A RUELLE-MCMULLEN FORMULA FOR THE VOLUME DIMENSION OF SKEW PRODUCTS IN $\mathbb{C}^2$

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Abstract. Ruelle gave an explicit second-order expansion at $c = 0$ of the Hausdorff dimension of the Julia set of the quadratic family $f_c(z) = z^2 + c$. McMullen later extended this result to polynomial perturbations of z^d for arbitrary degree $d \geq 2$. In this paper we study an analogue of this problem for skew products in $\mathbb{C}^2$. Since holomorphic dynamical systems in higher dimensions are non-conformal, we replace the Hausdorff dimension by the *volume dimension*, a dynamically defined notion we introduced in our earlier work and characterized as the zero of a natural pressure function. We consider families of holomorphic skew products of the form

$$f_t(z,w) = (z^d, w^d + t(c_1(z)w^{d-1} + c_2(z)w^{d-2} + \cdots + c_d(z))).$$

Our main result gives an explicit second-order expansion of the volume dimension of the Julia set $J(f_t)$ as $t \to 0$ in terms of the coefficients $c_k(z)$.

1. Introduction

In the early 1980s, Ruelle [Rue82] initiated the study of the variation of the Hausdorff dimension of Julia sets in holomorphic families. For the quadratic family $f_c(z) = z^2 + c$, he proved that the Hausdorff dimension of the Julia set admits the expansion

$$\text{H.dim}(J_c) = 1 + \frac{|c|^2}{4\log 2} + O(|c|^3), \qquad |c| \to 0.$$

In particular, the Hausdorff dimension function has a strict local minimum at the monomial map $z \mapsto z^2$. McMullen [McM08] generalized Ruelle's result to polynomials of arbitrary degree, obtaining an explicit formula for the second derivative of the Hausdorff dimension at the monomial $z \mapsto z^d$. Namely, he showed that the Hausdorff dimension of the Julia set of the map $f_t(z) = z^d + t(c_2 z^{d-2} + c_3 z^{d-3} + \cdots + c_d)$ satisfies

$$\text{H.dim}(J_t) = 1 + \frac{|t|^2}{4d^2 \log d} \sum_{k=2}^{d} k^2 |c_k|^2 + O(|t|^3), \qquad |t| \to 0.$$

The purpose of this paper is to investigate an analogue of this problem for polynomial-like skew products in $\mathbb{C}^2$. More precisely, for t near 0 we consider families of the form

$$f_t(z,w) = (z^d,\ w^d + t(c_1(z)w^{d-1} + c_2(z)w^{d-2} + \cdots + c_d(z))), \tag{1}$$

where the c_j's are holomorphic functions in z, and we study the variation of the dimension of their Julia sets. Examples of these maps are the regular polynomial skew products, i.e., polynomial endomorphisms of $\mathbb{C}^2$ of the form (1) which extend holomorphically to endomorphisms of $\mathbb{P}^2(\mathbb{C})$, see [BJ00; Jon99].

Contrary to the one-dimensional case, in higher-dimensional holomorphic dynamics the Hausdorff dimension is no longer well adapted to the dynamics. Holomorphic maps in dimension greater than one are non-conformal, and basic tools such as Koebe's distortion theorem are no longer available.

Date: March 10, 2026.

As a consequence, the Hausdorff dimension may fail to be dynamically meaningful, and its behaviour in parameter space is poorly understood.

To address this issue, in an earlier work [BH24] we introduced the notion of *volume dimension* for invariant sets and measures in higher-dimensional holomorphic dynamical systems. The volume dimension coincides (up to a factor of 2) with the Hausdorff dimension in complex dimension one, but incorporates the non-conformal geometry of higher-dimensional systems. In particular, it satisfies a Mañé–Manning type formula relating volume dimension, entropy, and Lyapunov exponents, and can be characterized as the zero of a natural pressure function for expanding invariant measures. For hyperbolic maps, this zero coincides (up to the same factor of 2) with the volume dimension of the Julia set.

In this paper we study the real-analytic function

$$t \longmapsto \mathrm{VD}_{f_t}(J(f_t)),$$

where $J(f_t)$ denotes the Julia set of f_t. Our main result gives an explicit second-order expansion of the volume dimension of $J(f_t)$ at $t = 0$.

Theorem 1.1. *Let f_t be the family as in* (1). *Then*

$$\mathrm{VD}_{f_t}(J(f_t)) = \frac{1}{2} + \frac{|t|^2}{16d^2 \log d} \int_{S^1} \sum_{k=1}^{d} k^2 |c_k(z)|^2 \, d\,\mathrm{Leb}_1(z) + O(|t|^3), \qquad |t| \to 0$$

where Leb_1 *denotes the normalized Lebesgue probability measure on* S^1.

Main ingredients and structure of the proof. The proof follows the general strategy of McMullen, adapted to the non-conformal setting of polynomial skew products.

The starting point is the characterization of the volume dimension as the zero of a pressure function. Writing δ_t for this zero, the problem reduces to computing the second derivative of δ_t at $t = 0$. A general second-derivative formula allows one to express $\ddot{\delta}_0$ in terms of the variance of the infinitesimal deformation $\dot{\phi}_0$ of the geometric potential $\phi_t = -\log|\mathrm{Jac} f_t|$, see Proposition 3.3.

A key observation, inspired by McMullen's work [McM08], is that this infinitesimal deformation $\dot{\phi}_0$ is not an arbitrary Hölder function, but a *virtual coboundary*: it is the trace on the Julia set of a dynamically defined Hölder function h on the basin of infinity, which can be written as the difference of a function g and its pull-back by the dynamics. This structure allows us to express the variance we need in terms of the asymptotic growth of the function g near the boundary of the basin of infinity, see Proposition 3.5. A direct computation of this behaviour, see Proposition 3.7, then completes the proof of Theorem 1.1.

Acknowledgements. This project has received funding from the Programme Investissement d'Avenir (ANR QuaSiDy /ANR-21-CE40-0016, ANR PADAWAN /ANR-21-CE40-0012-01, ANR TIGerS /ANR-24-CE40-3604), from the MIUR Excellence Department Project awarded to the Department of Mathematics of the University of Pisa, CUP I57G22000700001. The first author is affiliated to the GNSAGA group of INdAM.

2. Preliminaries

2.1. **Holomorphic skew products.**

Throughout this paper, we work with holomorphic skew products $F \colon \mathbb{C}^2 \to \mathbb{C}^2$ of the form

$$F(z, w) = (z^d, q(z, w)) = \Big(z^d, w^d + \sum_{k=1}^{d} c_k(z) w^{d-k}\Big), \tag{2}$$

where $d \geq 2$ and each $c_k(z)$ is holomorphic in a neighbourhood of the closed unit disc $\overline{\mathbb{D}}^1$. We denote the set of holomorphic skew products as above by $\mathcal{S}(d)$. Restricting any $F \in \mathcal{S}(d)$ to $S^1 \times \mathbb{C}$, we obtain a *fibered polynomial map* over the map $z \mapsto z^d$, namely a continuous map

$$f \colon S^1 \times \mathbb{C} \longrightarrow S^1 \times \mathbb{C}, \qquad f(z,w) = \big(z^d, q_z(w)\big),$$

where $q_z := q(z, \cdot)$. For every such f, the *fiberwise Green function* is defined by

$$G(z,w) = G_z(w) \ := \lim_{n\to\infty} \frac{1}{d^n} \log^+ |q_z^n(w)|, \qquad \text{where } \log^+ |w| := \max\{\log |w|, 0\}.$$

The function G is continuous on $S^1 \times \mathbb{C}$. For each $z \in S^1$, the function G_z is subharmonic, and equals 0 on the set K_z of points with bounded orbit. It is harmonic on the open set $W_z := \mathbb{C} \setminus K_z = \{G_z > 0\}$. The *fiberwise Julia set* is $J_z(f) := \partial K_z = \partial W_z$, and the Julia set of f is

$$J(f) := \overline{\bigcup_{z \in S^1} \{z\} \times J_z(f)} \subset S^1 \times \mathbb{C}.$$

We also set $J(F) = J(f)$.

We will only consider fibered polynomial maps for which K_z is connected for every $z \in S^1$. Equivalently, for every $z \in S^1$, all critical points of q_z have bounded orbit. We call any f with this property *fiberwise connected.* In this case, every $W_z \cup \{\infty\}$ is simply connected and biholomorphic to the unit disc. We will denote by Δ the unit disc in $\mathbb{C}$ and by $1/\Delta$ the complement of the closed unit disc in $\mathbb{C}$. The following result is due to Sester [Ses99].

Proposition 2.1 ([Ses99, Proposition 2.7]). *For every fiberwise connected f as above and $z \in S^1$, there exists a unique holomorphic isomorphism $\varphi_z \colon W_z \longrightarrow 1/\Delta$ such that:*

(1) the diagram

$$\begin{tikzcd} W_z \arrow[r, "q_z"] \arrow[d, "\varphi_z"'] & W_{z^d} \arrow[d, "\varphi_{z^d}"] \\ 1/\Delta \arrow[r, "w^d"] & 1/\Delta \end{tikzcd}$$

commutes;

(2) the map $\varphi \colon W \to \mathbb{C}$ defined by $\varphi(z,w) = \varphi_z(w)$ is continuous;

(3) $G(z,w) = \log |\varphi_z(w)|$ for all $(z,w) \in W$;

(4) φ_z is tangent to the identity at infinity.

Remark 2.2. Let $F(z,w) = (z^d, q(z,w))$ be a holomorphic skew product as in (2). Then there exist domains $U \Subset V \subset \mathbb{C}^2$ such that the restriction $F \colon U \to V$ is a proper holomorphic map. In particular, F admits a polynomial-like restriction in the sense of Dinh–Sibony [DS03], whose topological degree equals d^2, and the results of [BH24] apply in this context, see in particular [BH24, Remark 1.4].

A particularly relevant special case of holomorphic skew products as in (2) is given by maps for which each coefficient $c_j(z)$ is a polynomial of degree at most $d - j$. In this case, F extends to a holomorphic endomorphism of $\mathbb{P}^2$ (i.e., it is a *regular polynomial skew product* in the sense of [BJ00; Jon99]). We will denote by $\mathcal{P}(d)$ the space of such polynomial skew products endowed with the topology of uniform convergence of coefficients on compact subsets of $\mathbb{C}$.

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[1] All the results of this section are still valid when z^d is replaced by a hyperbolic polynomial p, not necessarily of degree d. We just describe the case $p(z) = z^d$ for simplicity and since we will only need this case in the sequel.

2.2. **Uniform hyperbolicity and vertical expansion.** Following Jonsson [Jon99], we say that a holomorphic skew product $F(z,w) = (z^d, q(z,w))$ is *vertically expanding* if there exist a neighbourhood U of the Julia set $J(F)$ and constants $c > 0$ and $\lambda > 1$ such that

$$\|D_w F^n(z,w)\| \geq c\,\lambda^n \quad \text{for all } (z,w) \in U \text{ and all } n \geq 1.$$

In this case, the dynamics is uniformly expanding in the vertical direction near the Julia set. The following result follows from standard arguments in hyperbolic dynamics. We refer to [Jon99] for several further equivalent characterizations of vertical expansion.

Proposition 2.3. *A skew product F as in* (2) *is vertically expanding if and only if its Julia set $J(F)$ is a uniformly hyperbolic repeller.*

Vertical expansion is an open condition in $\mathcal{S}(d)$ (and $\mathcal{P}(d)$) and defines connected components, which are called *hyperbolic components*. By [AB23], these components coincide with the stability components in the sense of [BBD18; Bia19] as soon as one parameter inside them is hyperbolic.

For maps belonging to a hyperbolic component, the Julia sets vary continuously with the map and the dynamics is structurally stable: all maps in the same component are topologically conjugate on their Julia sets. Moreover, the conjugacies respect the skew-product structure and depend continuously on the parameter.

In particular, the skew product

$$F_0(z,w) = (z^d, w^d)$$

belongs to a distinguished hyperbolic component of $\mathcal{S}(d)$ (and $\mathcal{P}(d)$). This component plays a role analogous to the main cardioid of the Mandelbrot set for quadratic polynomials.

2.3. **Conjugacies and regularity inside a hyperbolic component.** Let $\mathcal{H} \subset \mathcal{S}(d)$ be a hyperbolic component consisting of fiberwise connected maps, and fix a reference map $F_0 \in \mathcal{H}$. For any $F \in \mathcal{H}$, we denote by $J(F)$ the Julia set of F, and recall from Section 2.2 that all maps in $\mathcal{H}$ are topologically conjugate on their Julia sets by conjugacies respecting the skew-product structure. In this subsection, we use Böttcher coordinates to study the regularity of this canonical conjugacy between $J(F_0)$ and $J(F)$.

Fix $F \in \mathcal{H}$. Since F is fiberwise connected, for every $z \in S^1$ the basin of infinity $W_{F,z} = \mathbb{C} \setminus K_{F,z}$ admits a Böttcher coordinate

$$\varphi_{F,z} \colon W_{F,z} \longrightarrow 1/\Delta$$

conjugating $q_{F,z}$ to $w \mapsto w^d$ and tangent to the identity at infinity, see Proposition 2.1. We define a fibered map

$$H_F \colon W_{F_0} \longrightarrow W_F, \qquad H_F(z,w) = \big(z, H_{F,z}(w)\big),$$

by setting

$$H_{F,z} := \varphi_{F,z}^{-1} \circ \varphi_{F_0,z} \quad \text{on } W_{F_0,z}.$$

The following lemma is an immediate consequence of the defining properties of the Böttcher coordinates.

Lemma 2.4. *The map H_F is well-defined and satisfies:*

(1) $H_{F_0} = \mathrm{id}$*;*
(2) for each $z \in S^1$, the map $H_{F,z}$ is conformal on $W_{F_0,z}$;
(3) H_F conjugates F_0 and F on the basin of infinity:

$$F \circ H_F = H_F \circ F_0 \quad \text{on } W_{F_0}.$$

We now consider the extension of H_F to the Julia set. The basins of infinity $\{W_{F,z}\}_{z \in S^1}$ form a family of John domains [DH85; Pom92], uniformly in $z \in S^1$ (see, for instance, [Sum06]), and locally uniformly in $F \in \mathcal{H}$ (as the maximal expansion $\sup_{J(F)} \|DF\|$ and the minimal expansion

$\inf_{J(F)} \|DF^{-1}\|^{-1}$ depend continuously on F). As a consequence, the Böttcher coordinates admit Hölder continuous boundary extensions with uniform Hölder exponent and constant [CJY94].

Lemma 2.5. *There exist $\alpha \in (0,1)$ and $C > 0$ such that for every $F \in \mathcal{H}$, the conjugacy H_F extends uniquely to a map*

$$\hat{H}_F \colon W_{F_0} \cup J(F_0) \longrightarrow W_F \cup J(F)$$

which is α-Hölder continuous with Hölder constant bounded by C. Moreover, $\hat{H}_F$ conjugates the dynamics on the Julia sets and respects the skew-product structure.

Since both the extension given by Lemma 2.5 and structural stability provide conjugacies on $J(F_0)$, these maps coincide on $J(F_0)$ (by uniqueness of the conjugacy inside a hyperbolic component). In particular, the conjugacies among Julia sets are Hölder continuous, uniformly in z and locally uniformly in F.

We now strengthen the (local) compactness of the maps $\{\hat{H}_F\}_F$ in the Hölder norm, thanks to the fact that, for every $(z,w) \in J(F_0)$ the conjugacy map $F \mapsto H_F(z,w)$ is actually *holomorphic* in F. Let $G_t, t \in (-\varepsilon, \varepsilon)$ be a parametrization of a C^2 path in $\mathcal{H}$ with $G_0 = F_0$. Recall that, given a path $(\psi_t)_{t \in (-\varepsilon,\varepsilon)}$ in $C^\alpha\big(J(F_0)\big)$, we can define

$$\dot{\psi}_t := \frac{d}{ds}\bigg|_{s=t} \psi_s \qquad \text{and} \qquad \ddot{\psi}_t := \frac{d^2}{ds^2}\bigg|_{s=t} \psi_s. \tag{3}$$

We say that $(\psi_t)_{t\in(-\varepsilon,\varepsilon)}$ is C^1 (resp. C^2) if $\dot{\psi}_t$ (resp. $\ddot{\psi}_t$) defines a continuous path in $C^\alpha\big(J(F_0)\big)$.

Proposition 2.6. *The map $t \mapsto \hat{H}_{G_t}$ is C^2 as a map from $(-\varepsilon, \varepsilon)$ into the Hölder space $C^\alpha\big(W_{F_0} \cup J(F_0)\big)$.*

Proof. For every given (z,w), the map $F \mapsto H_F(z,w)$ is constructed as a uniform limit of holomorphic functions. Hence, it depends holomorphically on F. It is standard that the uniform control given by Lemma 2.5 allows one to apply Cauchy estimates to obtain uniform bounds on derivatives, yielding smoothness in the Hölder norm, see, for instance, [SU10] for similar arguments. $\square$

3. PROOF OF THEOREM 1.1

3.1. **From $\ddot{\delta}_0$ to the variance of $\dot{\phi}_0$.** We denote by F_0 the map $(z,w) \mapsto (z^d, w^d)$ and by $m_0 := \mathrm{Leb}_1 \times \mathrm{Leb}_1$ its unique measure of maximal entropy. Let $\mathcal{H} \subset \mathcal{S}(d)$ be the hyperbolic component of $\mathcal{S}(d)$ containing F_0. Consider a C^2 curve $(\varepsilon, \varepsilon) \ni t \mapsto F_t \in \mathcal{H}$ with $F_{t=0} = F_0$. By Section 2.3, for each t there exists a conjugacy

$$\hat{H}_t \colon J(F_0) \longrightarrow J(F_t)$$

which is Hölder continuous in (z,w) and depends C^2 on t with respect to the Hölder topology of $C^\alpha(J(F_0))$ for some $\alpha \in (0,1)$.

Consider the family of geometric potentials on $J(F_0) = S^1 \times S^1$ given by

$$\phi_t := -\log\big|\mathrm{Jac}\, F_t \circ \hat{H}_t\big|.$$

By Proposition 2.6 and the smooth dependence of $(z,w) \mapsto \log|\,\mathrm{Jac}\, F_t(z,w)|$ on t, the path $\{\phi_t\}_t$ is C^2 in $C^\alpha(J(F_0))$ for some $\alpha \in (0,1)$. In particular, $\dot{\phi}_0$ and $\ddot{\phi}_0$ are well-defined by (3) and Hölder continuous on $J(F_0) = S^1 \times S^1$.

Lemma 3.1. *We have $\int \dot{\phi}_0 \, dm_0 = 0$ and $\int \ddot{\phi}_0 \, dm_0 = 0$*

Proof. Observe that the function $t \mapsto L(t) := -\int \phi_t dm_0$ is the sum of the Lyapunov exponents of F_t. The assertion follows from the fact that this function is constant (and equal to $2\log d$) in a neighbourhood of F_0, see [Jon99, Theorem 5.3]. $\square$

Let δ_t denote the unique positive real number such that

$$P(\delta_t \phi_t) = 0,$$

where $P(\cdot)$ denotes the topological pressure. By real-analyticity of the pressure for Hölder potentials and the implicit function theorem, the map $t \mapsto \delta_t$ is real-analytic on a neighbourhood of 0 (see [Rue82]).

Lemma 3.2. *We have $\delta_0 = 1$ and $\dot{\delta}_0 = 0$.*

Proof. We have $\phi_0 \equiv -2 \log d$ on $J(F_0) = S^1 \times S^1$. Therefore, we have $P(s\phi_0) = \log(d^2) - s \log(d^2)$ for every $s \in \mathbb{R}$. Hence, the unique solution to $P(s\phi_0) = 0$ is $s = 1$, which proves the first assertion.

To prove the second one, differentiate the identity $P(\delta_t \phi_t) = 0$ at $t = 0$. By standard arguments in thermodynamic formalism (see e.g., [PU10]), we have

$$0 = \frac{d}{dt}\bigg|_{t=0} P(\delta_t \phi_t) = \int (\dot{\delta}_0 \, \phi_0 + \delta_0 \dot{\phi}_0) \, dm_0,$$

since m_0 is the equilibrium state of $\delta_0 \phi_0 = \phi_0 \equiv -2 \log d$. Since ϕ_0 is constant and $\delta_0 = 1$, this gives

$$0 = \dot{\delta}_0 \int \phi_0 \, dm_0 + \int \dot{\phi}_0 \, dm_0.$$

Since $\int \dot{\phi}_0 \, dm_0 = 0$ by Lemma 3.1, the assertion follows. ☐

Recall that, given a continuous function $\psi \colon J(F_0) \to \mathbb{R}$ with $\int \psi dm_0 = 0$, the (asymptotic) *variance* of ψ with respect to m_0 is defined by

$$\mathrm{Var}(\psi, m_0) := \lim_{n \to \infty} \frac{1}{n} \int_{J(F_0)} \left(\sum_{j=0}^{n-1} \psi \circ F_0^j \right)^2 dm_0 = \lim_{n \to \infty} \frac{1}{n} \| S_n \psi \|^2_{L^2(m_0)} \in [0, +\infty].$$

Proposition 3.3. *We have*

$$\ddot{\delta}_0 = \frac{\mathrm{Var}\big(\dot{\phi}_0, m_0\big)}{2 \log d}.$$

Proof. Set $\psi_t := \delta_t \phi_t$. By the regularity established above, $\{\psi_t\}$ is a C^2 path in $C^\alpha(J(F_0))$. As $P(\psi_t) \equiv 0$, it follows from [McM08, Theorem 2.2] (see also [PP90]) that

$$0 = \mathrm{Var}(\dot{\psi}_t, m_0) + \int \ddot{\psi}_0 \, dm_0.$$

A direct development using $\dot{\psi}_0 = \delta_0 \dot{\phi}_0 + \dot{\delta}_0 \phi_0$ and $\ddot{\psi}_0 = \delta_0 \ddot{\phi}_0 + 2 \dot{\delta}_0 \dot{\phi}_0 + \ddot{\delta}_0 \phi_0$ gives

$$\ddot{\delta}_0 = \frac{\mathrm{Var}\big(\delta_0 \dot{\phi}_0 + \phi_0 \dot{\delta}_0, m_0\big) + \delta_0 \int \ddot{\phi}_0 \, dm_0 + 2 \int \dot{\delta}_0 \dot{\phi}_0 \, dm_0}{-\int \phi_0 \, dm_0}.$$

This identity reduces to the one in the statement thanks to Lemmas 3.1 and 3.2 and the fact that $\phi_0 \equiv -2 \log d$ on $S^1 \times S^1$. ☐

3.2. **Variance of $\dot{\phi}_0$ as an asymptotic energy of $\partial_w v$.** In order to better describe $\dot{\phi}_0$, we consider the infinitesimal deformation of the conjugacies on the basin of infinity. Let

$$\widetilde{v} = \frac{d}{dt}\bigg|_{t=0} \hat{H}_t$$

denote the derivative of the conjugation given by the Böttcher coordinates, see Section 2.3. Since the conjugacies are fibered, $\widetilde{v}$ is vertical and can be written as

$$\widetilde{v}(z,w) = v(z,w)\,\frac{\partial}{\partial w} \tag{4}$$

for every $z \in S^1$, where $v(z,\cdot)$ is holomorphic on the basin of infinity $W_{F_0,z} = 1/\Delta$. Moreover, $\widetilde{v}$ is smooth in w on W_{F_0}.

Lemma 3.4. *We have*

$$\dot{\phi}_0(z,w) = -\Re\bigg(\frac{\partial v}{\partial w}(z,w)\bigg) + \Re\bigg(\frac{\partial v}{\partial w}\big(F_0(z,w)\big)\bigg).$$

Proof. From $F_t \circ H_t = H_t \circ F_0$ we obtain

$$\operatorname{Jac} F_t(H_t) = (\operatorname{Jac} H_t)^{-1}\,(\operatorname{Jac} H_t \circ F_0)\ \operatorname{Jac} F_0.$$

Taking logarithms and differentiating at $t = 0$, using $\operatorname{Jac} H_0 \equiv 1$ yields the stated identity. ☐

Proposition 3.5. *With the above notation, we have*

$$\operatorname{Var}(\dot{\phi}_0, m_0) = \frac{\log d}{2} \cdot I\left(\frac{\partial v}{\partial w}\right)$$

where

$$I\left(\frac{\partial v}{\partial w}\right) := \lim_{r\to 1^+} \frac{1}{|\log(r-1)|} \int_{S^1}\int_{|w|=r} \left|\frac{\partial v}{\partial w}(z,w)\right|^2 d\operatorname{Leb}_r(w)\, d\operatorname{Leb}_1(z)$$

and Leb_r *is the normalized Lebesgue probability measure on the circle* $|w| = r$*.*

Proof. For simplicity, we will denote $g := \Re\left(\frac{\partial v}{\partial w}\right) = \Re(\partial_w v)$, and observe that we have $\dot{\phi}_0 = -g + g \circ F_0$ on $W_{F_0} = S^1 \times (1/\Delta)$. Since $\partial_w v(z,\cdot)$ is holomorphic on $\{|w| > 1\}$ and vanishes at infinity, we have

$$\int_{|w|=r} \left|\Re\left(\frac{\partial v}{\partial w}(z,w)\right)\right|^2 d\operatorname{Leb}_r(w) = \frac{1}{2}\int_{|w|=r} \left|\left(\frac{\partial v}{\partial w}(z,w)\right)\right|^2 d\operatorname{Leb}_r(w)$$

for every $r > 1$ and $z \in S^1$. Therefore, it is enough to show the identity

$$\operatorname{Var}(\dot{\phi}_0, m_0) = (\log d) \cdot \lim_{r\to 1^+} \frac{1}{|\log(r-1)|} \int_{S^1}\int_{|w|=r} |g|^2 d\operatorname{Leb}_r(w)\, d\operatorname{Leb}_1(z).$$

We will show the assertion for a specific choice of $r_n \to 1$, the argument can be easily adapted to handle the general limit. Recall that we have

$$\operatorname{Var}(\dot{\phi}_0, m_0) = \lim_{n\to\infty} \frac{1}{n} \int_{J(F_0)} \left|\sum_{j=0}^{n-1} \dot{\phi}_0 \circ F_0^j\right|^2 dm_0.$$

Set $r_n := e^{d^{-n}} \sim 1 + d^{-n}$ and $Z_n(z,w) := (z, r_n w) \in W_{F_0}$. Using the fact that $Z_n(z,w) \to (z,w)$ exponentially fast and $\dot{\phi}_0$ is Hölder continuous in a neighbourhood of $S^1 \times S^1$, for every $(z,w) \in S^1 \times S^1$ we obtain

$$\sum_{j=0}^{n-1} \dot{\phi}_0 \circ F_0^j(z,w) = \sum_{j=0}^{n-1} \dot{\phi}_0 \circ F_0^j(Z_n(z,w)) + O(1), \qquad n \to \infty.$$

We obtain from Lemma 3.4 that

$$\sum_{j=0}^{n-1} \dot{\phi}_0 \circ F_0^j(Z_n(z,w)) = -g(Z_n(z,w)) + g \circ F_0^n(Z_n(z,w)) = -g(Z_n(z,w)) + O(1), \qquad n \to \infty,$$

where $g \circ F_0^n(Z_n(z,w))$ is uniformly bounded as all the points $F_0^n(Z_n(z,w))$ belong to the circle of radius $(e^{d^{-n}})^{d^n} = e$. It follows that

$$\begin{aligned}\mathrm{Var}(\dot\phi_0, m_0) &= \lim_{n\to\infty} \frac{1}{n} \int_{J(F_0)} |g(Z_n(z,w))|^2 dm_0 \\ &= \lim_{n\to\infty} \frac{1}{n} \int_{S^1} \int_{|w|=r_n} |g(z,w)|^2 d\,\mathrm{Leb}_{r_n}(w) d\,\mathrm{Leb}_1(z).\end{aligned}$$

The assertion follows taking into account the asymptotic $|\log(r_n - 1)| \sim n \log d$. ☐

Remark 3.6. Following the terminology of [McM08], we can say that $\dot\phi_0$ is the *virtual coboundary* of $\Re(\partial_w v)$ on $W_{F_0} = S^1 \times (1/\Delta)$. The above arguments show more generally that, if h is a virtual coboundary of a continuous function g on W_{F_0} with $\int h\, dm_0 = 0$, then

$$\mathrm{Var}(h, m_0) = (\log d) \cdot I(g),$$

where

$$I(g) = \lim_{r\to 1^+} \frac{1}{|\log(r-1)|} \int_{S^1} \int_{|w|=r} |g(z,w)|^2 \, d\,\mathrm{Leb}_r(w) \, d\,\mathrm{Leb}_1(z).$$

3.3. A computation of $I(\partial_w v)$.

We now consider the explicit path

$$F_t(z,w) = (z^d,\ w^d + t(c_1(z)w^{d-1} + \cdots + c_d(z))), \qquad t \in (-\varepsilon, \varepsilon).$$

By Proposition 3.5, the term $\mathrm{Var}(\delta_0 \dot\phi_0, m_0)$ appearing in Proposition 3.3 can be expressed in terms of the asymptotic energy $I(\partial_w v)$ of the infinitesimal conjugacy. The next proposition gives an explicit expression for $I(\partial_w v)$ in terms of the functions c_k.

Proposition 3.7. *We have*

$$I\left(\frac{\partial v}{\partial w}\right) = \frac{1}{d^2 \log d} \int_{S^1} \sum_{k=1}^{d} k^2 \, |c_k(z)|^2 \, d\,\mathrm{Leb}_1(z).$$

For simplicity, we will set $v_z(\cdot) = v(z, \cdot)$, where $v(z,\cdot)$ is as in (4).

Lemma 3.8. *For each $z \in S^1$, we have*

$$v_z(w) = -\frac{w}{d} \sum_{k=1}^{d} \sum_{n=0}^{\infty} \frac{c_k(z^{d^n})}{d^n} \, w^{-kd^n}. \tag{5}$$

Observe that the series above converges normally on $\{|w| > 1 + \eta\}$ for every $\eta > 0$.

Proof. Differentiating the conjugacy relation $F_t \circ H_t = H_t \circ F_0$ at $t = 0$, we obtain that, for every $z \in S^1$, v_z is the solution of the functional equation

$$v_{z^d}(w^d) = d\, w^{d-1} v_z(w) + \sum_{k=1}^{d} c_k(z) w^{d-k}$$

satisfying $v_z(w) \to 0$ as $w \to \infty$. A direct computation shows that the series (5) satisfies these conditions. ☐

Proof of Proposition 3.7. Differentiating (5) term by term gives

$$\frac{\partial v}{\partial w}(z,w) = \frac{1}{d} \sum_{k=1}^{d} \sum_{n=0}^{\infty} \left(\frac{c_k(z^{d^n})}{d^n} \, w^{-kd^n} + k\, c_k(z^{d^n}) \, w^{-kd^n - 1} \right).$$

The terms involving the denominator d^n are uniformly summable and negligible in the logarithmic limit defining $I(\partial_w v)$. Hence only the second sum contributes, and we obtain

$$I\left(\frac{\partial v}{\partial w}\right) = \lim_{r\to 1^+} \frac{1}{|\log(r-1)|} \int_{S^1} \int_{|w|=r} \left| \frac{1}{d} \sum_{k=1}^{d} \sum_{n=0}^{\infty} k\, c_k(z^{d^n})\, w^{-kd^n-1} \right|^2 d\,\mathrm{Leb}_r(w)\, d\,\mathrm{Leb}_1(z).$$

Since all exponents $-kd^n-1$ are distinct, the functions $w \mapsto w^{-kd^n-1}$ are orthogonal in $L^2(\mathrm{Leb}_r)$. Therefore, for every $z \in S^1$, we have

$$\int_{|w|=r} \left| \frac{1}{d} \sum_{k=1}^{d} \sum_{n=0}^{\infty} k\, c_k(z^{d^n})\, w^{-kd^n-1} \right|^2 d\,\mathrm{Leb}_r(w) = \frac{1}{d^2} \sum_{k=1}^{d} \sum_{n=0}^{\infty} k^2 |c_k(z^{d^n})|^2\, |r|^{-2kd^n-2}.$$

For any fixed $r > 1$, the factor $|r|^{-2kd^n-2}$ is close to 1 for $n \le N(r)$ and exponentially small for $n > N(r)$, where

$$N(r) \sim \frac{|\log(r-1)|}{\log d}.$$

Hence,

$$\int_{|w|=r} \left| \frac{1}{d} \sum_{k=1}^{d} \sum_{n=0}^{\infty} k\, c_k(z^{d^n})\, w^{-kd^n-1} \right|^2 d\,\mathrm{Leb}_r(w) = \frac{1}{d^2} \sum_{k=1}^{d} \sum_{n=0}^{N(r)} k^2 |c_k(z^{d^n})|^2 + o(|\log(r-1)|).$$

Dividing by $|\log(r-1)|$ and letting $r \to 1$, we may apply the Birkhoff ergodic theorem to the sequence $\{z^{d^n}\}_{n\in\mathbb{N}}$. For Leb_1-almost every $z \in S^1$ we obtain

$$\lim_{r\to 1^+} \frac{1}{|\log(r-1)|} \int_{|w|=r} \left| \frac{\partial v}{\partial w}(z,w) \right|^2 d\,\mathrm{Leb}_r(w) = \frac{1}{d^2 \log d} \sum_{k=1}^{d} k^2 \int_{S^1} |c_k(z')|^2\, d\,\mathrm{Leb}_1(z')$$

Integrating with respect to z (observe that the right hand side is constant) yields the assertion. □

3.4. **End of the proof of Theorem 1.1.** We can now conclude the proof of Theorem 1.1. By [BH24, Theorem 1.3], for every $t \in (-\varepsilon, \varepsilon)$ we have $2\ \mathrm{VD}_{f_t}(J_t) = \delta_t$. The assertion then follows from Lemma 3.2 and Propositions 3.3, 3.5, and 3.7.

Remark 3.9. We observe that the same proof applies when the first component of F_t is $z^{d'}$. In this case, the factor $2\log d$ appearing in the denominator of the coefficient in Theorem 1.1 is replaced by $\log d + \log d'$. This change comes from the denominator in the second-derivative formula of Proposition 3.3, where $\phi_0 = -\log|\operatorname{Jac} F_0| = -(\log d + \log d')$.

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