^2, i.e., for any interval $I \subset \mathbb{R}$ we have

$$\lim_{n\to\infty} \mu\Big\{\frac{S_n(u) - n\langle \mu, u\rangle}{\sqrt{n}} \in I\Big\} = \begin{cases} 1 \text{ when } I \text{ is of the form } I = (-\delta, \delta) & \text{if } \sigma^2 = 0, \\ \frac{1}{\sqrt{2\pi\sigma^2}} \displaystyle\int_I e^{-t^2/(2\sigma^2)} dt & \text{if } \sigma^2 > 0. \end{cases}$$

By a result of Björklund and Gorodnik [BG20], the following is then a consequence of Theorem 2.3.2. We refer to [BD22; DPU96; DS06a; Dup10; PR07; SUZ14; SUZ15] for other cases where the CLT for Hölder observables was established in holomorphic dynamics. As is the case for Theorem 2.3.2, this is the first time that this is done for systems with both positive and negative Lyapunov exponents.

Corollary 2.3.3. *Let f be a complex Hénon map and μ its measure of maximal entropy. Then all Hölder observables u satisfy the Central Limit Theorem with respect to μ with*

$$\sigma^2 = \sum_{n\in\mathbb{Z}} \langle \mu, \widetilde{u}(\widetilde{u} \circ f^n)\rangle = \lim_{n\to\infty} \frac{1}{n} \int_X (\widetilde{u} + \widetilde{u} \circ f + \ldots + \widetilde{u} \circ f^{n-1})^2 d\mu,$$

where $\widetilde{u} := u - \langle \mu, u\rangle$.

Theorem 2.3.2 and Corollary 2.3.3 in particular apply to any real Hénon map of maximal entropy [BS04], i.e., complex Hénon maps with real coefficients and whose measure of maximal entropy is supported in $\mathbb{R}^2$. By Friedland-Milnor [FM89], they apply to all automorphisms of $\mathbb{C}^2$ which are not conjugated to a map preserving a fibration. They hold also in the larger settings of

Hénon-Sibony automorphisms (sometimes called regular, or regular in the sense of Sibony) of $\mathbb{C}^k$ in any dimension [Sib99], and invertible horizontal-like maps in any dimension [DNS08; DS06c].

Our method to prove Theorem 2.3.2 relies on pluripotential theory and on the theory of positive closed currents. The idea is as follows. Using the classical theory of interpolation [Tri95], we can reduce the problem to the case $\gamma = 2$. For simplicity, assume that $\|g_j\|_{\mathcal{C}^2} \leq 1$ for all j. The measure of maximal entropy μ of a Hénon map f of $\mathbb{C}^2$ of algebraic degree $d \geq 2$ is the intersection $\mu = T_+ \wedge T_-$ of the two Green currents T_+ and T_- of f [BS91; Sib99]. If we identify $\mathbb{C}^2$ to an affine chart of $\mathbb{P}^2$ in the standard way, these currents are the unique positive closed $(1,1)$-currents of mass 1 on $\mathbb{P}^2$, without mass at infinity, satisfying $f^*T_+ = dT_+$ and $f_*T_- = dT_-$.

Consider the automorphism F of $\mathbb{C}^4$ given by $F := (f, f^{-1})$. Such automorphism also admits Green currents $\mathbb{T}_+ = T_+ \otimes T_-$ and $\mathbb{T}_- = T_- \otimes T_+$. These currents satisfy $(F^n)^*\mathbb{T}_+ = d^2\mathbb{T}_+$ and $(F^n)_*\mathbb{T}_- = d^2\mathbb{T}_-$. Under mild assumptions on their support, other positive closed $(2,2)$-currents S of mass 1 of $\mathbb{P}^4$ satisfy the estimate

$$|\langle d^{-2n}(F^n)_*(S) - \mathbb{T}_-, \Phi\rangle| \leq c_{S,\Phi} d^{-n} \tag{2.16}$$

when Φ is a sufficiently smooth test form. Here, $c_{S,\Phi}$ is a constant depending on S and Φ.

We show that proving the exponential mixing for $\kappa + 1$ observables $g_0, \ldots, g_\kappa$ with $\|g_j\|_{\mathcal{C}^2} \leq 1$ can be reduced to proving the convergence (we assume that n_1 is even for simplicity)

$$|\langle d^{-n_1}(F^{n_1/2})_*[\Delta] - \mathbb{T}_-, \Theta_{\{g_j\},\{n_j\}}\rangle| \lesssim d^{-\min_{0\leq j\leq \kappa-1}(n_{j+1}-n_j)/2}, \tag{2.17}$$

where

$$\Theta_{\{g_j\},\{n_j\}} := g_0(w)g_1(z)(g_2 \circ f^{n_2-n_1}(z)) \ldots (g_\kappa \circ f^{n_\kappa - n_1}(z))\mathbb{T}_+,$$

$[\Delta]$ denotes the current of integration on the diagonal Δ of $\mathbb{C}^2 \times \mathbb{C}^2$, and (z, w) denote the coordinates on $\mathbb{C}^2 \times \mathbb{C}^2$. A crucial point here is that the estimate should not only be uniform in the g_j's, but also in the n_j's. Note also that the current $[\Delta]$ is singular and the dependence of the constant $c_{S,\Phi}$ in (2.16) from S makes it difficult to employ regularization techniques to deduce the convergence (2.17) from (2.16).

The key point here is to notice that, when $dd^c\Phi \geq 0$ (on a suitable open set), one can also get the following variation of (2.16):

$$\langle d^{-2n}(F^n)_*(S) - \mathbb{T}_-, \Phi\rangle \leq c_\Phi d^{-n}. \tag{2.18}$$

With respect to (2.16), only the bound from above is present, but the constant c_Φ is now independent of S. This permits to regularize Δ and work as if this current were smooth. Note also that, although $\Theta_{\{g_j\},\{n_j\}}$ is not smooth, we can handle it using a similar regularization.

Working by induction, we show that it is possible to replace both $\Theta_{\{g_j\},\{n_j\}}$ and $-\Theta_{\{g_j\},\{n_j\}}$ in (2.17) with currents $\Theta^\pm$ satisfying $dd^c\Theta^\pm \geq 0$. This permits to deduce the estimate (2.17) from two upper bounds given by (2.18) for $\Theta^\pm$, completing the proof.

2.3.2 Automorphisms of compact Kähler manifolds

In [BD23b], we explained how to adapt the strategy above to get the exponential mixing of all orders and the CLT for the measure of maximal entropy of every automorphism of a compact Kähler manifold with *simple action on cohomology*. As non-trivial plurisubharmonic functions do not exist on compact Kähler manifolds, the approach described above cannot work here. Instead, we use more refined estimates on the regularity of the currents involved. We prove that if $\Theta_{\{g_j\},\{n_j\}}$ is *Hölder continuous* in a precise sense (i.e., when seen as a function on the space of positive exact (k,k)-currents, endowed with a suitable metric), then the convergence (2.17) holds. This is done by exploiting the theory of *super-potentials* for positive closed currents, as developed by Dinh and Sibony [DS09; DS10c].

Let us be more precise. Let (X,ω) be a compact Kähler manifold of dimension k and f a holomorphic automorphism of X. We refer to [BK09; FT23; McM02; McM07; Ogu09; OT15] for interesting examples of such maps, and to [dD12; Din05b; DS05a; DS10c] for their general properties, see also [Can01; CD20; DH22; FT21; Gue10; Ogu14]. We denote by f^n the iterate of order n of f. For $0 \le q \le k$, the dynamical degree d_q is defined as the spectral radius of the pull-back operator acting on the cohomology group $H^{q,q}(X,\mathbb{R})$ (see also Section 3.3.1). We have $d_0 = d_k = 1$ and $d_q(f^n) = d_q(f)^n$ for all $n \in \mathbb{N}$.

By a fundamental result of Khovanskii [Kho79], Teissier [Tei79], and Gromov [Gro90], the sequence $q \mapsto \log d_q$ is concave, see also [DN06] and Section 3.3.1. This implies that there exist integers $0 \le p \le p' \le k$ such that

$$1 = d_0 < d_1 < \ldots < d_p = \ldots = d_{p'} > \ldots > d_k = 1.$$

We say that f has *simple action on cohomology* if $p = p'$ and if moreover the action of f^* on $H^{p,p}(X,\mathbb{R})$ admits a unique eigenvalue of maximal modulus. Such eigenvalue is then necessarily equal to d_p. We denote in this case by $\delta = \delta(f)$ the maximum between $\max_{q\neq p} d_q$ and the moduli of the other eigenvalues for the action of f^* on $H^{p,p}(X,\mathbb{R})$. We call d_p the *main dynamical degree* and δ the *auxiliary dynamical degree* of f.

From now on, we assume that f has simple action on cohomology. It admits a unique probability measure of maximal entropy μ, which is the intersection of a positive closed (p,p)-current T_+ and a positive closed $(k-p,k-p)$-current T_- (the main Green currents of f), see [DS05a; DS10c]. Such measure is also called the *equilibrium measure* of f, and is mixing and hyperbolic. It was shown in [DS10a] that it is exponentially mixing for Hölder observables, see also [Din05a; Vig15; Wu22].

Theorem 2.3.4 (Bianchi-Dinh [BD23b]). *Let f be a holomorphic automorphism of a compact Kähler manifold (X,ω) with simple action on cohomology. Let d_p be its main dynamical degree and δ its auxiliary dynamical degree. Then, for every $\delta < \delta' < d_p$ and $0 < \gamma \le 2$, the equilibrium probability measure μ of f is exponentially mixing of all orders $\kappa \in \mathbb{N}^*$ for all C^γ observables, with $\theta_\kappa = (d_p/\delta')^{-(\gamma/2)^{\kappa+1}/2}$, see Definition 2.3.1.*

Using the classical theory of interpolation [Tri95], it is enough to prove the theorem in the case where $\gamma = 2$. Consider the compact Kähler manifold $\mathbb{X} := X \times X$. The automorphism $F := (f, f^{-1})$ of $\mathbb{X}$ and its inverse have simple action on cohomology, with the largest dynamical degree being that of order k, which is equal to d_p^2. We can define the main Green currents $\mathbb{T}_+$ and $\mathbb{T}_-$ for F as $\mathbb{T}_+ := T_+ \otimes T_-$ and $\mathbb{T}_- := T_- \otimes T_+$. They satisfy $(F^n)^*(\mathbb{T}_+) = d_p^2 \mathbb{T}_+$ and $(F^n)_*(\mathbb{T}_-) = d_p^2 \mathbb{T}_-$. As in the case of Hénon maps in the previous section and [BD24], proving the exponential mixing of order κ for the $\kappa+1$ observables $g_0, \dots, g_\kappa$ can be reduced to proving the estimate

$$|\langle d_p^{-n_1} (F^{n_1/2})_*[\Delta] - \mathbb{T}_-, \Theta_{\{g_j\},\{n_j\}} \rangle| \lesssim \prod_{j=0}^{\kappa} \|g_j\|_{\mathcal{C}^2} d^{-\min_{0 \le j \le \kappa-1} (n_{j+1}-n_j)/2}, \tag{2.19}$$

where $[\Delta]$ denotes the current of integration on the diagonal $\Delta \subset X \times X$, (z, w) the coordinates on $X \times X$, we set

$$\Theta_{\{g_j\},\{n_j\}} := g_0(w) g_1(z) (g_2 \circ f^{n_2 - n_1}(z)) \dots (g_\kappa \circ f^{n_\kappa - n_1}(z)) \mathbb{T}_+,$$

and we assumed for simplicity that n_1 is even.

In the case of Hénon maps, we saw that the above convergence can be established by proving that $\Theta_{\{g_j\},\{n_j\}}$ can be replaced by suitable currents $\Theta^\pm$ with $dd^c \Theta^\pm \ge 0$, for which the estimate above can be proved thanks to the properties of plurisubharmonic functions. As non-trivial plurisubharmonic functions do not exist on compact Kähler manifolds, that approach cannot work here. Instead, we use more refined estimates on the regularity of the currents involved. We prove that if $\Theta_{\{g_j\},\{n_j\}}$ is *Hölder continuous* in a precise sense (i.e., when seen as a function on the space of positive exact (k, k)-currents, endowed with a suitable metric), then the convergence (2.19) holds. This is done by exploiting the theory of *super-potentials* for positive closed currents, as developed by Dinh and Sibony.

The main task becomes to prove the Hölder continuity of the current $\Theta_{\{g_j\},\{n_j\}}$. As before, the estimate needs to be uniform in the n_j's and in the g_j's (assuming $\|g_j\|_{\mathcal{C}^2} \le 1$ for all j), in order for the implicit constant in (2.17) not to depend on such parameters. Observe also that this problem does not exist when just proving the mixing of order $\kappa = 1$, see [DS10a]. This is the main technical point in [BD23b].

By means of a general comparison principle for the super-potentials of positive closed currents [DNV18], we show that it is enough to find a positive closed current Ξ with a Hölder continuous super-potential and such that

$$|dd^c \Theta_{\{g_j\},\{n_j\}}| \le \Xi \quad \text{for all } n_j \text{ and all } g_j \text{ with } \|g_j\|_{\mathcal{C}^2} \le 1.$$

Finding such Ξ and establishing such an estimate rely on the gap between d_p and the auxiliary dynamical degree of f and on Hölder estimates for the action on f^* on $(p+1, p+1)$-currents, that we also develop in [BD23b].

As in the previous section, the following is a consequence of Theorem 2.3.4 and [BG20].

Corollary 2.3.5. *Let f be a holomorphic automorphism of a compact Kähler manifold (X, ω) with simple action on cohomology. Then all Hölder observables $u \colon X \to \mathbb{R}$ satisfy the Central Limit Theorem with respect to the measure of maximal entropy μ of f with*

$$\sigma^2 = \sum_{n \in \mathbb{Z}} \langle \mu, \widetilde{u}(\widetilde{u} \circ f^n) \rangle = \lim_{n \to \infty} \frac{1}{n} \int_X (\widetilde{u} + \widetilde{u} \circ f + \dots + \widetilde{u} \circ f^{n-1})^2 d\mu,$$

where $\widetilde{u} := u - \langle \mu, u \rangle$.

Let now X be a compact Kähler surface and f an automorphism of positive entropy. By [Gro03; Yom87], the topological entropy is equal to $\log d_1$, see also [DS04]. In particular, d_1 is strictly larger than 1. A result by Cantat [Can01] says that all the eigenvalues of the action of f^* on $H^{1,1}(X, \mathbb{R})$ have modulus 1, except for two eigenvalues d_1 and $1/d_1$, which have multiplicity 1. In particular, every automorphism of positive entropy of a Kähler surface has simple action on cohomology.

Corollary 2.3.6. *Let f be a holomorphic automorphism of positive entropy on a compact Kähler surface X. Then, for every $1 < d' < d_1$ and $0 < \gamma \leq 2$, the equilibrium probability measure μ of f is exponentially mixing of all orders $\kappa \in \mathbb{N}^*$ for all $\mathcal{C}^\gamma$ observables, with $\theta_\kappa = (d')^{-(\gamma/2)^{\kappa+1}/2}$, see Definition 2.3.1. Moreover, all Hölder observables satisfy the Central Limit Theorem with respect to the measure of maximal entropy of f.*

3 Other related works

3.1 Weighted equidistribution results

In this section I present the equidistribution of repelling periodic points with respect to the equilibrium states as in Theorem 2.2.1 and a version of this result for the motions of the repelling periodic points in stable families of endomorphisms of $\mathbb{P}^k$ as in Theorem 1.1.1.

3.1.1 Repelling points and equilibrium states

We use in this section the notations introduced in Section 2.2. The following result, obtained in [BD23a], is contained in Theorem 2.2.1 and completes its proof.

Theorem 3.1.1 (Bianchi-Dinh [BD23a]). *Let $f\colon \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic endomorphism of $\mathbb{P}^k$ of algebraic degree $d \geq 2$ and satisfying Assumption **(Hf)**. Let $\varphi\colon \mathbb{P}^k \to \mathbb{R}$ satisfy $\|\varphi\|_{\log^q} < \infty$ for some $q > 2$ and $\Omega(\varphi) < \log d$. Let μ_φ be the unique equilibrium state associated to φ, and λ the scaling ratio. Then for every $n \in \mathbb{N}$ there exists a set P'_n of repelling periodic points of period n in the small Julia set such that*

$$\lim_{n\to\infty} \lambda^{-n} \sum_{y\in P'_n} e^{\varphi(y)+\varphi(f(y))+\cdots+\varphi(f^{n-1}(y))}\delta_y = \mu_\varphi. \tag{3.1}$$

A related equidistribution property for Hölder continuous weights was proved by Comman-Rivera-Letelier [CR11] for (hyperbolic and) topologically Collect-Eckmann rational maps on $\mathbb{P}^1$.

To prove Theorem 3.1.1, we followed a now classical strategy due to Briend-Duval [BD99] for the measure of maximal entropy (which corresponds to the case $\varphi \equiv 0$), which permitted to generalize to any dimension the equidistribution of the periodic points with respect to this measure, first established by Lyubich [Lyu82; Lyu83a] in dimension 1. We employ a trick due to Buff [Buf04] which simplifies the original proof. An extra difficulty with respect to the case $\varphi \equiv 0$ is due to the fact that there is no a priori upper bound for the mass of the left hand side of (3.1) when we replace P'_n with the set of all repelling periodic points of period n.

Given any point $x \in \mathbb{P}^k$ we denote by $\mu_{x,n}$ the measure

$$\mu_{x,n} := \lambda^{-n}\rho(x)^{-1} \sum_{f^n(a)=x} e^{\varphi(a)+\varphi(f(a))+\cdots+\varphi(f^{n-1}(a))}\rho(a)\delta_a.$$

It follows from Theorem 2.2.1 and Section 2.2 that, for every continuous function $g\colon \mathbb{P}^k \to \mathbb{R}$, we have

$$\langle \mu_{x,n}, g\rangle = \lambda^{-n}\rho(x)^{-1} \sum_{f^n(a)=x} e^{\varphi(a)+\varphi(f(a))+\dots+\varphi(f^{n-1}(a))}\rho(a)g(a) \to \rho(x)^{-1}\langle \rho(x)m_\varphi, \rho g\rangle = \langle \mu_\varphi, g\rangle$$

as $n\to\infty$. This means that, for all $x\in\mathbb{P}^k$, we have $\mu_{x,n}\to\mu_\varphi$ as $n\to\infty$.

We denote by $0 < L_1 \le \dots \le L_k$ the Lyapunov exponents of μ_φ (we recall that they are strictly positive since the measure-theoretic entropy of μ_φ is larger than $(k-1)\log d$). We fix in what follows a constant $0 < L_0 < L_1$. Given $x \in X$, a ball B of center x, and $n\in\mathbb{N}$, we say that $g\colon B\to B'$ is an *m-good inverse branch of f of order n on B* if

$$g\circ f^n = \mathrm{Id}_{|B'} \quad \text{and} \quad \operatorname{diam} f^l(B') \le e^{-m-(n-l)L_0} \text{ for all } 0\le l\le n.$$

Notice that the definition in particular implies that $\operatorname{diam}(B)\le e^{-m}$. We denote by $\mu_{B,n}^{(m)}$ the measure

$$\mu_{B,n}^{(m)} := \lambda^{-n}\rho(x)^{-1}\sum_{a=g(x)} e^{\varphi(a)+\varphi(f(a))+\dots+\varphi(f^{n-1}(a))}\rho(a)\delta_a,$$

where the sum is taken on the m-good inverse branches g of f of order n on B. Since we have $\mu_{B,n}^{(m)} \le \mu_{x,n}$ for all $n\ge 0$, it follows that any limit value μ'_B of the sequence $\{\mu_{B,n}^{(m)}\}$ satisfies $\mu'_B\le\mu_\varphi$. In particular, we have $\|\mu'_B\|\le 1$.

Given $m>1$ we say that a ball B centred at x is *m-nice* if

1. $\inf_B\rho > (1-1/m)\sup_B\rho$;
2. $\|\mu_{B,n}^{(m)}\| \ge 1-1/m$ for every n sufficiently large.

Observe that the second condition implies that $\operatorname{diam}(B)\le e^{-m}$ for every m-nice ball B. Moreover, we have $\|\mu'_B\|\ge 1-1/m$ for every limit value μ'_B of the sequence $\mu_{B,n}^{(m)}$. The following is a key step in getting Theorem 3.1.1.

Lemma 3.1.2. *For μ_φ-almost every $x\in\mathbb{P}^k$, every sufficiently small ball centred at x is m-nice.*

The proof of Lemma 3.1.2 is elementary but makes uses of the natural extension of the system $(\mathbb{P}^k, f, \mu_\varphi)$, see for instance [CFS12, Section 10.4]. We denote by X_0, C_f, PC_f the small Julia set, the critical set and the postcritical set $PC_f := \cup_{n\ge0} f^n(C_f)$ of f, respectively. We also set $X := X_0\setminus \cup_{m\in\mathbb{N}} f^{-m}(PC_f)$. By Theorem 2.2.1 we have $\mu_\varphi(f^{-m}(PC_f))=0$ for every $m\in\mathbb{N}$, hence $\mu(X)=1$. We denote by $\widehat{X}$ the set

$$\widehat{X} := \{\,\widehat{x} := (x_n)_{n\in\mathbb{Z}}\colon x_n\in X, f(x_n)=x_{n+1}\,\},$$

by $\pi_n : \widehat{x}\mapsto x_n$ the natural projection from $\widehat{X}$ to X and by $\widehat{f}\colon\widehat{X}\to\widehat{X}$ the map

$$\widehat{f}(\dots,x_{-1},x_0,x_1,\dots) := (\dots, f(x_{-1}), f(x_0), f(x_1),\dots) = (\dots,x_0,x_1,x_2,\dots).$$

Observe that $\pi_n\circ\widehat{f} = f\circ\pi_n$ for all $n\in\mathbb{Z}$. Let us consider on $\widehat{X}$ the σ-algebra $\widehat{\mathcal{B}}$ generated by all *cylinders*, i.e., the sets of the form

$$A_{n,B} := \pi_n^{-1}(B) = \{\widehat{x}\colon x_n\in B\} \text{ for } n\le 0 \text{ and } B\subseteq\mathbb{P}^k \text{ a Borel set}$$

and set

$$\widehat{\mu}_\varphi(A_{n,B}) := \mu_\varphi(B) \text{ for all } A_{n,B} \text{ as above.}$$

It follows from the invariance of μ_φ and the fact that $x_n \in B$ if and only if $x_{n-m} \in f^{-m}(B)$ (with $m \geq 0$) that $\widehat{\mu}_\varphi$ is well defined on the collection of the sets $A_{n,B}$ and

$$\widehat{\mu}_\varphi(A_{n,B}) = \widehat{\mu}_\varphi(A_{n-m,B}) \text{ for all } m \geq 0.$$

Similarly, for every $m > 0$ and Borel sets $B_0, B_{-1}, \ldots, B_{-m} \subseteq \mathbb{P}^k$ we then have

$$\begin{aligned}\widehat{\mu}_\varphi(\{\widehat{x}\colon x_0 \in B_0, x_{-1} \in B_{-1}, \ldots, x_{-m} \in B_{-m}\}) \\ &= \widehat{\mu}_\varphi(\{\widehat{x}\colon x_{-m} \in f^{-m}(B_0) \cap f^{-(m-1)}(B_{-1}) \cap \cdots \cap B_{-m}\}) \\ &= \mu_\varphi(f^{-m}(B_0) \cap f^{-(m-1)}(B_{-1}) \cap \cdots \cap B_{-m}).\end{aligned}$$

We then extend $\widehat{\mu}_\varphi$ to a probability measure, still denoted by $\widehat{\mu}_\varphi$, on $\widehat{\mathcal{B}}$. Observe that $\widehat{\mu}_\varphi$ is $\widehat{f}$-invariant by construction and satisfies $(\pi_0)_* \widehat{\mu}_\varphi = \mu_\varphi$.

For $n > 0$ we denote by $f_{\widehat{x}}^{-n}$ the inverse branch of f^n defined in a neighbourhood of x_0 and such that $f_{\widehat{x}}^{-n}(x_0) = x_{-n}$. This branch exists for all $x_0 \in X$. We have the following lemma, which is again a consequence of the fact that μ_φ has strictly positive Lyapunov exponents, see also [BD99] and [BDM08] for the case of $\varphi = 0$.

Lemma 3.1.3. *For every $0 < L < L_1$ there exist two measurable functions $\eta_L\colon \widehat{X} \to (0,1]$ and $S_L\colon \widehat{X} \to (1, +\infty)$ such that, for $\widehat{\mu}_\varphi$-almost every $\widehat{x} \in \widehat{X}$, the map $f_{\widehat{x}}^{-n}$ is defined on $\mathbb{B}_{\mathbb{P}^k}(x_0, \eta_L(\widehat{x}))$ with $\mathrm{Lip}(f_{\widehat{x}}^{-n}) \leq S_L(\widehat{x}) e^{-nL}$ for every $n \in \mathbb{N}$.*

We can now give an idea of the proof of Lemma 3.1.2.

Proof of Lemma 3.1.2. Since ρ is continuous and strictly positive, we only need to check that, for μ_φ-almost every $x \in \mathbb{P}^k$, every sufficiently small ball B centred at x satisfies $\|\mu_{B,n}^{(m)}\| \geq 1 - 1/m$ for every n sufficiently large.

Let us consider the disintegration of the measure $\widehat{\mu}_\varphi$ with respect to μ_φ and the projection π_0. We denote by $\widehat{\mu}_\varphi^x$ the conditional measure on $\{x_0 = x\}$. The measure $\widehat{\mu}_\varphi^x$ is uniquely defined for μ_φ-almost all $x \in X$ and characterized by the identity

$$\langle \widehat{\mu}_\varphi, g \rangle = \langle \mu_\varphi, u(x) \rangle, \text{ where } u(x) := \left\langle \widehat{\mu}_\varphi^x, g \right\rangle$$

for all bounded measurable functions $g\colon \widehat{X} \to \mathbb{R}$. Since $(\pi_0)_* \widehat{\mu}_\varphi = \mu_\varphi$, $\widehat{\mu}_\varphi^x$ is a probability measure for μ_φ-almost every x.

We will need a more explicit description of the conditional measures $\widehat{\mu}_\varphi^x$. For $n > 0$ and $x \in X$ we consider the measure $\widehat{\mu}_n^x$ on $\widehat{X}$ defined as follows. First, let us consider the projection $\widehat{X} \to X^{n+1}$ given by

$$\widehat{\pi}^n := (\pi_{-n}, \ldots, \pi_{-1}, \pi_0).$$

For every element $(y_{-n}, \ldots, y_0) \in X^{n+1}$ we choose a representative $\widehat{z} \in \widehat{X}$ such that $z_j = y_j$ for all $-n \leq j \leq 0$. For any given y_0 and any $n > 0$ we then have d^{kn} distinct such representatives, and we denote by $\widehat{Z}_n$ their collection. We then set

$$\widehat{\mu}_n^x := \lambda^{-n} \rho(x)^{-1} \sum_{\widehat{z} \in \widehat{Z}_n\colon z_0 = x} e^{\varphi(z_{-n}) + \varphi(z_{-n+1}) + \cdots + \varphi(z_{-1})} \rho(z_{-n}) \delta_{\widehat{z}}.$$

Since this is a finite sum, the measures $\widehat{\mu}_n^x$ are well defined on $\widehat{X}$. Moreover, we can prove that

$$\lim_{n\to\infty} \widehat{\mu}_n^x = \widehat{\mu}_\varphi^x \quad \text{for } \mu_\varphi - \text{almost every } x \in X. \tag{3.2}$$

The proof of (3.2) is elementary and is omitted here.

Let us now fix an integer $m > 0$, a constant $L_0 < L < L_1$, and a second positive integer γ. For every integer $N > 0$ we set

$$\widehat{X}_N := \{\widehat{x} \in \widehat{X} : \eta_L(\widehat{x}) \geq N^{-1} \text{ and } S_L(\widehat{x}) \leq N\}.$$

Observe that $\widehat{\mu}_\varphi(\widehat{X}_N) \to 1$ as $N \to \infty$. In particular, there exists $N_0 = N_0(m, \gamma)$ such that, for every $N > N_0$, we have $\widehat{\mu}_\varphi(\widehat{X}_N) > 1 - 1/(2m^{\gamma+1})$. It follows by Markov inequality that there exists a subset $X_\gamma \subset X$ with $\mu_\varphi(X_\gamma) > 1 - 1/m^\gamma$ such that, for all $N > N_0$,

$$\widehat{\mu}_\varphi^x(\widehat{X}_N \cap \{x_0 = x\}) > 1 - 1/(2m) \text{ for all } x \in X_\gamma.$$

It is enough to prove the property in the lemma for all $x \in X_\gamma$. Let us fix one such x. By Lemma 3.1.3 and the definition of $\widehat{X}_N$, for every $\widehat{x} \in \widehat{X}_N$ and $n \geq 0$ the inverse branch $f_{\widehat{x}}^{-n}$ is defined on the ball $B_{\mathbb{P}^k}(x_0, N^{-1})$ with $\mathrm{Lip}(f_{\widehat{x}}^{-n}) \leq Ne^{-nL}$. In particular, $\mathrm{diam}(f_{\widehat{x}}^{-n}(B_{\mathbb{P}^k}(x_0, e^{-m}/(2N)))) \leq e^{-m-nL_0}$ for all $n \geq 0$. It follows that all inverse branches on $B_{\mathbb{P}^k}(x, e^{-m}/(2N))$ corresponding to elements $\widehat{x} \in \widehat{X}_N \cap \{x_0 = x\}$ are m-good for all n.

By (3.2), we have

$$\widehat{\mu}_n^x(\widehat{X}_N \cap \{x_0 = x\}) > 1 - 1/m \text{ for all } n \text{ large enough.}$$

This precisely means that, for all n sufficiently large, we have $\|\mu_{B,n}^{(m)}\| > 1 - 1/m$, where $B = B_{\mathbb{P}^k}(x, e^{-m}/(2N))$. This implies that such a ball B is m-nice and completes the proof of Lemma 3.1.2. □

Lemma 3.1.4. *There exists a positive constant $C = C(L_0, q)$ such that, for all $n \in \mathbb{N}, m > 0$, and every m-good inverse branch $g \colon B \to B'$ of f of order n on a ball B, and for all sequences of points $\{x_l\}, \{y_l\}$ with $0 \leq l \leq n-1$ and $x_l, y_l \in f^l(B')$ we have*

$$\sum_{l=0}^{n-1} |\varphi(x_l) - \varphi(y_l)| \leq Cm^{-(q-1)}.$$

Proof. Since g is m-good, we have $\mathrm{dist}(x_l, y_l) \leq e^{-m-(n-l)L_0}$ for all $0 \leq l \leq n-1$. Hence, since $\sum_{l=l_0>0}^\infty l^{-r} \lesssim l_0^{-(r-1)}$ for $r > 2$, we have

$$\begin{aligned}\sum_{l=0}^{n-1} |\varphi(x_l) - \varphi(y_l)| &\leq \sum_{l=0}^{n-1} \|\varphi\|_{\log^q} |\log^\star \mathrm{dist}(x_l, y_l)|^{-q} \leq \|\varphi\|_{\log^q} \sum_{l=0}^{n-1} |1 + m + (n-l)L_0|^{-q} \\ &= \|\varphi\|_{\log^q} \sum_{l=1}^{n} |1 + m + lL_0|^{-q} \leq \|\varphi\|_{\log^q} \sum_{l=1}^{\infty} |1 + m + lL_0|^{-q} \lesssim m^{-(q-1)},\end{aligned}$$

where the implicit constant depends on L_0, q and we used the assumption that $q > 2$. □

Lemma 3.1.5. *Let $\mathcal{U}$ be a finite collection of disjoint open subsets of $\mathbb{P}^k$. For every $m > 0$ there exists $n(m,\mathcal{U}) > m$ and, for every $n \geq n(m,\mathcal{U})$, a set $Q_{m,n}$ of repelling periodic points of period n in the intersection of the union of the sets in $\mathcal{U}$ with the small Julia set such that, for all $U \in \mathcal{U}$,*

$$(1 - 1/m)\mu_\varphi(U) \leq \lambda^{-n} \sum_{y \in Q_{m,n} \cap U} e^{\varphi(y)+\varphi(f(y))+\cdots+\varphi(f^{n-1}(y))} \leq (1 + 1/m)\mu_\varphi(U).$$

Proof. We can assume that $\mathcal{U}$ consists of a single open set U, the general case follows by taking $n(m,\mathcal{U})$ to be the maximum of the $n(m,U)$, for $U \in \mathcal{U}$. We can also assume that $\mu_\varphi(U) > 0$ because otherwise we can choose $n(m,U) = m+1$ and $Q_{m,n} = \varnothing$. Fix integers $m_2 \gg m_1 \gg m$. By Lemma 3.1.2, for μ_φ-almost every point a, every ball of sufficiently small radius centred at a is m_2-nice. Hence, we can find a finite family of disjoint m_2-nice balls $B_i \Subset U$, such that $\mu_\varphi(U \setminus \cup B_i) < \mu_\varphi(U)/m_2$. It is then enough to prove the lemma for each B_i instead of U. More precisely, let $B = B_{\mathbb{P}^k}(a,r)$ be an m_2-nice ball. It is enough to find an $n(m_2) > m_2$ and, for all $n \geq n(m_2)$, a set Q of repelling periodic points of period n in $B \cap \operatorname{supp}(\mu_\varphi)$ such that

$$(1 - 1/m_1)\mu_\varphi(B) \leq \lambda^{-n} \sum_{y \in Q} e^{\varphi(y)+\varphi(f(y))+\cdots+\varphi(f^{n-1}(y))} \leq (1 + 1/m_1)\mu_\varphi(B). \tag{3.3}$$

We fix in what follows an integer $m_3 \gg m_2/\mu_\varphi(B)$ and a second ball $B^\star = B_{\mathbb{P}^k}(a, r^\star)$, with $r^\star < r$, such that $\mu_\varphi(B^\star) > (1 - 1/m_2)\mu_\varphi(B)$. Choose a finite family of disjoint m_3-nice balls D_i with the property that $\mu_\varphi(\cup D_i) > 1 - 1/m_3$. We set $D := \cup D_i$ and let b_i be the center of D_i. We also fix balls $D_i^\star \Subset D_i$ centred at b_i and such that $\mu_\varphi(\cup D_i^\star) > 1 - 1/m_3$ and set $D^\star := \cup D_i^\star$.

We have the following two estimates, which are a consequence of the definitions of the objects involved, the fact that we work with nice balls, and the equidistribution of preimages in Theorem 2.2.1. The proofs are omitted here.

Claim 1. There is an integer $M_1 = M_1(m_2, B, B^\star, D_i)$ such that, for all $N \geq M_1$, we have

$$(1 - 4/m_2)\mu_\varphi(B) \leq \mu^{(m_3)}_{D_i,N}(B^\star) \leq (1 + 4/m_2)\mu_\varphi(B) \quad \text{for all } i.$$

Claim 2. There is an integer $M_2 = M_2(m_2, B, D^\star)$ such that, for all $N \geq M_2$, we have

$$1 - 4/m_2 \leq \mu^{(m_2)}_{B,N}(D^\star) \leq 1 + 4/m_2.$$

For every N_1 sufficiently large, every point in the support of $\mathbb{1}_{B^\star}\mu^{(m_3)}_{D_i,N_1}$ corresponds to an m_3-good inverse branch of f of order N_1 mapping D_i to a relatively compact subset of B. Similarly, for every N_2 sufficiently large every point in the support of $\mathbb{1}_{D^\star}\mu^{(m_2)}_{B,N_2}$ corresponds to an m_2-good inverse branch of f of order N_2 mapping B to a relatively compact subset of D. Composing such inverse branches we get inverse branches g_j of $f^{N_1+N_2}$ defined on B whose images are relatively compact in B. In what follows, we only consider these inverse branches g_j. We also write g_j as $g_j^{(1)} \circ g_j^{(2)}$, where $g_j^{(2)}$ is the corresponding inverse branch of f^{N_2} on B (whose image is then in D) and $g_j^{(1)}$ is the corresponding inverse branch of f^{N_1} on $g_j^{(2)}(B)$. We also set $i = i(j)$, where $g_j^{(2)}(B) \subset D_i$.

Each inverse branch g_j as above contracts the Kobayashi metric of B, and thus admits a unique fixed point y_j, which is attracting for g_j and hence repelling for $f^{N_1+N_2}$. Up to possibly increasing the integers M_1 and M_2 given by the Claims above, we can assume that the above properties hold for $N_1 = M_1$ and $N_2 = M_2$. We set $n(m) := M_1(m_2) + M_2(m_2)$ for a fixed choice of sufficiently large m_1, m_2, m_3 and, for all $n \geq n(m)$, we define the set Q as the union of all such fixed points constructed as above with $N_1 = M_1(m_2)$ and $N_2 = n - N_1 \geq M_2(m_2)$. The points in Q are then repelling periodic points of period $n = N_1 + N_2$ for f. Observe that, for all j and all $z \in B$, since $g_j(B) \Subset B$ we have $g_j^l(z) \to y_j$ as $l \to \infty$. Since B intersects the small Julia set, by taking z in the small Julia set we see that y_j belongs to the small Julia set. To conclude, we need to prove (3.3) for this choice of Q. We set

$$\mu_n := \lambda^{-n} \sum_{y \in Q} e^{\varphi(y)+\varphi(f(y))+\cdots+\varphi(f^{n-1}(y))} \delta_y = \lambda^{-n} \sum_j e^{\varphi(y_j)+\varphi(f(y_j))+\cdots+\varphi(f^{n-1}(y_j))} \delta_{y_j}$$

and

$$\widetilde{\mu}_n := \lambda^{-n} \sum_j \Big(e^{\varphi(g_j^{(1)}(b_{i(j)}))+\varphi(f\circ g_j^{(1)}(b_{i(j)}))+\cdots+\varphi(f^{N_1-1}\circ g_j^{(1)}(b_{i(j)}))} \frac{\rho(g_j^{(1)}(b_{i(j)}))}{\rho(b_{i(j)})} \times$$
$$\times\, e^{\varphi(g_j^{(2)}(a))+\varphi(f\circ g_j^{(2)}(a))+\cdots+\varphi(f^{N_2-1}\circ g_j^{(2)}(a))} \frac{\rho(g_j^{(2)}(a))}{\rho(a)} \delta_{g_j(a)} \Big).$$

Observe that there is a correspondence between the terms in μ_n and those in $\widetilde{\mu}_n$. Moreover, since all the balls B and D_i are m_2-nice, we have

$$|\rho(g_j^{(1)}(b_{i(j)}))/\rho(a) - 1| \lesssim m_2^{-1} \text{ and } |\rho(g_j^{(2)}(a))/\rho(b_{i(j)}) - 1| \lesssim m_2^{-1} \text{ for all } i \text{ and } j.$$

It follows from these inequalities and Lemma 3.1.4 that $|\mu_n(B) - \widetilde{\mu}_n(B)| \lesssim \widetilde{\mu}_n(B) m_2^{-1}$. Hence, in order to conclude it is enough to prove that

$$(1 - 1/(2m_1))\mu_\varphi(B) \leq \widetilde{\mu}_n(B) \leq (1 + 1/(2m_1))\mu_\varphi(B)$$

because m_2 is chosen large enough. By construction, we have

$$\widetilde{\mu}_n(B) = \sum_i \mu_{B,N_2}^{(m_2)}(D_i^\star) \cdot \mu_{D_i,N_1}^{(m_3)}(B^\star).$$

By Claim 1, this implies that

$$(1 - 4/m_2)\mu_\varphi(B) \sum_i \mu_{B,N_2}^{(m_2)}(D_i^\star) \leq \widetilde{\mu}_n(B) \leq (1 + 4/m_2)\mu_\varphi(B) \sum_i \mu_{B,N_2}^{(m_2)}(D_i^\star).$$

The assertion then follows from Claim 2 and the fact that $\sum_i \mu_{B,N_2}^{(m_2)}(D_i^\star) = \mu_{B,N_2}^{(m_2)}(D^\star)$, by taking m_2 large enough. □

We can now conclude the proof of Theorem 3.1.1. As mentioned at the beginning of the section, this also completes the proof of Theorem 2.2.1.

End of the proof of Theorem 3.1.1. For every $i \in \mathbb{N}$ we construct a finite family of disjoint open sets $\mathcal{U}_i := \{U_{i,j}\}_{1 \leq j \leq J_i}$ with the following properties:

1. $\mu_\varphi(\cup_{1 \leq j \leq J_i} U_{i,j}) = 1;$

2. for all $1 \leq j \leq J_i$ we have $\mathrm{diam}(U_{i,j}) < 1/i$;
3. for all $i \geq 2$ and $1 \leq j \leq J_i$ there exists $1 \leq j' \leq J_{i-1}$ such that $U_{i,j} \subset U_{i-1,j'}$.

We can construct these sets using local coordinates and generic real hyperplanes which are parallel to the coordinate hyperplanes. Observe also that, by the first condition, we have $\mu_\varphi(\partial U_{i,j}) = 0$ for all i and $1 \leq j \leq J_i$.

For every n, we define $i_n := \max\{m \leq n \colon n \geq n(m, \mathcal{U}_m)\}$, where $n(m, \mathcal{U}_m)$ is given by Lemma 3.1.5. Observe that $i_n \to \infty$ as $n \to \infty$. We define $P'_n \subset \cup_j U_{i_n,j}$ as the union of the sets of repelling periodic points of period n in the small Julia set obtained by applying Lemma 3.1.5 to the collection $\mathcal{U}_{i_n}$ instead of $\mathcal{U}$, and set

$$\mu'_n := \lambda^{-n} \sum_{y \in P'_n} e^{\varphi(y)+\varphi(f(y))+\cdots+\varphi(f^{n-1}(y))} \delta_y.$$

By Properties (i) and (ii) of the open sets $U_{i,j}$ and Lemma 3.1.5, any limit μ' of the sequence $\{\mu'_n\}$ has mass 1. So, since $\mu_\varphi(\cup_j U_{i_n,j}) = 1$ for all n and $\mathrm{diam}(U_{i,j}) < 1/i$ for all i, it is enough to prove that

$$\liminf_{n\to\infty} \mu'_n(U_{i^\star,j^\star}) \geq \mu_\varphi(U_{i^\star,j^\star}) \text{ for all } i^\star \in \mathbb{N} \text{ and } 1 \leq j^\star \leq J_{i^\star}. \tag{3.4}$$

Indeed, given any open set $A \subseteq \mathbb{P}^k$, we can write A as a countable union of compact sets of the form $\overline{U}_{i,j} \Subset A$, overlapping only on their boundaries. We then see that (3.4) implies that $\mu_\varphi(A) \leq \mu'(A)$ for every open set A, and the facts that $\|\mu_\varphi\| = \|\mu'\|$ and $\mu_\varphi(\partial U_{i,j}) = 0$ for all i, j imply that $\mu_\varphi = \mu'$.

We can then fix $i^\star, j^\star$ as in (3.4) and a positive number ε, and it is enough to prove that

$$\mu'_n(U_{i^\star,j^\star}) \geq \mu_\varphi(U_{i^\star,j^\star}) - \varepsilon \quad \text{for all } n \text{ sufficiently large.}$$

We only consider in what follows integers n such that $i_n > i^\star$ and the sets $U_{i_n,j}$ which are contained in $U_{i^\star,j^\star}$. For all such n, we have $\mu_\varphi(U_{i^\star,j^\star}) = \sum_j \mu_\varphi(U_{i_n,j})$ and $\mu'_n(U_{i^\star,j^\star}) = \sum_j \mu'_n(U_{i_n,j})$. It follows by the definition of μ'_n and Lemma 3.1.5 that

$$|\mu'_n(U_{i^\star,j^\star}) - \mu_\varphi(U_{i^\star,j^\star})| \leq \sum_j |\mu'_n(U_{i_n,j}) - \mu_\varphi(U_{i_n,j})| \leq i_n^{-1} \sum_j \mu_\varphi(U_{i_n,j}) = i_n^{-1} \mu_\varphi(U).$$

The assertion follows. □

3.1.2 Holomorphic motions of weighted periodic points

In this section we come back to the topic of bifurcations. We present here the result obtained in [BB23b], together with my PhD student Maxence Brévard. We postponed the description of this work since it relies on Theorems 2.2.1 and 3.1.1.

As we saw in Chapter 1, it is still an open question whether the stability of a general family of endomorphisms in any dimension is equivalent to the motion of all the repelling periodic cycles. On the other hand, we saw in Theorem 1.1.1 that the stability of a family implies at least a weaker version of the motion of the repelling cycles, see Definition 1.1.4. The cycles in Definition 1.1.4 can be seen, in some sense, as generic with respect to the measure of maximal entropy. More precisely, with the terminology introduced in Section 1.1.1, and owing to the equidistribution of the repelling periodic cycles with respect to the measure of maximal entropy [BD99], we see in particular that the fact that the existence of $\mathcal{P}$ as in Definition 1.1.4 is equivalent to the other conditions of stability as in Theorem 1.1.1 leads to the following equidistribution result for periodic graphs in any stable family.

Theorem 3.1.6 (Bianchi [Bia19a]). *Let M be an open connected and simply connected manifold and $(f_\lambda)_{\lambda\in M}$ a stable family of endomorphisms of $\mathbb{P}^k$ of algebraic degree $d \geq 2$. Then, for every $M' \Subset M$ and every $n \in \mathbb{N}^*$, there exists a non-empty subset $\mathcal{P}_n \subset \mathcal{J}$ of motions γ of n-periodic points such that $\gamma(\lambda)$ is repelling for all $\lambda \in M'$ and*

$$\lim_{n\to\infty} d^{-kn} \sum_{\gamma\in\mathcal{P}_n} \delta_\gamma = \mathcal{M},$$

where $\mathcal{M}$ is an equilibrium web.

In [BB23b], we addressed the question of the motion of cycles which similarly equidistribute invariant measures which are equilibrium states for some suitable weight φ, as constructed in Section 2.2 and [BD23a].

Recall from Section 2.2 that if f is an endomorphism of $\mathbb{P}^k$ of algebraic degree $d \geq 2$ satisfying condition **(Hf)** and φ is a real continuous function satisfying condition **(Hφ)** in Theorem 2.2.1, the existence and uniqueness of the equilibrium state μ_φ for f follow from Theorem 2.2.1. By the definition of pressure and the assumption on $\Omega(\varphi)$, all these equilibrium states satisfy $h_{\mu_\varphi} > \log d^{k-1}$. We saw in Section 1.4 that, given a stable family $(f_\lambda)_{\lambda\in M}$ and any $\lambda_0 \in M$, for any f_{λ_0}-invariant measure ν satisfying $h_\nu > \log d^{k-1}$, it is possible to construct an associated web $\mathcal{M}_{\lambda_0,\nu}$ with the property that

$$(p_{\lambda_0})_* \mathcal{M}_{\lambda_0,\nu} = \nu,$$

as well as the associated lamination. This in particular applies to the equilibrium states as above. The following is the main result of [BB23b].

Theorem 3.1.7 (Bianchi-Brévard [BB23b]). *Let M be an open connected and simply connected manifold and $(f_\lambda)_{\lambda\in M}$ a stable family of endomorphisms of $\mathbb{P}^k$ of algebraic degree $d \geq 2$. Take $\lambda_0 \in M$ and assume that f_{λ_0} satisfies condition **(Hf)** . Let $\varphi : \mathbb{P}^k \to \mathbb{R}$ satisfy **(Hφ)** and let μ_φ be the equilibrium state for f_{λ_0} associated to φ. Then, for every $M' \Subset M$ and every $n \in \mathbb{N}$, there exists a non-empty subset $\mathcal{P}_{\varphi,n} \subset \mathcal{J}$ of motions γ of n-periodic points such that $\gamma(\lambda)$ is repelling for all $\lambda \in M'$ and*

$$\lim_{n\to\infty} e^{-nP(\varphi)} \sum_{\gamma\in\mathcal{P}_{\varphi,n}} e^{\varphi(\gamma(\lambda_0))+\cdots+\varphi(f_{\lambda_0}^{n-1}(\gamma(\lambda_0)))} \delta_\gamma = \mathcal{M}_{\lambda_0,\mu_\varphi}.$$

Observe that, in particular, Theorem 2.2.1 generalizes to general weights φ Theorem 3.1.6 , the latter corresponding to the case $\varphi = 0$. It gives a holomorphic motion of a large set of repelling points, related to the equilibrium state μ_φ, in an asymptotic sense in the spirit of Definition 1.1.4 for the case of the measure of maximal entropy.

At the parameter λ_0, the equidistribution of repelling periodic points with respect to the equilibrium state μ_φ is given by Theorem 2.2.1. As we saw in Section 3.1.1, the proof follows the now classical strategy by Briend-Duval [BD99], who showed this result for the measure of maximal entropy, which corresponds to the case $\varphi = 0$. When $\varphi \neq 0$, as the Jacobian of μ_φ is not constant, the proof requires more precise estimates on the contraction along generic inverse branches for μ_φ, which in turn follow from delicate distortion estimates along inverse branches due to Berteloot-Dupont-Molino [BD19; BDM08]. In [BB23b], we adapted this strategy in the setting of the dynamical system $(\mathcal{J}, \mathcal{F}, \mathcal{M}_{\lambda_0,\mu_\varphi})$. This requires to precisely control the contraction of f on tubes (i.e., tubular neighbourhoods, of uniform radius in λ, of the graphs in $M' \times \mathbb{P}^k$ of elements of $\mathcal{J}$) centered at $\mathcal{M}_{\lambda_0,\mu_\varphi}$-generic elements of $\mathcal{J}$. The following version of Lemma 3.1.2 is the key point of the construction. The definition of m-nice tubes is given after the statement.

Lemma 3.1.8. *For every open subset $\Omega \subset M$ and for $\mathcal{M}_{\lambda_0,\mu_\varphi}$-almost every $\gamma \in \mathcal{X}$, there exists $\eta_0 > 0$ such that, for every $0 < \eta < \eta_0$, the tube $T_\Omega(\gamma, \eta)$ is m-nice.*

Given $\Omega \subset M$, $\gamma \in \mathcal{X}$ and $\eta > 0$, we denote by $T_\Omega(\gamma, \eta)$ the η-neighbourhood of the graph Γ_γ of γ in $\Omega \times \mathbb{P}^k$, i.e.,

$$T_\Omega(\gamma, \eta) := \{(\lambda, z) \in \Omega \times \mathbb{P}^k \colon \operatorname{dist}_{\mathbb{P}^k}(z, \gamma(\lambda)) < \eta\}.$$

We call such neighbourhood a *tube* at γ over Ω. Observe that a tube $T_\Omega(\gamma, \eta)$ corresponds to the ball $\mathcal{B}_\Omega(\gamma, \eta)$ in the metric space $(\mathcal{J}, \operatorname{dist}_\Omega)$, where the distance $\operatorname{dist}_\Omega$ is given by

$$\operatorname{dist}_\Omega(\gamma_1, \gamma_2) := \sup_{\lambda \in \Omega} \operatorname{dist}_{\mathbb{P}^k}(\gamma_1(\lambda), \gamma_2(\lambda)).$$

Given a tube $T = T_\Omega(\gamma, \eta)$, the *slice* $T_{|\lambda}$ is the ball $B(\gamma(\lambda), \eta) = T \cap (\{\lambda\} \times \mathbb{P}^k)$. More generally, given the image of a tube T by a holomorphic map $g \colon T \to \Omega \times \mathbb{P}^k$ fibered over M, we define the slice $g(T)_{|\lambda}$ of $g(T)$ at λ as $g(T) \cap (\{\lambda\} \times \mathbb{P}^k)$.

We fix in what follows two constants $0 < A_0 < A_1$, where A_1 is a lower bound for the Lyapunov exponents of $(p_\lambda)_* \mathcal{M}_{\lambda_0,\mu_\varphi}$. Observe that this lower bound is positive as all these measure the same measure-theoretic entropy of μ_φ, which is strictly larger than $(k-1)\log$. Given $\Omega \subset M$, $\gamma \in \mathcal{X}$, a tube T at γ over Ω, and $n \in \mathbb{N}$, we say that a map $g : T \to g(T)$ is an *m-good inverse branch of f of order n on T* if

1. $g \circ f^n = id_{g(T)}$;
2. $\operatorname{diam} f^l_\lambda(g(T)_{|\lambda}) \leq e^{-m-(n-l)A_0}$ for all $\lambda \in \Omega$ and $0 \leq l \leq n$.

Given an inverse branch g of f of order n defined on a tube $T = T(\gamma_0, \eta)$, given any $\gamma \in \mathcal{J}$ with $\Gamma_\gamma \cap (\Omega \times \mathbb{P}^k) \subset T$ we can in particular associate to such inverse branch a map γ'_g such that $\Gamma_{\gamma'_g} \subset g(T)$ and $\mathcal{F}(\gamma'_g) = \gamma|_\Omega$. In particular, the association $\gamma \mapsto \gamma'_g$ defines a map $\mathcal{G}$ on the ball $\mathcal{B}_\Omega(\gamma_0, \eta)$, that we can see as an inverse branch for $\mathcal{F}$ over such ball.

Given $\Omega \subset M$ and a tube T at $\gamma \in \mathcal{X}$ over Ω, we denote by $\mathcal{M}^{(m)}_{T,n}$ the measure

$$\mathcal{M}^{(m)}_{T,n} := \theta(\gamma)^{-1} \sum_{\gamma'_g} e^{\psi(\gamma'_g) + \ldots + \psi(\mathcal{F}^{n-1}(\gamma'_g))} \theta(\gamma'_g) \delta_{\gamma'_g},$$

where the sum is over the preimages γ'_g of γ associated to m-good inverse branches g of f of order n on T. Given $\Omega \subset M$ and $m > 1$, we say that a tube T at $\gamma \in \mathcal{X}$ over Ω is *m-nice* if $\|\mathcal{M}^{(m)}_{T,n}\| \geq 1 - 1/m$ for all n sufficiently large. We say that a ball $\mathcal{B}_\Omega(\gamma, \eta)$ is m-nice if the tube $T_\Omega(\gamma, \eta)$ is m-nice.

The idea to prove Theorem 3.1.2 in Section 3.1.1 was to consider the natural extension $(\widehat{\mathbb{P}^k}, \widehat{f}, \widehat{\mu}_\varphi)$ of the system $(\mathbb{P}^k, f, \mu_\varphi)$ and to construct good approximations of the disintegration of the measure $\widehat{\mu}_\varphi$ with respect to a projection $\widehat{\mathbb{P}^k} \to \mathbb{P}^k$. This was mainly achieved through Lemma 3.1.3, which gave a control on the contraction of generic inverse branches of f with respect to μ_φ. In the current setting, we need to work with the natural extension $(\widehat{\mathcal{J}}, \widehat{\mathcal{F}}, \widehat{\mathcal{M}}_{\lambda_0,\mu_\varphi})$ of the dynamical system $(\mathcal{J}, \mathcal{F}, \mathcal{M}_{\lambda_0,\mu_\varphi})$. This can be defined in a similar way as $(\mathbb{P}^k, f, \mu_\varphi)$, but now the underlying space $\widehat{\mathcal{J}}$ is much more complicated than $\widehat{\mathbb{P}^k}$. This is due to the fact that it is a priori not true that every element of $\mathcal{J}$ has d^k preimages under $\mathcal{F}$.

Apart from the technical issues above, the main point is to get an exponential contraction along ($\mathcal{M}_{\lambda_0,\mu_\varphi}$-generic) inverse branches of $\mathcal{F}$. This is achieved through the following proposition, which is the counterpart of Lemma 3.1.3 in this setting. I had established a version of it in the case of the measure of maximal entropy in my PhD thesis (see [BBD18, Propositions 4.2 and 4.3]) as a key point to prove the existence of an equilibrium lamination, and the general case has essentially the same proof. We had already used it with Rakhimov in the following form in the proof of Theorem 1.4.3.

Proposition 3.1.9. *For every open set $\Omega \Subset M$ and $0 < A < A_1$ there exists a Borel subset $\widehat{\mathcal{Y}} \subseteq \widehat{\mathcal{J}}$ with $\widehat{\mathcal{M}}_{\lambda_0,\mu_\varphi}(\widehat{\mathcal{Y}}) = 1$, and two measurable functions $\widehat{\eta}_A \colon \widehat{\mathcal{Y}} \to]0, 1]$ and $\widehat{l}_A \colon \widehat{\mathcal{Y}} \to [1, +\infty[$ which satisfy the following properties.*

For every $\widehat{\gamma} \in \widehat{\mathcal{Y}}$ and every $n \in \mathbb{N}^*$ the iterated inverse branch $f^{-n}_{\widehat{\gamma}}$ is defined and Lipschitz-continuous on the tubular neighbourhood $T_\Omega(\gamma_0, \widehat{\eta}_A(\widehat{\gamma}))$ of the graph $\Gamma_{\gamma_0} \cap (\Omega \times \mathbb{P}^k)$ of γ_0, and we have

$$f^{-n}_{\widehat{\gamma}}(T_\Omega(\gamma_0, \widehat{\eta}_A(\widehat{\gamma}))) \subset T_\Omega(\gamma_{-n}, e^{-nA}) \quad \text{and} \quad \widetilde{\mathrm{Lip}}(f^{-n}_{\widehat{\gamma}}) \leq \widehat{l}_A(\widehat{\gamma}) e^{-nA},$$

where $\widetilde{\mathrm{Lip}}(f^{-n}_{\widehat{\gamma}}) := \sup_{\lambda \in \Omega} \mathrm{Lip}\left((f^{-n}_{\widehat{\gamma}})_{|B(\gamma_0(\lambda), \widehat{\eta}_A)}\right)$.

Once Lemma 3.1.8 is established, the proof of Theorem 3.1.7 follows the same lines as that of Theorem 3.1.1 (up to a couple of further technical issues due to the more complicated underlying space $\mathcal{J}$).

3.2 Degeneration of quadratic polynomial endomorphisms to a Hénon map

The work [BO20], joint with Yûsuke Okuyama, originated from a questions by Charles Favre. The original goal was to study an algebraic family of regular quadratic polynomial endomorphisms of $\mathbb{C}^2$ parametrized by a punctured open disk and degenerating to a Hénon map at the puncture. For one dimensional meromorphic families of rational functions, such degenerations towards rational functions of lower (topological) degrees had already been intensively studied, see for instance [DeM16; FG18; DO18]. Favre had recently introduced a general framework to study such degenerations for holomorphic endomorphisms of $\mathbb{P}^k$ in [Fav20] and we provided the first concrete study in dimension k higher than 1. As a consequence of our estimates, we could also study the geometry of the bifurcation locus in the space of quadratic holomorphic endomorphisms of $\mathbb{P}^2$, in the sense of Theorem 1.1.1 and Definition 1.1.2, near the degeneration parameters.

The geometry of the bifurcation locus near the line at infinity of the moduli space $\mathcal{M}_2 \cong \mathbb{C}^2$ of quadratic rational functions on $\mathbb{P}^1$ had been studied in [BG15a], and I had also just studied the parallel problem near the hyperplane at infinity of the natural parameter space $\cong \mathbb{C}^3$ of quadratic polynomial skew products on $\mathbb{C}^2$ in [AB23] with Matthieu Astorg, as was described in Section 1.3, see Theorem 1.3.1.

In the rest of this section, we fix a constant $c \in \mathbb{C}^*$ and a polynomial $p(w) = w^2 + c_1 w + c_2 \in \mathbb{C}[w]$. For each ordered pair

$$(g,h) \in \mathbb{C}[z,w] \times \mathbb{C}[z,w] \quad \text{such that } \deg g = 2,\ g_{zz} \in \mathbb{C}^*, \text{ and } \deg h \le 2, \tag{3.5}$$

we focus on the algebraic family

$$f_t(z,w) = f_t(z,w;g,h) = \begin{pmatrix} w \\ cz + p(w) \end{pmatrix} + t \begin{pmatrix} g(z,w) \\ h(z,w) \end{pmatrix}, \quad t \in \mathbb{D}, \tag{3.6}$$

of quadratic polynomial endomorphisms of $\mathbb{C}^2$ parametrized by $\mathbb{D}$; we also set the constants $g_{zz}/2! =: G_g = G \in \mathbb{C}^*$ and $h_{zz}/2! =: H_h = H \in \mathbb{C}$, respectively, and set

$$\widetilde{g}(z,w) := g(z,w) - Gz^2.$$

For $t = 0$, the map $f_0(z,w) = (w, cz + p(w))$ is a Hénon map and is independent of (g,h). For every $0 < |t| \ll 1$, f_t is a *regular* quadratic polynomial endomorphism of $\mathbb{C}^2$, that is, it extends to a holomorphic endomorphism of $\mathbb{P}^2$ (see, e.g., [BJ00]). In particular, if $0 < |t| \ll 1$, then f_t admits a unique maximal entropy measure μ_{f_t}, whose support J_{f_t} is compact in $\mathbb{C}^2$, and the sum $L(f_t)$ of the two *individual* Lyapunov exponents $\chi_1(f_t) \ge \chi_2(f_t)$(indeed $\ge \log\sqrt{2}$ [BD99]) of f_t with respect to μ_{f_t} is given by

$$L(f_t) = \int_{\mathbb{C}^2} \log|\det(Df_t)|\mu_{f_t} \in \mathbb{R}.$$

For simplicity, in this section we will call $L(f_t)$ *the Lyapunov exponent* of f_t with respect to μ_{f_t}. Here and below, we fix the trivialization of the tangent bundle $T\mathbb{C}^2$ of $\mathbb{C}^2$ induced by the orthonormal frame (∂_z, ∂_w) of $T\mathbb{C}^2$, and identify the derivative df of a polynomial endomorphism $f \in (\mathbb{C}[z,w])^2$ of $\mathbb{C}^2$ with the $M(2,\mathbb{C})$-valued function $p \mapsto (Df)_p$ on $\mathbb{C}^2$, where $(Df)_p$ is the Jacobian matrix of f at $p \in \mathbb{C}^2$, by convention.

We regard the set of all (g,h) as in (3.5) as $(\mathbb{C}^* \times \mathbb{C}^5) \times \mathbb{C}^6$, parametrizing it by the coefficients of g,h to mention the local uniformity of estimates on (g,h). Set $\mathbb{D}_r := \{|t| < r\}$ and $\mathbb{D}_r^* := \{0 < |t| < r\}$ for each $r > 0$, as usual.

3.2.1 Degeneration of the Lyapunov exponent

Our first interest was in the asymptotic behaviour of $L(f_t)$ as $t \to 0$, where $f_t = f_t(z,w;g,h)$, for each (g,h) as in (3.5). Such a behaviour had been studied for meromorphic families of rational functions on $\mathbb{P}^1$ by DeMarco [DeM16]. In our situation, it follows from Favre's generalization [Fav20] of DeMarco's estimate that there is a non-negative constant α such that

$$L(f_t) = \alpha \log|t|^{-1} + o(\log|t|^{-1}) \quad \text{as } t \to 0$$

and that the constant α is characterized as the *non-archimedean* Lyapunov exponent associated to the meromorphic family $(f_t)_{t\in\mathbb{D}^*}$, regarding this family as a single rational function defined over a valued field of formal Laurent series at $t=0$. The function $t\mapsto L(f_t)-\alpha\log|t|^{-1}$ is continuous and subharmonic on $0<|t|\ll 1$ (see, e.g., [DS10a] for more details), and is a potential of the *bifurcation current* (indeed *measure*) on $0<|t|\ll 1$ associated to the family (f_t) in the sense of Theorem 1.1.1 and [BBD18].

In the following, it is convenient to say that the pair (g,h) (or the associated $(G,H)=(G_g,H_h)$) is *non-exceptional* if

$$\left|\frac{H}{G}\right|\neq|c|.$$

Our first main result answered affirmatively Favre's general question [Fav20, Problem 1] in our context by establishing the continuous (and indeed harmonic) extendibility of the potential $t\mapsto L(f_t)-\alpha\log|t|^{-1}$ across $t=0$, with the concrete value $\alpha=1/2$, for any non-exceptional (g,h).

Theorem 3.2.1 (Bianchi-Okuyama [BO20]). *The following assertions hold.*

(i) Pick (g_0,h_0) as in (3.5). *If (g_0,h_0) is non-exceptional, i.e., $|H_{h_0}/G_{g_0}|\neq|c|$, then for every $\beta\in(0,1)$ so small that*

$$\begin{cases}|H_{h_0}/G_{g_0}|<C(\beta;g_0,h_0)|c| & \text{if } |H_{h_0}/G_{g_0}|<|c|\\ |H_{h_0}/G_{g_0}|>C(\beta;g_0,h_0)|c| & \text{if } |H_{h_0}/G_{g_0}|>|c|,\end{cases}$$

there exist $r_0\in(0,1)$ and an open neighborhood Ω_0 of (g_0,h_0) in $(\mathbb{C}^*\times\mathbb{C}^5)\times\mathbb{C}^6$ such that

$$\log\frac{(1-\beta)\cdot 4\big||H/G|-C(\beta;g,h)|c|\big|^{1/2}}{|G|^{1/2}}\leq L(f_t)-\frac{1}{2}\log|t|^{-1}\leq\log\frac{(1+\beta)\cdot 4\big(|H/G|+|c|\big)^{1/2}}{|G|^{1/2}}$$

for every $(g,h)\in\Omega_0$ and every $t\in\mathbb{D}^*_{r_0}$. We recall here that $f_t=f_t(z,w;g,h)$ and $(G,H)=(G_g,H_h)$. Moreover, for every $\beta\in(0,1)$ and every non-exceptional (g,h), we set

$$C(\beta;g,h):=\begin{cases}(1+\beta)^{-1/2} & \text{if } |H/G|<|c|,\\ (1-\beta)^{-1/2} & \text{if } |H/G|>|c|.\end{cases}$$

In particular, for every non-exceptional (g,h), the non-archimedean Lyapunov exponent α associated to the meromorphic family $(f_t)_{t\in\mathbb{D}^*}$ equals $1/2$.

(ii) Pick a non-exceptional (g,h). Then for the algebraic family $(f_t)_{t\in\mathbb{D}^*}$ in (3.6) *associated to this (g,h), the continuous and subharmonic function $t\mapsto L(f_t)-(1/2)\log|t|^{-1}$ is harmonic on $0<|t|\ll 1$ and extends harmonically across $t=0$, satisfying*

$$\lim_{t\to 0}\Big(L(f_t)-\frac{1}{2}\log|t|^{-1}\Big)=\log\frac{4\max\{|c|,|H/G|\}^{1/2}}{|G|^{1/2}}.$$

Notice that a similar continuous extendability result had already been obtained for meromorphic families of polynomials in one variable by Favre–Gauthier [FG18] (see also [GY17] for cubic polynomials in one variable) and that examples of discontinuity at $t = 0$ had been obtained in [DO18] for meromorphic families of rational functions on $\mathbb{P}^1$.

The main idea to prove Theorem 3.2.1 is to get quantitative estimates on the distance between the critical set and the Julia set as the family degenerates. This, in turn, gives estimates of the Lyapunov exponent. I give below an overview of the estimates that lead to the first item of the Theorem for an explicit choice of the parameter involved (possibly the simplest ones). The estimates for the general case and for the regularity in the second item are more refined, but essentially similar.

Let us first recall a simple computation from dimension 1. Consider the family $f_t(z) = z^2 + t^{-1}$, and its degeneration as $t \to 0$ (in particular, we assume here that t can be taken arbitrarily close to 0). Classical estimates give that the Julia set of f_t is contained in the ball $\mathbb{B}(0, |2t|^{-1})$. Moreover, for $|t|$ sufficiently small, it is also possible to prove that every point in the ball $\mathbb{B}(0, |2t|^{-1/2})$ escapes to infinity. Hence, the Julia set J_t of f_λ is contained in the annulus $\{|2t|^{-1} \leq |z| \leq |2t|^{-1/2}\}$. As $f_t'(z) = 2z$ for every t, we have

$$\log|2t| \leq \min_{J_\lambda} \log|f_t'| \leq L(f_t) = \int \log|f_t'| \mu_\lambda \leq \max_{J_t} \log|f_t'| \leq -\frac{1}{2}\log|2t|,$$

which gives the estimates

$$\log|t| \lesssim L(f_\lambda) \lesssim \frac{1}{2}\log|t|.$$

where the inequalities are up to additive constants. In particular, the non Archimedean Lyapunov exponents of the family is at least $1/2$ (more precise estimates, for instance a computation of the Green function at 0, show that this is indeed the value).

Let us now consider the 2-dimensional family $f_t(z, w) = (w + tz^2, z + w^2)$, for $t \in \mathbb{C}^*$. We can see that f_t degenerates to the map $f_0(z, w) := (w, z + w^2)$, which is a Hénon map. The following lemma gives some estimates on the position of the Julia set of f_t. In this case, the lemma can also be deduced from [Duj17, Lemma 5.2].

Lemma 3.2.2. *Fix two constants $0 < \gamma < 1/2 < \alpha < 1$ and set*

$$V_t := \left\{(z, w) \in \mathbb{C}^2 \colon \frac{1}{2|t|} < |z| < \frac{3}{2|t|}, \frac{1}{|t|^\gamma} < |w| < \frac{1}{|t|^\alpha}\right\}.$$

Then for t sufficiently small we have $f_t^{-1}(V_t) \Subset V_t$.

It follows from this lemma that the Julia set of f_t is contained in V_t. A straightforward computation also shows that the critical set does not intersect V_t and thus also the Julia set.

We can now show that, for every $0 < \gamma < 1/2 < \alpha < 1$, we have

$$\gamma \log\frac{1}{|t|} \lesssim L(f_t) \lesssim \alpha \log\frac{1}{|t|},$$

where again the inequalities are up to additive constants. Indeed, since the Julia set is contained in the set V_t and the Jacobian is given by $4zwt - 1$, we have (for every t sufficiently small),

$$L(f_t) = \int \log \operatorname{Jac}(f_t)\mu_t \leq \sup_{V_t} \log|4zwt - 1| \leq \log\left(\left|4\frac{3}{2|t|}\frac{1}{|t|^\alpha}|t|\right| + 1\right) \leq \alpha \log\frac{1}{|t|} + C$$

and

$$L(f_t) = \int \log \operatorname{Jac}(f_t)\mu_t \geq \inf_{V_t} \log |4zwt - 1| \geq \log \left(\left| 4\frac{1}{2\,|t|}\frac{1}{|t|^{\gamma}}\,|t| \right| - 1 \right) \geq \gamma \log \frac{1}{|t|} - C'.$$

for some constants C, C'. The asymptotic behaviour $L(f_t) \sim \frac{1}{2}\log\frac{1}{|t|}$ follows.

3.2.2 Accumulation of the bifurcation locus to the Hénon locus

Let us now focus on the family

$$f_{t,G,H}(z,w) := \begin{pmatrix} w \\ cz + p(w) \end{pmatrix} + t \begin{pmatrix} Gz^2 \\ Hz^2 \end{pmatrix}, \quad (t, G, H) \in \mathbb{D}^* \times \mathbb{C}^* \times \mathbb{C},$$

of quadratic polynomial endomorphisms of $\mathbb{C}^2$, which are regular for every $t \in \mathbb{D}^*$ since the leading homogeneous term $(tGz^2, w^2 + tHz^2)$ of $f_{t,G,H}$ maps only $(0,0)$ to $(0,0)$.

Theorem 3.2.3 (Bianchi-Okuyama [BO20]). *The bifurcation locus in the parameter space $\mathbb{D}^* \times \mathbb{C}^* \times \mathbb{C}$ of the family $(f_{t,G,H})$ accumulates to $\{t = 0\}$ in $\mathbb{D} \times \mathbb{C}^* \times \mathbb{C}$ tangentially to the locus $|H/G| = |c|$ in $\mathbb{D}^* \times \mathbb{C}^* \times \mathbb{C}$.*

The set $\operatorname{Hol}_2(\mathbb{P}^2)$ of all quadratic holomorphic endomorphisms of $\mathbb{P}^2$ is a Zariski open subset in $\mathbb{P}^{N_2}$, where $N_2 = 3 \cdot 4!/(2!2!) - 1 = 17$, in the coefficients parametrization, and in turn is regarded as the parameter space of the holomorphic family of (all) quadratic holomorphic endomorphisms of $\mathbb{P}^2$. We also note that all Hénon maps live in $\mathbb{P}^{N_2} \setminus \operatorname{Hol}_2(\mathbb{P}^2)$ in the coefficients parametrization.

The following immediate consequence of Theorem 3.2.3 answered affirmatively a question posed by Johan Taflin.

Corollary 3.2.4. *The whole Hénon locus in $\mathbb{P}^{N_2} \setminus \operatorname{Hol}_2(\mathbb{P}^2)$ is accumulated by the bifurcation locus of $\operatorname{Hol}_2(\mathbb{P}^2)$.*

The proof of Theorem 3.2.3 is based on Theorem 3.2.1(i) and is purely analytical. It essentially follows from the estimate in the second item of Theorem 3.2.1, which allows us to show explicitly that the Lyapunov function is not harmonic in any neighbourhood of the degenerating parameter. In former studies, the presence of bifurcations had been established by means of more geometric arguments, usually by creating Misiurewicz parameters, see, e.g., [BT17; Duj17; BB18a; Taf21; AB23; Bie19].

3.2.3 Individual Lyapunov exponents

It follows from a result by Pham [Pha05] that, for any holomorphic family $(f_t)_{t\in M}$ parametrized by a complex manifold M of holomorphic endomorphisms of $\mathbb{P}^k$, if we denote by $\chi_1(f_t) \geq \cdots \geq \chi_k(f_t)$ all the individual Lyapunov exponents of f_t with respect to μ_{f_t} for each $t \in M$, then for every $j \in \{1, \ldots, k\}$, the function $t \mapsto \sum_{\ell=1}^{j} \chi_\ell(f_t)$ on M is plurisubharmonic.

Let us focus on our family (f_t) as in (3.6). Recall that the function $t \mapsto L(f_t) \equiv \chi_1(f_t) + \chi_2(f_t)$ is continuous and subharmonic on $0 < |t| \ll 1$, so by the above result by Pham, the function $t \mapsto \chi_1(f_t)$ is subharmonic and the function $t \mapsto \chi_2(f_t)$ is lower semicontinuous, in general. We conclude this introduction with the following precision of Theorem 3.2.1(ii).

Theorem 3.2.5 (Bianchi-Okuyama [BO20]). *Pick a non-exceptional (g,h). Then for the algebraic family $(f_t)_{t\in\mathbb{D}^*}$ in* (3.6) *associated to this (g,h), the functions $t\mapsto \chi_1(f_t)-\frac{1}{2}\log|t|^{-1}$ and $t\mapsto\chi_2(f_t)$ are harmonic on $0<|t|\ll 1$ and extend harmonically across $t=0$, satisfying*

$$\lim_{t\to 0}\Big(\chi_1(f_t)-\frac{1}{2}\log|t|^{-1}\Big)=\log\frac{2\max\{|c|,|H/G|\}^{1/2}}{|G|^{1/2}} \quad \textit{and} \quad \lim_{t\to 0}\chi_2(f_t)=\log 2.$$

The proof of the harmonicity of $\chi_1(f_t),\chi_2(f_t)$ is based on the estimates developed so far and on the full strength of Berteloot–Dupont–Molino's approximations, that is, approximations of not only $\chi_1(f_t)+\chi_2(f_t)(=L(f_t))$ but also $\chi_1(f_t)$ [BDM08, Theorem 1.5].

3.3 Dynamical degrees of Hénon-like and polynomial-like maps

This section mainly describes the content of the paper [BDR23], written in collaboration with Tien-Cuong Dinh and Karim Rakhimov, see Section 3.3.1. In Section 3.3.2, I also describe the result from [BR22] that was postponed in Section 1.4.

3.3.1 Monotonicity of dynamical degrees

Let $f\colon X\to X$ be dominant rational self-map of a complex projective manifold, or more generally a dominant meromorphic self-map of a compact Kähler manifold of dimension k. Let ω be a Kähler form on X. For any $0\le s\le k$ one can define the sequence $\lambda^+_{s,n}:=\|(f^n)_*(\omega^{k-s})\|_X$ of the masses of the positive closed $(k-s,k-s)$-currents $(f^n)_*(\omega^{k-s})$, and the *dynamical degree of order s of f* as

$$\lambda^+_s:=\limsup_{n\to\infty}(\lambda^+_{s,n})^{1/n}.$$

For cohomological reasons, ω^{k-s} could be replaced by any smooth form or some positive closed current in the same cohomology class. In particular, the sequence $(\lambda^+_{s,n})_{n\in\mathbb{N}}$ detects the volume growth of s-dimensional subvarieties (whenever they exist) under the action of f^n, for $n\in\mathbb{N}$.

It turns out that the sequences $(\lambda^+_{s,n})_{n\in\mathbb{N}}$ are (almost) sub-multiplicative, hence the $\limsup$ in the definition above is actually a limit [DS04; DS05b; Gue05; RS97; Ves92], see also [Dan20; Tru20]. For a precise behaviour of these sequences in a number of settings, see also [BFJ08; DF21; FW12; Ngu06; Tru14] and references therein. It is also a consequence of the fundamental Khovanskii-Teissier inequalities that the sequence of the dynamical degrees λ^+_s is log-concave, i.e., the function $s\mapsto\log\lambda^+_s$ is concave [DN06; Gro90; Kho79; Tei79]. An immediate consequence of this property is that there exists $1\le p\le k$ such that

$$\lambda^+_0\le\lambda^+_1\le\ldots\le\lambda^+_p\ge\ldots\ge\lambda^+_{k-1}\ge\lambda^+_k. \tag{3.7}$$

When X is projective, this means that the growth rate under the action of f of the volumes of s-dimensional analytic subsets, for $s\le p$, dominates that of $(s-1)$-dimensional analytic subsets, and the reversed property is true for $s>p$. Moreover, the so-called *algebraic entropy* $\log\lambda^+_p$ of f is larger than or equal to the (topological) entropy of the system [DS04; DS05b; Gro03; Vu21], see also [dV10; Yom87] for the reversed inequality.

Let us stress that the proof of the log-concavity of the sequence of the degrees $\{\lambda_s^+\}_{0\leq s\leq k}$, and as a consequence that of (3.7), deeply relies on the algebraic setting and on cohomological arguments (Hodge-Riemann theorem). In particular, it breaks down when considering non-compact or local situations, even when λ_k^+ is the degree of maximal value (i.e., when the system is somehow geometrically expanding). We addressed this problem in [BDR23], with new tools coming from pluripotential theory.

We consider in this section invertible horizontal-like maps in any dimensions and polynomial-like maps in any dimensions. In dimension 2, horizontal-like maps were introduced by Hubbard and Oberste-Vorth in [HO95] and systematically studied by Dujardin in [Duj04], as a generalization of Hénon automorphisms [BLS93; FS92; Sib99]. For these maps, the properties of dynamical degrees that we consider in this section are obvious. We will focus on horizontal-like maps in any dimensions which have been studied in [DNS08; DS06c]. *Horizontal-like maps* are essentially holomorphic maps, defined on some bounded (convex, for simplicity) subset D of $\mathbb{C}^k$, that have an expanding behaviour in p directions and contracting behaviour in the remaining $k-p$ directions. Such expansion and contraction are of global nature, and these maps are in general not uniformly hyperbolic. Assuming $D = M \times N$ (with $M \Subset \mathbb{C}^p$ and $N \Subset \mathbb{C}^{k-p}$ bounded convex domains), the map f sends a vertical open subset of D to a horizontal one and, roughly speaking, the vertical (resp. horizontal) part of the boundary of the first to the vertical (resp. horizontal) part of the boundary of the second. As a particular case, when $p = k$ the set N reduces to a point and one recovers the notion of *polynomial-like maps*, proper holomorphic maps of the form $f\colon U \to V$, for some open bounded subsets $U \Subset V \Subset \mathbb{C}^k$, with V convex [DS03; DS10a]. Polynomial-like maps in dimension 1 were first introduced by Douady and Hubbard in [DH85]. Properties of dynamical degrees in dimension 1 are obvious.

Horizontal-like and polynomial-like maps can be seen as the building blocks of larger systems and, in particular, give a good setting to study local dynamical problems in larger dynamical systems. Small perturbations of such maps still belong to these classes (up to slightly shrinking the domain of definition), hence we get large classes of examples, and the families are infinite-dimensional. As examples, perturbations of lifts to $\mathbb{C}^{k+1}$ of holomorphic endomorphisms of $\mathbb{P}^k(\mathbb{C})$ give examples of polynomial-like maps. Perturbations of complex Hénon automorphisms of $\mathbb{C}^2$ [BLS93; FS92; Sib99] give horizontal-like maps with $k = 2$ and $p = 1$. Such maps were for instance considered in [Duj04]. More generally, we call any invertible horizontal-like map a *Hénon-like map*. It is known that not all polynomial-like maps are conjugated to, or are small perturbations of, polynomial endomorphisms, see for instance [DS10a, Example 2.25]. One can deduce a similar property for horizontal-like maps considering maps of the form $(z, w) \mapsto (h(z) + \varepsilon w, \varepsilon' z)$ on suitable domains, with h a polynomial-like map, see [Duj04]. In particular, one can construct horizontal-like maps with both attracting and repelling points, something which is not possible for Hénon maps.

Given a Hénon-like map or a polynomial-like map, one can introduce dynamical degrees as above. Denoting again by ω the standard Kähler form on $\mathbb{C}^k$, one can roughly define

$$\lambda_s^+ := \limsup_{n\to\infty} (\lambda_{s,n}^+)^{1/n}, \quad \text{where} \quad \lambda_{s,n}^+ := \|(f^n)_*\omega^{k-s}\|_{M'\times N'}.$$

Here $M' \Subset M$ and $N' \Subset N$ are open convex sets slightly smaller than M and N respectively. In fact, we can show that λ_s^+ is independent of the choice of $D' := M' \times N'$. Because of the geometry of the problem, these definitions are only given for $0 \leq s \leq p$. On the other hand, for Hénon-like maps, since $p < k$, one can also define the remaining degrees as $\lambda_s^-(f) := \lambda_s^+(f^{-1})$ for $0 \leq s \leq k-p$ (since f^{-1} is a "vertical-like" map with $k-p$ expanding directions).

Observe that, a priori, the sequences $(\lambda_{s,n}^+)_{n\in\mathbb{N}}$ above do not need to be sub-multiplicative this time. More importantly, the lack of a Hodge theory means that, a priori, the resulting degree may change if one replaces ω^s with the integration on a given analytic set of dimension $k-s$, or more generally with a (positive closed) current of bi-degree (s,s). Hence, for $0 \leq s \leq p$ it is natural to also introduce the degree

$$d_s^+ := \limsup_{n\to\infty}(d_{s,n}^+)^{1/n}, \quad \text{where} \quad d_{s,n}^+ := \sup_S \|(f^n)_*(S)\|_{M'\times N}$$

and S runs over the set of all horizontal positive closed currents of bi-dimension (s,s) and of mass 1 on $D' := M' \times N'$. These definitions are also independent of the choice of D'. As for λ_s^-, for Hénon-like maps we can define the remaining dynamical degrees as $d_s^-(f) := d_s^+(f^{-1})$ for every $0 \leq s \leq k-p$. By [DNS08], we have $\lambda_p^+ = \lambda_{k-p}^- = d_p^+ = d_{k-p}^- =: d \in \mathbb{N}$. This is the main dynamical degree, which is crucial in the construction of Green currents, in particular, to control the mass of vertical positive closed (p,p)-currents and horizontal positive closed $(k-p,k-p)$-currents using an adapted intersection theory, see [DNS08; DS06c]. The other degrees $\{d_s^+\}_{0\leq s\leq p-1}$ also play an important role in [DNS08] to control the norm of the pushforwards of the other horizontal positive closed currents.

The following theorems are our main results, which in particular answer [DNS08, Question 6.3].

Theorem 3.3.1 (Bianchi-Dinh-Rakhimov [BDR23]). *Let $1 \leq p < k$ be integers and f be a Hénon-like map from a vertical open subset of a bounded convex domain $D = M \times N \subset \mathbb{C}^p \times \mathbb{C}^{k-p}$ to a horizontal open subset of D. Then, the sequences $\{\lambda_s^+\}_{0\leq s\leq p}$, $\{\lambda_s^-\}_{0\leq s\leq k-p}$, $\{d_s^+\}_{0\leq s\leq p}$, and $\{d_s^-\}_{0\leq s\leq k-p}$ satisfy*

$$\lambda_0^+ \leq \lambda_1^+ \leq \ldots \leq \lambda_p^+ = d = \lambda_{k-p}^- \geq \ldots \geq \lambda_1^- \geq \lambda_0^-$$

and

$$1 = d_0^+ \leq d_1^+ \leq \ldots \leq d_p^+ = d = d_{k-p}^- \geq \ldots \geq d_1^- \geq d_0^-.$$

Moreover, we have $\lambda_p^+ = d_p^+ \in \mathbb{N}$.

Since $\log d_p^+$ is equal to the topological entropy $h_t(f)$ of f [DNS08], we deduce the following immediate consequence of Theorem 3.3.1.

Corollary 3.3.2. *Let p, k, f be as in Theorem 3.3.1. Then, for all $0 \leq s^+ \leq p$ and $0 \leq s^- \leq k-p$ we have*

$$\log \lambda_{s^\pm}^\pm \leq \log d_{s^\pm}^\pm \leq \log d_p^+ = h_t(f).$$

Theorem 3.3.1 also shows that the assumption $d > \max(\max_{0\leq s\leq p-1} d_s^+, \max_{0\leq s\leq k-p} d_s^-)$ can be weakened to $d > \max(d_{p-1}^+, d_{k-p-1}^-)$ in [DNS08, Theorem 5.4]. This implies that the measure of maximal entropy for f constructed in [DNS08] is hyperbolic as soon as this last assumption is satisfied.

The proof of the monotonicity of the sequence $\{d_s^+\}_{0\leq s\leq p}$ in Theorem 3.3.1 can be applied also when f is not invertible. On the other hand, our proof of Theorem 3.3.1 does not give the monotonicity of the sequence $\{d_s^-\}_{0\leq s\leq k-p}$ when f is not invertible, as well as the monotonicity of the sequences $\{\lambda_s^+\}_{0\leq s\leq p}$ and $\{\lambda_s^-\}_{0\leq s\leq k-p}$ when $p\neq k$. The reason is essentially that the pushforward of smooth forms is not smooth when f is not invertible, which implies that the definition of the degrees λ_s^+ and λ_s^- may a priori depend of the domains of definition. Our proof of Theorem 3.3.1 also does not give any information on the possible log-concavity of the sequence $\{d_s^+\}_{0\leq s\leq p}$.

When $p=k$, i.e., f is a polynomial-like map, the problems above can be avoided and, in particular, the degrees do not depend on the domain of definition [DS03]. As a consequence, we have the following version of Theorem 3.3.1 for polynomial-like maps.

Theorem 3.3.3 (Bianchi-Dinh-Rakhimov [BDR23]). *Let $k\geq 1$ be an integer and take open sets $U\Subset V\Subset \mathbb{C}^k$ with V convex. Let $f\colon U\to V$ be a polynomial-like map of topological degree d_t. Then the sequences $\{\lambda_s^+\}_{0\leq s\leq k}$ and $\{d_s^+\}_{0\leq s\leq k}$ satisfy*

$$\lambda_0^+\leq\lambda_1^+\leq\ldots\leq\lambda_k^+=d_t \quad \textit{and} \quad 1=d_0^+\leq d_1^+\leq\ldots\leq d_k^+=d_t.$$

In particular, all the dynamical degrees of f are smaller than or equal to d_t.

As far as we are aware of, the only (non trivial) cases where the monotonicity of the dynamical degrees could be established until now are of algebraic nature, and such monotonicity is a consequence of the log-concavity property mentioned above. In a nutshell, in order to establish the monotonicity of the sequence $\{d_s^+\}_{0\leq s\leq p}$, given a (horizontal positive closed) current S of bi-dimension (s,s) with $s<p$, one needs to find another current, of bi-dimension $(s+1,s+1)$ whose mass growth under iteration bounds the mass growth of S under iteration. This is not a big problem when S is smooth (which essentially gives the monotonicity of the sequence $\{\lambda_s^+\}_{0\leq s\leq p}$, whose proof is considerably easier and does not rely on the arguments below). The main problem arises when S is not smooth, and already in the case where S is given by the integration on an analytic set. Even considering an analytic set containing the first one, it is not clear at all why the iterates of the first should behave nicely inside the iterates of the second.

Our solution to the problem can be roughly explained as follows. Given a (horizontal positive closed) current S in D, we first construct a "holomorphic" family of positive closed currents S_θ parametrized by $\theta\in\mathbb{D}$ and with $S_0\geq S$. We then consider all these currents as the slices of a unique current $\mathcal{R}$, of bi-dimension $(s+1,s+1)$, on the space $D\times\mathbb{D}$, using the slices $D\times\{\theta\}$. The candidate to the role of current of bi-dimension $(s+1,s+1)$ in D would then be $R:=(\pi_D)_*(\mathcal{R})$, where $\pi_D\colon D\times\mathbb{D}\to D$ is the natural projection. Two difficulties arise here. First, we need to make sure that R is well-defined and horizontal in D. In order to do this, we suitably modify $\mathcal{R}$ in the space $D\times\mathbb{D}=M\times N\times\mathbb{D}$, in order to make it become *horizontal in* $M\times(N\times\mathbb{D})$, i.e., to have support contained in $M\times K$, for some compact subset K of $N\times\mathbb{D}$. This makes both the projection $(\pi_D)_*(\mathcal{R})$ well-defined (since now the projection π_D is proper on the support of $\mathcal{R}$), and horizontal there (since the projection of the support of R on N is relatively compact). In order to do this, we exploit some results in the theory of the $d,\overline{\partial}$, and dd^c equations. Observe that, in order to get all these controls, it is crucial to work with the extra flexibility given by (positive closed) currents, and not only with analytic subsets.

Once R is well-defined, we still need to make sure that the growth of the mass of $(f^n)_*(R)$ dominates the growth of the mass of $(f^n)_*(S)$. In order to get this, we need to pay extra attention, and get further estimates, during the construction of $\mathcal{R}$ and R. More precisely, we make sure that the family of deformations S_θ, and the family of the *slices* $\mathcal{R}_\theta$ of $\mathcal{R}$ with $D \times \{\theta\}$ are sufficiently continuous in a suitable sense. Studying the growth of the mass of $(f^n)_*(R)$ in D amounts to study the mass growth of $(F^n)_*(\mathcal{R})$, where $F := (f, \mathrm{id})$ on $D \times \mathbb{D}$. We prove that the sequence of functions φ_n on $\mathbb{D}$, where $\varphi_n(\theta)$ is the mass of $(f^n)_*(\mathcal{R}_\theta)$ (suitably normalized), is bounded with respect to a suitable norm (the DSH norm). A now-classical theorem by Skoda then implies that a large growth of this sequence at $\theta = 0$ must imply a large growth of this sequence for θ sufficiently close to 0. Going back to currents, this implies a bound (from below) on the growth of the mass of $(f^n)_*(\mathcal{R}_\theta)$, hence of $(F^n)_*(\mathcal{R})$, and hence of $(f^n)_*(R)$. The assertion then follows.

3.3.2 Entropy and dynamical degrees for polynomial like-maps

In this section I briefly come back to the work [BR22] that was described in the Section 1.4. As we mentioned at the end of that section, the main results of that paper still hold in the more general setting of families of polynomial-like maps of large topological degree. With the terminology of the previous section, this means that $d_t > d_{k-1}^+$. By Theorem 3.3.3, this implies that $d_t > d_s^+$ for all $0 \leq s \leq k-1$.

As mentioned in Section 1.4, Theorem 1.4.3 applies also to families of polynomial-like maps. However, the proof of Corollary 1.4.4 relies on a result by de Thélin and Dupont, stating that every measure whose entropy is strictly larger than $(k-1)\log d$ has strictly positive Lyapunov exponents. We proved in [BR22] the following generalization of such result to the setting of polynomial like maps.

Theorem 3.3.4 ([Bianchi-Rakhimov [BR22]). *] Let $f\colon U \to V$ be a polynomial-like map of large topological degree. Let ν be an ergodic f-invariant probability measure satisfying $h_\nu(f) > \log d_{k-1}^+(f)$. Then all the Lyapunov exponents of ν are larger than or equal to $(h_\nu(f) - \log d_{k-1}^+)/(2m) > 0$, where m is the multiplicity of the smallest Lyapunov exponent of ν.*

In the case of endomorphisms of $\mathbb{P}^k$, this result is due to de Thélin [deT08] and Dupont [Dup12]. When ν is the measure of maximal entropy, it is a result of Dinh-Sibony [DS03], see Briend-Duval [BD99] for the case of endomorphisms of $\mathbb{P}^k$. Although the proof of Theorem 3.3.4 follows the strategy of the proof of de Thélin and Dupont, because of the lack of a Hodge theory in this setting, we needed to replace some cohomological arguments when working with polynomial-like maps. We also could not exploit the linearity of the sequence $\{d_p^*\}_{1\leq p\leq k}$ of the dynamical degrees.

Theorem 3.3.4 relies on Theorem 3.3.3 in the following sense. A priori, setting $d^*(f) := \max_{0\leq s\leq k-1} d_s^+(f)$, our proof gives that if a measure ν satisfies $h_\nu(f) > \log d^*(f)$, then all the Lyapunov exponents of ν are larger than or equal to $(h_\nu(f) - \log d^*(f)/(2m) > 0$. Theorem 3.3.4 in the form above then follows from the equality $d^*(f) = d_{k-1}^+(f)$ given by Theorem 3.3.3.

Once Theorem 3.3.4 is established, thanks to it we can get the following version of Corollary 1.4.4 for general families of polynomial-like maps of large topological degree.

Corollary 3.3.5. *Let M be a connected and simply connected complex manifold and let $(f_\lambda)_{\lambda\in M}$ be a stable family of polynomial-like maps of large topological degree. Fix $\lambda_0 \in M$ and $h \in \mathbb{R}$ such that*

$\log d_{k-1}^*(f_{\lambda_0}) < h < \log d_t$ and let $M_{\lambda_0,h}$ be a simply connected open neighbourhood of λ_0 such that $\log d_{k-1}^*(f_\lambda) < h$ for any $\lambda \in M_{\lambda_0,h}$. Then

1. *for every ergodic f_{λ_0}-invariant probability measure ν_0 on J_{λ_0} such that $h_{\nu_0}(f_{\lambda_0}) > h$, the properties **(S4')** and **(S5')** hold on $M_{\lambda_0,h}$;*
2. *there exists a dynamical lamination $\mathcal{L}$ satisfying $\nu(\{\gamma(\lambda) : \gamma \in \mathcal{L}\}) = 1$ for every $\lambda \in M_{\lambda_0,h}$ and every f_λ-invariant measure ν such that $h_\nu(f_\lambda) > h$.*

3.4 A Mañé-Manning formula for expanding measures for endomorphisms of $\mathbb{P}^k$

In the paper [BH23], written with Yan Mary He, we proved a version of the Mañé-Manning formula in several complex variables. The content of this section is dynamical, but does not rely on pluripotential techniques.

Let $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ be a rational map of degree $d \geq 2$ and ν an ergodic f-invariant probability measure whose Lyapunov exponent is strictly positive. Such a measure is necessarily supported on the Julia set $J(f)$ of f. There is a well-known relation between the Hausdorff dimension $\mathrm{HD}(\nu)$, the measure-theoretic entropy $h_\nu(f)$, and the Lyapunov exponent $\chi_\nu(f)$ of ν; namely, we have

$$\mathrm{HD}(\nu) = \frac{h_\nu(f)}{\chi_\nu(f)}. \tag{3.8}$$

This formula is usually referred to as the Mañé-Manning formula; see [Man84; Mañ88]. Hofbauer and Raith [HR92] proved a version of (3.8) for piecewise monotone maps on the unit interval with bounded variation; see also [Led81]. The fact that (3.8) holds in one-dimensional complex dynamics crucially relies on distortion estimates for univalent holomorphic maps coming from Koebe's theorem; see Section 3.4.2.

For smooth dynamical systems in higher dimensions, related formulas are known to hold in a number of settings. If $f : M \to M$ is a diffeomorphism of a compact manifold M and ν is an ergodic probability measure on M which is absolutely continuous with respect to the Lebesgue measure, Pesin [Pes77] proved that

$$h_\nu(f) = \chi_\nu^+(f)$$

where $\chi_\nu^+(f)$ is the sum of the non-negative Lyapunov exponents of f counted with multiplicity; see also [Mañ81]. When M is a surface, Young [You82] proved that

$$\mathrm{HD}(\nu) = \frac{h_\nu(f)}{\chi_+} + \frac{h_\nu(f)}{|\chi_-|}$$

when ν is ergodic and $\chi_- < 0 < \chi_+$ are its Lyapunov exponents. This formula has been generalized to the case of diffeomorphisms in any dimension; see [LY85] and [BPS99]. Such systems display attracting and repelling directions, and one decomposes the problem into two problems, one for f (along unstable manifolds) and one for f^{-1} (along stable manifolds). The Mañé-Manning formula (3.8) can be seen as a version of Young's result in (complex) dimension 1 where the system is not invertible. In [BH23], we addressed the validity of (3.8) in several complex variables, and more specifically for expanding measures for (non-invertible) holomorphic endomorphisms of projective spaces in any dimension.

Let $k \geq 1$ be an integer and denote $\mathbb{P}^k := \mathbb{P}^k(\mathbb{C})$. If $f : \mathbb{P}^k \to \mathbb{P}^k$ is a holomorphic endomorphism of algebraic degree $d \geq 2$, it is not hard to find examples where (3.8), with χ_ν replaced by the sum of the Lyapunov exponents of ν (the natural generalization of the expansion rate along generic orbits), does not hold. For instance, one can consider product self-maps of $\mathbb{C}^2$ of the form $(z, w) \mapsto (z^2 + a_1, w^2 + a_2)$, where $a_i \in \mathbb{C}$ are such that the measures of maximal entropy of each component have different Hausdorff dimensions.

In [BD03], Binder-DeMarco proposed a conjectural formula for the Hausdorff dimension of the measure of maximal entropy μ of an endomorphism of $\mathbb{P}^k$ as follows:

$$\mathrm{HD}(\mu) = \frac{\log d}{\chi_1} + \cdots + \frac{\log d}{\chi_k}.$$

This conjecture has been partially settled [BD03; DD04; Dup11], and also versions of it have been proposed (and partially proved) for more general invariant measures [DD04; Dup11; Dup12; dV15; DR20]. In [BH23], we introduced a natural dimension $\mathrm{VD}(\nu)$ for ergodic f-invariant measures ν with strictly positive Lyapunov exponents and show that this dimension satisfies a natural generalization of (3.8), where χ_ν is replaced by (two times) the sum of the Lyapunov exponents.

3.4.1 Results

Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic endomorphism of algebraic degree $d \geq 2$. The *Julia set* $J(f)$ of f is the support of the unique measure of maximal entropy of f [Lyu83a; BD01; DS10a]. Let $\mathcal{M}^+(f)$ (resp. $\mathcal{M}^+_J(f)$) be the set of ergodic invariant probability measures on $\mathbb{P}^k$ (resp. on $J(f)$) with strictly positive Lyapunov exponents. The set $\mathcal{M}^+_J(f)$ contains the set $\mathcal{M}^+_e(f)$ of all ergodic probability measures whose measure-theoretic entropy is strictly larger than $(k-1)\log d$ [deT08; Dup12], which are the natural generalization of the ergodic measures with strictly positive entropy in dimension 1. We saw in Section 2.2 how to construct large classes of examples of measures in $\mathcal{M}^+_e(f)$, see also [Dup12; UZ13; SUZ14].

We introduced a *v*olume dimension for measures $\nu \in \mathcal{M}^+(f)$; see Section 3.4.2 for an overview of the construction. The volume dimension is dynamical in nature and generalizes the notion of Hausdorff dimension in dimension 1 to higher dimensions to incorporate the absence of an analogue of Koebe's theorem and the non-conformality of holomorphic endomorphisms.

For $\nu \in \mathcal{M}^+(f)$, we denote by $\mathrm{VD}(\nu)$ the volume dimension, $h_\nu(f)$ the measure-theoretic entropy, and $L_\nu(f)$ the *s*um of the Lyapunov exponents of ν. The main result of [BH23] relates these three quantities and generalizes the Mañé-Manning formula to any $k \geq 1$.

Theorem 3.4.1 (Bianchi-He [BH23]). *Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic endomorphism of algebraic degree $d \geq 2$. For every $\nu \in \mathcal{M}^+(f)$ we have*

$$\mathrm{VD}(\nu) = \frac{h_\nu(f)}{2L_\nu(f)}.$$

When $k = 1$, Theorem 3.4.1 reduces to the Mañé-Manning formula (3.8), as in this case we can prove that $2\,\mathrm{VD}(\nu) = \mathrm{HD}(\nu)$. The factor $2 = 2k/k$ is due to the fact that we weight open sets of covers by their volume instead of their diameter and we have k Lyapunov exponents, counting multiplicities.

As an application of Theorem 3.4.1, we can study a number of natural dimensions and quantities associated to an endomorphism f. In dimension 1, these quantities are already defined and well studied; see for example [DU91b; DU91c; PU10; McM00a]. We first define a *d*ynamical dimension $\mathrm{DD}_J^+(f)$ of f as

$$\mathrm{DD}_J^+(f) := \sup \left\{ \mathrm{VD}(\nu) \colon \nu \in \mathcal{M}_J^+(f) \right\}.$$

For $k = 1$, recall that the *pressure function* is defined as

$$P(t) := \sup_{\nu} \left\{ h_\nu(f) - t\chi_\nu(f) \right\},$$

where $t \in \mathbb{R}$ and the supremum is taken over the set of invariant probability measures on $J(f)$. In fact, the supremum can be taken over $\nu \in \mathcal{M}_J^+(f) = \mathcal{M}^+(f)$. This can be seen by combining Ruelle's inequality [Rue78a] with a theorem of Przytycki [Prz93] stating that all invariant measures supported on the Julia set of a rational map have non-negative Lyapunov exponent.

For any $k \geq 1$, we define in a similar way a pressure function $P_J^+(t)$ as

$$P_J^+(t) := \sup \left\{ h_\nu(f) - tL_\nu(f) \colon \nu \in \mathcal{M}_J^+(f) \right\}.$$

By the above, we have $P_J^+(t) = P(t)$ when $k = 1$. We remark that, for any $k \geq 2$, there may exist ergodic probability measures ν on $J(f)$ with $L_\nu(f) < 0$. However, as in the case of $k = 1$, the pressure function $P_J^+(t)$ is still non-increasing and convex for all $k \geq 1$. We define

$$p_J^+(f) := \inf \left\{ t > 0 \colon P_J^+(t) \leq 0 \right\}.$$

As a consequence of Theorem 3.4.1, we have the following result which generalizes a theorem due to Denker-Urbański [DU91b; DU91c] in the case of rational maps to any dimension.

Theorem 3.4.2 (Bianchi-He [BH23]). *Let $f \colon \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic endomorphism of algebraic degree $d \geq 2$. Then we have*

$$2\,\mathrm{DD}_J^+(f) = p_J^+(f).$$

Finally, in the spirit of the celebrated Bowen-Ruelle formula for hyperbolic maps [Bow79; Rue82], we give an interpretation of $p_J^+(f)$ when f is hyperbolic (i.e., uniformly expanding on $J(f)$) in terms of (volume-)conformal measures. Given $t \geq 0$, we say that a probability measure ν on $J(f)$ is *t-volume-conformal on* $J(f)$ if, for every Borel subset $A \subset J(f)$ on which f is invertible, we have

$$\nu(f(A)) = \int_A |\operatorname{Jac} f|^t d\nu$$

and define

$$\delta_J(f) := \inf \left\{ t \geq 0 \colon \text{ there exists a } t\text{-volume-conformal measure on } J(f) \right\}.$$

For $k = 1$, the definitions of t-volume-conformal measures and $\delta_J(f)$ reduce to those of conformal measures and conformal dimension for rational maps; see [DU91b; DU91c; McM00a; PU10]. In this case, owing to Bowen [Bow79], one sees that

$$\delta_J(f) = p_J^+(f) = \mathrm{HD}(J(f))$$

for every hyperbolic rational map f on $\mathbb{P}^1(\mathbb{C})$, and that there exists a unique ergodic measure ν on $J(f)$ such that $\mathrm{HD}(\nu) = \mathrm{HD}(J(f))$. We have here the following result in any dimension, which further motivates the definition of the volume dimension as a natural generalization of the Hausdorff dimension for all $k \geq 1$. Observe that, if f is hyperbolic, every invariant probability measure ν on $J(f)$ belongs to $\mathcal{M}^+(f)$.

Theorem 3.4.3 (Bianchi-He [BH23]). *Let $f\colon \mathbb{P}^k \to \mathbb{P}^k$ be a hyperbolic holomorphic endomorphism of algebraic degree $d \geq 2$. Then we have*

$$\delta_J(f) = p_J^+(f) = 2\,\mathrm{VD}(J(f))$$

and there exists a unique ergodic measure ν on $J(f)$ such that $\mathrm{VD}(\nu) = \mathrm{VD}(J(f))$.

As in dimension 1, the measure ν in this statement can be characterized as a unique equilibrium state with respect to the weight $-\delta_J(f)|\operatorname{Jac} f|$.

3.4.2 Volume dimensions and strategy of the proofs

Let us first recall the idea of the proof of the Mañé-Manning formula (3.8) in dimension 1. It essentially consists of two steps.

1. The first step consists of defining a *local dimension* at a point x by setting

$$\delta_x := \lim_{r \to 0} \frac{\log \nu(B(x,r))}{\log r}$$

   (whenever the limit exists), where $B(x,r)$ denotes the balls of radius r centred and x, and proving that the limit is well-defined and equal to the ratio $h_\nu(f)/\chi_\nu(f)$ for ν-almost every x. In particular, ν is *exact-dimensional*.

2. The second step is to prove that the Hausdorff dimension of ν must be equal to the common value of the local dimensions found in the first step [You82].

Let us describe how the one-dimensional setting plays a crucial role in Step (1). By [BK83] and [Mañ81], for ν-almost every x we have

$$h_\nu(f) = \lim_{\kappa \to 0} \lim_{n \to \infty} \frac{-\log \nu(B_n(x,\kappa))}{n},$$

where $B_n(x,\kappa)$ is the *Bowen ball of radius κ and depth n*. This is defined as

$$B_n(x,\kappa) := \left\{ y\colon |f^j(y) - f^j(x)| < \kappa, 0 \leq j \leq n \right\}.$$

The crucial observation is that, for large n, the Bowen ball $B_n(x,\kappa)$ is comparable (up to precisely quantifiable errors) to the ball $B(x, \kappa\, e^{-n\chi_\nu(f)})$ of the same center and radius $\kappa\, e^{-n\chi_\nu(f)}$. Fixing a κ_0 for simplicity, and setting $n(r) \sim |\log r|/\chi_\nu(f)$, it then follows that

$$\lim_{r \to 0} \frac{\log \nu(B(x,r))}{\log r} = \lim_{r \to 0} \frac{-\log \nu(B_{n(r)}(x,\kappa_0))}{n(r)} \frac{n(r)}{-\log r} = \frac{h_\nu(f)}{\chi_\nu},$$

which in particular shows that δ_x is well-defined. The precise relation between geometric balls and Bowen balls is a consequence of Koebe's theorem and related distortion estimates, which imply that images of balls by holomorphic maps (and in particular by their inverse branches) are still comparable to balls. As a consequence, in complex dimension 1, there is a natural interplay between the Hausdorff dimension and the dynamics of a rational map. Observe in particular that one may define the Hausdorff dimension of ν by using covers consisting of Bowen balls, indexed over their depth n, and sending n to infinity; see also [CPZ19; Pes88].

All the above is in sharp contrast with the higher-dimensional situation, where one cannot expect to obtain a general relation between geometric balls and Bowen balls. As a result, the notion of Hausdorff dimension is not intrinsic anymore with respect to the dynamics. If $f : M \to M$ is a C^2-diffeomorphism of a smooth compact Riemannian manifold and ν is an ergodic f-invariant probability measure with non-zero Lyapunov exponents, Pesin [Pes88] introduced a dimension $\dim_f(\nu)$ relative to the map f and proved that

$$\dim_f(\nu) = \frac{h_\nu(f)}{L_\nu^+(f)}, \tag{3.9}$$

where $L_\nu^+(f)$ is the sum of positive Lyapunov exponents of ν; see [Pes88, Theorem 6.4]. Such dimension is constructed using an analogue of the Bowen balls inside the unstable manifolds of f. Although it is likely that such formula could be generalized to smooth, not necessarily invertible, endomorphisms of M (see for instance [QXZ09] for adapted techniques) and expanding measures (in which case the whole M is an unstable manifold), such Bowen balls carry only a dynamical information but, a priori, no geometric one. We aimed in [BH23] at proving a version of (3.9) for endomorphisms of $\mathbb{P}^k$ in any dimension $k \geq 1$, where furthermore our dimension has a more geometric interpretation. Our construction is independent of Pesin's [Pes88] and deeply exploits the complex setting.

As we mentioned above, in several complex variables, due to the lack of conformality of holomophic maps, preimages of balls can be arbitrarily distorted, and far from being balls. In the best possible scenario (e.g., for hyperbolic product maps), the preimages of balls are approximately ellipses whose axes reflect the contraction rate of the inverse branches in the different directions. On the other hand, when $\nu \in \mathcal{M}^+(f)$, the linearization result by Berteloot-Dupont-Molino [BDM08; BD19], already encountered several times in this manuscript, states that the best possible scenario described above is actually true, in an infinitesimal sense, for preimages of balls along generic orbits of ν. More precisely, there exists an increasing (as $\varepsilon \to 0$) measurable exhaustion $\{Z_\nu^\star(\varepsilon)\}_\varepsilon$ of a full-measure subset $Z_\nu^\star$ of the space of orbits for f such that the preimages of sufficiently small balls along orbits in $Z_\nu^\star(\varepsilon)$ are approximately ellipses, and the contraction rate for their volume is essentially given (up to further controllable error terms) by $e^{-nL_\nu(f)+nO(\varepsilon)}$. This is a consequence of very refined estimates on the convexity defect of such preimages.

Fix $\nu \in \mathcal{M}^+(f)$. Denote by $\pi\colon Z_\nu^\star \to \mathbb{P}^k$ the projection associating to any orbit $\widehat{z} = \{z_n\}_{n\in\mathbb{Z}}$ its element z_0. For $x \in \pi(Z_\nu^\star(\varepsilon))$, $\kappa > 0$, and $N \in \mathbb{N}$, we consider (when well-defined) the neighbourhood $U = U(N, x, \kappa, \varepsilon)$ of x satisfying

$$f^N(U) = B(f^N(x), \kappa\, e^{-NM\varepsilon})$$

where e^M is a bound for the expansion of f and we require that $f^N|_U$ is injective. It follows from the above result by Berteloot-Dupont-Molino that there exist some $r(\varepsilon)$ and $n(\varepsilon)$ such that, for all $x \in \pi(Z^\star_\nu(\varepsilon))$, $0 < \kappa < r(\varepsilon)$, and $N \geq n(\varepsilon)$ the sets $U(N, x, \kappa, \varepsilon)$ are indeed well-defined and approximately ellipses, of controlled geometry. We see these sets $U(N, x, \kappa, \varepsilon)$ as a suitable version of the Bowen balls $B_n(x, \kappa)$ in any dimension. Let us set

$$\delta_x(\varepsilon, \kappa, N) := \frac{\log \nu(U(N, x, \kappa, \varepsilon))}{\log \mathrm{Vol}(U(N, x, \kappa, \varepsilon))},$$

where Vol denotes the volume with respect to the Fubini-Study metric. As a first step (which corresponds to Step (1) above) towards proving Theorem 3.4.1, we show that every $\nu \in \mathcal{M}^+(f)$ is *exact (volume-)dimensional*; namely, for ν-almost every x, we have

$$\limsup_{\varepsilon \to 0} \limsup_{\kappa \to 0} \limsup_{N \to \infty} \delta_x(\varepsilon, \kappa, N) = \liminf_{\varepsilon \to 0} \liminf_{\kappa \to 0} \liminf_{N \to \infty} \delta_x(\varepsilon, \kappa, N) = \frac{h_\nu(f)}{2L_\nu(f)}.$$

We adapt here the approach of Mañé [Mañ88] in higher dimensions, thanks to the distortion estimates developed as a consequence of [BDM08; BD19].

Once the local dimension of every $\nu \in \mathcal{M}^+(f)$ is well-defined as above, we give a global interpretation of this quantity by defining a *volume dimension* for these measures. The idea is to use the sets $U(N, x, \kappa, \varepsilon)$ to cover the "slice" $X \cap \pi(Z^\star_\nu(\varepsilon))$ of every set $X \subseteq Z^\star_\nu$. More precisely, for every $X \subseteq \pi(Z^\star_\nu)$ and $\varepsilon > 0$, setting $X^\varepsilon := X \cap \pi(Z^\star_\nu(\varepsilon))$, we define the quantity $\mathrm{VD}^\varepsilon_\nu(X^\varepsilon)$ as

$$\mathrm{VD}^\varepsilon_\nu(X^\varepsilon) := \sup \{\alpha : \Lambda^\varepsilon_\alpha(X^\varepsilon) = \infty\} = \inf \{\alpha : \Lambda^\varepsilon_\alpha(X^\varepsilon) = 0\},$$

where

$$\Lambda^\varepsilon_\alpha(X^\varepsilon) := \lim_{\kappa \to 0} \lim_{N^\star \to \infty} \inf_{\{U_i\}} \sum_{i \geq 1} \mathrm{Vol}(U_i)^\alpha.$$

Here the infimum is taken over the covers consisting of sets U_i of the form $U_i = U(N_i, x, \kappa, \varepsilon)$, for some $x \in \pi(Z_\nu(\varepsilon))$ and $N_i \geq N^\star$. The *volume dimensions* of X and ν are then respectively defined as

$$\mathrm{VD}_\nu(X) := \limsup_{\varepsilon \to 0} \mathrm{VD}^\varepsilon_\nu(X^\varepsilon) \quad \text{and} \quad \mathrm{VD}(\nu) := \inf \{\mathrm{VD}_\nu(X) \colon X \subseteq \pi(Z_\nu), \nu(X) = 1\},$$

and the $\limsup_{\varepsilon \to 0}$ is actually a limit. We prove a version of Young's criterion [You82, Proposition 2.1], relating the local volume dimensions δ_x with the volume dimensions $\mathrm{VD}_\nu(X)$ and $\mathrm{VD}(\nu)$. This corresponds to Step (2) above and, together with the exact volume-dimensionality of ν proved in the first step, completes the proof of Theorem 3.4.1.

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