BIFURCATIONS IN THE ELEMENTARY DESBOVES FAMILY

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We give an example of a family of endomorphisms of $\mathbb{P}^2(\mathbb{C})$ whose Julia set depends continuously on the parameter and whose bifurcation locus has non empty interior.

1. Introduction and result

It is a classical result due to Lyubich [Lyu83] and Mané-Sad-Sullivan [MSS83] that, for a family of rational maps, the continuity of the Julia set (in the Hausdorff topology) with the parameter is equivalent to the holomorphic motion of the repelling periodic points and of the Julia set itself. This allows one to decompose the parameter space of the family into the *stability locus* (where such holomorphic motions exist) and its complement, the *bifurcation locus*. De Marco [DeM01] considered the canonical current $dd^c L$ on the parameter space (where L is the Lyapunov function, given by the integration of the logarithm of the Jacobian with respect to the equilibrium measure, i.e. the unique measure of maximal entropy) and proved that it is exactly supported on the bifurcation locus. Thus, stability is equivalent to the harmonicity of the Lyapunov function. An important feature of this picture is that the stability locus is dense in the parameter space.

This description was recently extended to higher dimension by Berteloot, Dupont, and the first author in [BBD15]. It is proved there that the complex Laplacian of the Lyapunov function still detects the bifurcation of the repelling cycles and that, up to a zero measure subset with respect to the equilibrium measure, the Julia set moves holomorphically precisely out of the support of $dd^c L$. This lead to a coherent definition of stability and bifurcation in general dimension. Two main questions were left open in that work: the equivalence between the above stability conditions and the Hausdorff continuity of the Julia set, and the possible density of stability. The goal of this paper is to present an example answering by the negative to both of the above questions. More precisely, we prove the following.

Main Theorem. *The family of endomorphisms of $\mathbb{P}^2$ given by*

$$f_\lambda = \left[-x(x^3 + 2z^3) : y(z^3 - x^3 + \lambda(x^3 + y^3 + z^3)) : z(2x^3 + z^3) \right]$$

with $\lambda \in \mathbb{C}^*$ satisfies the following properties:

(1) *the Julia set of f_λ depends continuously on λ, for the Hausdorff topology;*
(2) *the bifurcation locus coincides with $\mathbb{C}^*$.*

The family above (referred to as the *elementary Desboves family*) was studied by Bonifant-Dabija [BD02] and Bonifant-Dabija-Milnor [BDM07], who revealed the dynamical richness of these maps. One of their main properties is that the small Julia set $J_2(\lambda)$ of f_λ (the support of the equilibrium measure) and its large Julia set $J_1(\lambda)$ (the complement of the Fatou set, which coincides with the support of the

Green current) coincide (see Theorem 2.2). We can thus refer to any of them as the *Julia set*. It would be interesting to know whether the bifurcations in our example are *robust*. Let us mention that Romain Dujardin has announced the existence of several mechanisms leading to such phenomena.

The paper is organized as follows. In Section 2 we describe in detail the Desboves maps and collect (in Theorem 2.2) the properties that we shall need in the construction, deducing in particular the continuity of the Julia set and the presence of bifurcations. This already proves that continuous and holomorphic dependence of the Julia set on the parameter are not equivalent in general dimension. Then, in Section 3 we prove that the bifurcation locus coincides with $\mathbb{C}^*$, by proving that *Misiurewicz parameters* (see Definition 3.1) are dense in the parameter space. The appendix is devoted to the proof of Theorem 2.2. Since in our particular setting the proof can be simplified with respect to the work by Bonifant and Dabija, we include it for sake of completeness.

2. ELEMENTARY DESBOVES MAPS

In this section we define and give the main properties of the *elementary Desboves maps*. See [BD02] and [BDM07] for a more detailed description. We start considering the classical Desboves map given by

$$f_{\mathcal{D}} := \left[x(y^3 - z^3) : y(z^3 - x^3) : z(x^3 - y^3) \right].$$

This is a rational map on $\mathbb{P}^2$, with non-empty indeterminacy locus. The important feature we are interested in is the following: the map $f_{\mathcal{D}}$ leaves invariant a pencil of elliptic curves and in particular the *Fermat curve* $\mathcal{C} := \{x^3 + y^3 + z^3 = 0\}$. We will now consider a well-chosen perturbation of $f_{\mathcal{D}}$, still leaving the curve $\mathcal{C}$ invariant. A natural way to do this is to consider the following *Desboves family* (studied in [BDM07]), parametrized by $(a, b, c) \in \mathbb{C}^3$. We set $\Phi(x, y, z) := x^3 + y^3 + z^3$ for convenience. The Desboves family is thus given by

$$f_{a,b,c} := \left[x(y^3 - z^3 + a\Phi) : y(z^3 - x^3 + b\Phi) : z(x^3 - y^3 + c\Phi) \right].$$

By construction, every such map leaves invariant the Fermat curve $\mathcal{C}$. Moreover, $f_{a,b,c}$ is actually a well defined endomorphisms of $\mathbb{P}^2$ (i.e., its indeterminacy locus is empty) with (a, b, c) avoiding the union of the seven hyperplanes $abc(a + b + c)(a + 1 - b)(b + 1 - c)(c + 1 - a) = 0$.

We will now specialize the parameters in order to recover a last property that we shall need in the sequel: we want every element of our perturbation to preserve a pencil of lines (i.e., to be *elementary*, see [Dab00]). This can be done by specializing the coefficients $a = -1$ and $c = 1$. We thus get the following definition (we rename the parameter b to λ).

Definition 2.1. *Let $\lambda \in \mathbb{C}$. An* elementary Desboves map *is a map of the form*

$$f_{\lambda} = \left[-x(x^3 + 2z^3) : y(z^3 - x^3 + \lambda(x^3 + y^3 + z^3)) : z(2x^3 + z^3) \right].$$

The elementary Desboves family *is the holomorphic family* $f : \mathbb{C}^* \times \mathbb{P}^2 \to \mathbb{C}^* \times \mathbb{P}^2$ *given by* $f(\lambda, z) := (\lambda, f_{\lambda}(z))$.

When $\lambda \neq 0$, the map f_{λ} is an endomorphism of $\mathbb{P}^2$. With $\lambda = 0$, the map f_0 has an indeterminacy locus equal to the point $\rho_0 := [0 : 1 : 0]$. Moreover, we have $f_{\lambda}^{-1}(\{\rho_0\}) = (\{\rho_0\})$ for every $\lambda \in \mathbb{C}^*$. So, f can also be seen as a family of endomorphisms of $\mathbb{P}^2 \setminus \{\rho_0\}$, parametrized by $\lambda \in \mathbb{C}$.

We denote by $\mathcal{P}$ the pencil of lines passing through the point ρ_0. For every $\lambda \in \mathbb{C}$, f_λ preserves $\mathcal{P}$ and leaves invariant the coordinate lines $X := \{x = 0\}, Y := \{y = 0\}, Z := \{z = 0\}$, and the Fermat curve $\mathcal{C}$. The line Y is transversal to $\mathcal{P}$. Hence, the action on $\mathcal{P}$ can be identified to the restriction g of f_λ to Y, which is independent from λ. To be more precise, by construction the restriction h of f_λ to $\mathcal{C}$ is independent from λ. As all endomorphisms of an elliptic curve, it lifts to a map of the form $z \mapsto \alpha z + \beta$ on $\mathbb{C}$ and a simple computation of the multiplier at a fixed point gives that $\alpha = -2$. In particular, h has degree 4 and is uniformly expanding. If $\pi \colon \mathbb{P}^2 \setminus \{\rho_0\} \to Y$ denotes the projection along $\mathcal{P}$ on Y then π semiconjugates g to h, i.e. $\pi \circ h = g \circ \pi$. Therefore, the map g is a degree 4 Lattès map. An explicit computation gives that, in the coordinate $w := \frac{x}{z}$, the map g takes the form

$$(1) \qquad\qquad w \mapsto -w\frac{w^3 + 2}{2w^3 + 1}.$$

It has 3 critical points, all with double multiplicity, which are given by $[1 : 0 : \omega^j]$, where ω is a primitive third-root of 1 and $j \in \{0, 1, 2\}$. It is worth pointing out that these three points are mapped by g to the three (fixed) points of intersections between Y and $\mathcal{C}$, $[1 : 0 : -\omega^j]$. Since the intersection of $\mathcal{C}$ and Y are transverse, and the dynamics is expansive on both curves, each of these points is repelling for f_λ, $\lambda \in \mathbb{C}$.

In the next Theorem we collect the properties of the elementary Desboves maps that we shall need in the sequel.

Theorem 2.2. [BD02, Proposition 6.16] *Let $\lambda \in \mathbb{C}^*$. If $\overline{R}(\lambda)$ denotes the closure of the repelling periodic points of f_λ then*

(1) $J_1(\lambda) = J_2(\lambda) = \overline{R}(\lambda)$;
(2) *the Fatou set coincides with the basin of the superattracting fixed point $\rho_0 = [0 : 1 : 0]$.*

The first statement in the Main Theorem is then a consequence of Theorem 2.2.

Corollary 2.3. *Let $(f_\lambda)_{\lambda \in \mathbb{C}^*}$ be the elementary Desboves family. Then the Julia set depends continuously on λ for the Hausdorff topology.*

Proof. As the equilibrium measure depends continuously on λ, its support $J_2(\lambda)$ varies lower semicontinuously with the parameter for every family of endomorphisms of $\mathbb{P}^2$ (see [DS10]). It is thus enough to prove that, in the elementary Desboves family, this dependence is also upper semicontinuous. This is equivalent to prove that the set $\cup_\lambda \{\lambda\} \times J_2(\lambda)$ is closed in the product space $\mathbb{C}^* \times \mathbb{P}^2$. This immediately follows since its complement Ω is the basin of attraction of $\mathbb{C}^* \times \{\rho_0\}$ in the product space, which is open. More precisely, if $(\lambda_0, z_0) \in \Omega$ then by Theorem 2.2, z_0 is in basin of ρ_0 for f_{λ_0}. Therefore, there exists an open subset U of $\mathbb{P}^2$ included in this basin such that $\overline{f_{\lambda_0}(U)} \subset U$ and $z_0 \in U$. Hence, if $V \subset \mathbb{C}^*$ is a small neighborhood of λ_0 then $\overline{f_\lambda(U)} \subset U$ for all $\lambda \in V$, i.e. $V \times U \subset \Omega$. $\qquad \square$

Remark 2.4. *In our setting, the point ρ_0 has to be thought of as the point at infinity, and the Julia set shares a lot of similarities with the* filled Julia set *of a polynomial map. In particular, being the Julia set $J_2(\lambda)$ the complement of the "basin at infinity" of f_λ, the above proof is essentially the same of the classical fact that the filled Julia set depends upper semicontinuously on the parameter.*

We study now the critical set C_λ of the maps f_λ. This is a curve in $\mathbb{P}^2$ of degree 9. More precisely, in homogeneous coordinates the Jacobian of f_λ is equal to

$$\text{Jac}(f_\lambda) = 8(x^3 - z^3)^2(-x^3 + z^3 + \lambda(4y^3 + x^3 + z^3)).$$

The term $(x^3 - z^3)$ gives the three lines L_j passing thought ρ_0 and $[1:0:\omega^j]$, with $j \in \{0,1,2\}$, respectively, each with multiplicity 2. This corresponds to the critical set of the Lattès map g. The other terms gives a (possibly non irreducible) curve of degree 3, given in homogeneous coordinates by $C'_\lambda := \{-x^3 + z^3 + \lambda(4y^3 + x^3 + z^3) = 0\}$. We thus have $C_\lambda = C'_\lambda \cup (L_0 \cup L_1 \cup L_2)$. Since C'_λ has degree 3, it intersects Y in three points. These are given by the points $[x:0:z]$ with (x,z) solution of $(1+\lambda)z^3 + (\lambda - 1)x^3 = 0$. For $\lambda = 1$, we get the point $[1:0:0] = Z \cap Y$ (as a triple solution). Analogously, for $\lambda = -1$ we get the point $[0:0:1] = X \cap Y$. Otherwise, we have the three points $[w:0:1]$, where $w^3 = \frac{1+\lambda}{1-\lambda}$. So, every point $w \in Y$ except the six points $[1:0:\pm\omega^j]$ (i.e., the critical points for the Lattès map and their images, which are always repelling for f_λ, see above) there exists a $\lambda \neq 0$ such that $w \in C'_\lambda \cap Y$. Moreover, the description above implies the following observation.

Lemma 2.5. *Let V be an open subset of $\mathbb{C}$. Then the set $W := \cup_{\lambda \in V} C'_\lambda \cap Y$ is open in Y.*

For $\lambda = 0$ the critical set is very special: it is reduced to the union of the three lines L_i, each with multiplicity 3. Thus, the intersection of C'_0 with Y is again given by the three points $[1:0:\omega^j]$.

Finally, we study the two intersection points $x_0 := [0:0:1] = X \cap Y$ and $z_0 := [1:0:0] = Z \cap Y$. We focus on x_0, since the arguments are the same for z_0. Since both lines X and Y are fixed, the point x_0 is a fixed point. Since the restriction of f to Y is a Lattès map, the differential of f at x_0 has one repelling eigenvalue. To compute the other one, notice that the restriction of f_λ to X has the form (in the coordinate $t := \frac{y}{z}$)

$$(2) \qquad\qquad\qquad\qquad t \mapsto (1+\lambda)t + \lambda t^4.$$

In particular, the point ρ_0 corresponds to the superattracting fixed point at infinity. The differential at 0 (corresponding to x_0) being $(1+\lambda)$, we get that x_0 is a repelling fixed point for $|1 + \lambda| > 1$. The analogous computation on z_0 gives that z_0 is repelling if and only if $|1 - \lambda| > 1$. In particular,

$$(3) \qquad\qquad \text{for every } \lambda \neq 0 \text{ one among } x_0 \text{ and } z_0 \text{ is repelling.}$$

Remark 2.6. *Observe that the transverse multiplier $1 + \lambda$ at x_0 can almost be chosen arbitrarily. If $1 + \lambda$ satisfies the Brjuno condition, the system f_λ admits a Siegel disc (see [BBD15]) passing through x_0. By Theorem 2.2, this disc is contained in the support of the equilibrium measure. This answers to a question stated in [BBD15]. A different example of this phenomenon was given by the first author in [Bia16]. For other values of λ, it can give a parabolic petal or an attracting disc in $J_2(\lambda)$. It is even possible to have a set of positive Lebesgue measure covered by such attracting discs, see [BDM07, Theorem 6.3] and [Taf10]. In particular, the small Julia set has positive measure for these parameters.*

3. MISIUREWICZ PARAMETERS AND BIFURCATIONS

In this section we prove that the bifurcation locus of the elementary Desboves family coincides with the parameter space $\mathbb{C}^*$, completing the proof of the Main Theorem. By [BBD15], *Misiurewicz parameters* (as defined below) are dense in the bifurcation locus. We will thus prove that these parameter are dense in $\mathbb{C}^*$. Recall that C_f and C_λ denote the critical set of the family f and of f_λ, respectively. We shall set $C_f^n := f^n(C_f)$ and $C_\lambda^n := f_\lambda^n(C_\lambda)$.

Definition 3.1. *A parameter $\lambda_0 \in \mathbb{C}^*$ is called a* Misiurewicz parameter *if there exist a neighbourhood $N_{\lambda_0} \subset \mathbb{C}^*$ of λ_0 and a holomorphic map $\sigma \colon N_{\lambda_0} \to \mathbb{P}^2$ such that:*

(1) for every $\lambda \in N_{\lambda_0}$, $\sigma(\lambda)$ is a repelling periodic point;
(2) $\sigma(\lambda_0)$ is in the Julia set of f_{λ_0};
(3) there exists an n_0 such that $(\lambda_0, \sigma(\lambda_0)) \in f^{n_0}(C_f)$;
(4) $\sigma(N_{\lambda_0}) \nsubseteq f^{n_0}(C_f)$.

Theorem 3.2. *Let f be the elementary Desboves family. Then Misiurewicz parameters are dense in $\mathbb{C}^*$.*

Proof. Let $\lambda_0 \in \mathbb{C}^*$. If $V \subset \mathbb{C}^*$ is small enough open neighbourhood of λ_0, by (3) one of the two points $[0:0:1]$ and $[1:0:0]$ is repelling throughout V. Call N this point. By Lemma 2.5, the set $W := \cup_{\lambda \in V} C'_\lambda \cap Y$ is open in Y. Since the preimages of N by g are dense in Y, there exists a $\lambda_1 \in V$, a point $c_1 \in Y$ and a number n_1 such that:

(1) c_1 is a critical point for f_{λ_1};
(2) $f_{\lambda_1}^{n_1}(c_1) = N$.

Since N is repelling (and thus contained in the Julia set, by Theorem 2.2), to prove that λ_1 is a Misiurewicz parameter it is enough to check that N is not persistently contained in $f^{n_1}(C_f)$. This follows from Lemma 3.3 below, and the proof is completed as V can be chosen arbitrarily small. $\qquad\square$

Lemma 3.3. *The points $x_0 = [0:0:1]$ and $z_0 = [1:0:0]$ are not persistently contained in any set C_λ^n.*

Proof. Since for every $\lambda \in \mathbb{C}$ we have $f_\lambda^{-1}(\rho_0) = \{\rho_0\}$, we can see the family as a family of endomorphisms of $\mathbb{P}^2 \setminus \{\rho_0\}$, with parameter space $\mathbb{C}$. We still denote by C_λ^n and C_f^n the iterates of the critical sets.

Assume, for sake of contradition, that there exists a n_0 such that $x_0 \in C_\lambda^{n_0}$ for every $\lambda \in \mathbb{C}^*$. Since $C_f^{n_0}$ is closed, this would imply that $x_0 \in C_0^{n_0}$. But, for every $n \geq 1$, we have $C_0^n \cap Y = \{[1:0:-\omega^j] \colon j = 0,1,2\}$. This gives the desired contradiction. $\qquad\square$

Remark 3.4. *Theorem 4.6 in [BBD15] states that, in any holomorphic family f of endomorphisms of $\mathbb{P}^k$, the set of parameters λ for which $\mathbb{P}^k$ coincides with the closure of the postcritical set of f_λ is dense in any open subset of the bifurcation locus. It follows that, in the elementary Desboves family, this property holds for a dense subsets of parameters in $\mathbb{C}^*$.*

Remark 3.5. *It follows from [BBD15] that, for a dense subset of parameters $\lambda \in \mathbb{C}^*$, the map f_λ admits a Siegel disc contained in the support of the equilibrium measure.*

Appendix A. Proof of Theorem 2.2

This section consists of a simplified proof, in our setting, of Theorem 2.2. The original proof can be found in [Dab00, BD02].

Lemma A.1. *For every $\lambda \neq 0$, the exceptional $\mathcal{E}$ set of f_λ is reduced to the point ρ_0.*

Proof. Since the point ρ_0 is backward invariant, it is contained in the exceptional set. Let us prove the other inclusion. The intersection of $\mathcal{E}$ with the invariant line $X = \{x = 0\}$ must be reduced to ρ_0 (the exceptional set for the restriction, by (2)). So, if $\mathcal{E}$ were of dimension 1, it would have to be a line passing thought ρ_0. This is excluded since the exceptional set of the Lattès restriction g to Y is empty. Once we know that $\mathcal{E}$ has dimension 0, the fact that is reduced to ρ_0 again follows since the exceptional set of g is empty. $\square$

Lemma A.2. *There exists a repelling point r_0 on the curve $\mathcal{C}$ contained in the small Julia set $J_2(\lambda)$.*

Proof. Since the intersection of $\mathcal{C}$ and Y are transverse, and the dynamics is expansive of both curves, every intersection point of $\mathcal{C}$ and Y is repelling. Take as r_0 any such point (e.g., the fixed point $[1:0:-1]$). In order to prove that $r_0 \in J_2(\lambda)$ we prove that, given any sufficiently small open neighbourhood U of r_0, we have

$$(4) \qquad\qquad \mathbb{P}^2 \setminus \cup_{n\geq 0} f_\lambda^n(U) = \{\rho_0\}.$$

Indeed, take any small ball V disjoint from ρ_0 and intersecting the small Julia set $J_2(\lambda)$. By (4), preimages of V must accumulate r_0. Since $J_2(\lambda) \cap V \neq \emptyset$, this proves that $r_0 \in J_2(\lambda)$, as desired.

Let us thus prove the claim (4). Suppose that $E := \mathbb{P}^2 \setminus (\{\rho_0\} \cup \cup_n f_\lambda^n(U)) \neq \emptyset$ and take a line l_0 through ρ_0 intersecting E in a point e_0. Since the exceptional set of the Lattès restriction of f_λ to Y is empty, there exists a sequence of preimages of the line l_0 converging to the line l_r through ρ_0 and r_0. This gives a sequence of preimages e_n of e_0 that, up to a subsequence, converge to a point $\tilde{e} \in l_r$. Since $f_\lambda^{-1}(E) \subset E$ and E is closed in $\mathbb{P}^2 \setminus \{\rho_0\}$, we have $\tilde{e} \in l_r \cap E$. Moreover, $\tilde{e} \neq \rho_0$ since ρ_0 is attracting. On the other hand, the point $\tilde{e}$ must be contained in the exceptional set of the restriction of f_λ to l_r. An explicit computation yields that this exceptional set is reduced to ρ_0 and gives the desired contradiction. $\square$

We can now prove Theorem 2.2.

Proof of Theorem 2.2. The first assertion is a consequence of Lemma A.2. Indeed, since backward preimages (in $\mathcal{C}$) of any point in $\mathcal{C}$ are dense in this curve, this implies that all the curve $\mathcal{C}$ is included in the Julia set $J_2(\lambda)$. Since the exceptional set ρ_0 (see Lemma A.1) is not contained in $\mathcal{C}$, preimages of this curve equidistribute (see [DS10]) the Green current (supported on the large Julia set $J_1(\lambda)$). This gives that the preimages of $r_0 \in J_2(\lambda)$ are dense in $J_1(\lambda)$, yielding $J_1(\lambda) \subseteq J_2(\lambda)$. The assertion follows since the inclusions $J_2(\lambda) \subseteq \overline{R}(\lambda) \subseteq J_1(\lambda)$ always hold (the first by Briend-Duval equidistribution result [BD99]).

The equality between the Fatou set and the basin Ω of ρ_0 simply comes from the fact that Ω is totally invariant and f_λ is locally uniformly expanding in the transverse direction to $\mathcal{P}$ in $\mathbb{P}^2 \setminus \{\rho_0\}$. More precisely, for every small ball B centered at a point z in $\mathbb{P}^2 \setminus \Omega$, for n large enough we have $\pi \circ f_\lambda^n(B) = Y$ and $f_\lambda^n(z) \in \mathbb{P}^2 \setminus \Omega$. Thus, B cannot be contained in the Fatou set. $\square$

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