PRESSURE PATH METRICS ON PARABOLIC FAMILIES OF POLYNOMIALS

FABRIZIO BIANCHI AND YAN MARY HE

Abstract. Let Λ be a subfamily of the moduli space of degree $D \geq 2$ polynomials defined by a finite number of parabolic relations. Let Ω be a bounded stable component of Λ with the property that all critical points are attracted by either the persistent parabolic cycles or by attracting cycles in $\mathbb{C}$. We construct a positive semi-definite pressure form on Ω and show that it defines a path metric on Ω. This provides a counterpart in complex dynamics of the pressure metric on cusped Hitchin components recently studied by Kao and Bray-Canary-Kao-Martone.

1. Introduction

Let S be a closed surface of genus at least 2. The Teichmüller space $T(S)$ of S, which parametrizes the hyperbolic structures on S, carries a number of natural metrics defined from different perspectives, e.g., the Teichmüller metric, the Weil-Petersson metric, and the Thurston metric; see [Hub06; IT92]. In [Bri10] and [McM08], Bridgeman and McMullen have respectively shown that the Weil-Petersson metric on $T(S)$ can be reconstructed via thermodynamic formalism. More precisely, they proved that the Weil-Petersson metric is a constant multiple of the so-called *pressure metric*.

From the perspective of Sullivan's dictionary [Sul85], the space of degree $D \geq 2$ Blaschke products $\mathcal{B}_D$ can be viewed as a counterpart of the Teichmüller space $T(S)$ in complex dynamics; see for instance [Luo23; McM09a; McM09b; McM10]. In [McM08] McMullen, using thermodynamic formalism, introduced a counterpart of the Weil-Petersson metric on $\mathcal{B}_D$, whose construction is analogous to that of the pressure metric on the Teichmüller space $T(S)$. Nie and the second author constructed pressure metrics on certain hyperbolic components in the moduli space of degree $D \geq 2$ rational maps [HN23a] and polynomial shift loci [HN23b].

If S is a *punctured* surface of negative Euler characteristic, the Teichmüller space $T(S)$ of S parametrizes *complete* hyperbolic structures on S. Denote by $\pi_1 S$ the fundamental group of S. A hyperbolic structure on S can be identified with a discrete faithful representation $\rho : \pi_1 S \to \mathrm{PSL}(2, \mathbb{R})$ such that the element $g \in \pi_1 S$ representing a puncture is mapped to a *parabolic* matrix, i.e., whose trace squares to 4. Kao [Kao20] constructed pressure metrics on such Teichmüller spaces. In higher Teichmüller theory, Bray-Canary-Kao-Martone [Bra+23] have studied pressure metrics for *cusped Hitchin components*, which are generalizations of Teichmüller spaces of punctured surfaces. These metrics are expected to be the induced metric on the strata at infinity of the metric completion of the Hitchin component of a closed surface with its pressure metric [Bra+23; Mas76].

In this paper, as a natural counterpart of cusped Hitchin components in complex dynamics, we consider Λ-*hyperbolic components* (see below) Ω in an algebraic family Λ of conjugacy classes of degree $D \geq 2$ polynomials or rational maps defined by a finite number of parabolic relations. We construct a positive semi-definite pressure form on Ω and show that it defines a path metric on Ω whenever Λ is polynomial and $\Omega \subset \Lambda$ is bounded.

1.1. **Statement of results.** Let $D \geq 2$ be an integer. We denote by rat_D^{cm} (resp. poly_D^{cm}) the moduli space of degree D rational maps (resp. polynomials) with marked critical points $c_1, \dots, c_{2D-2}$ (resp. $c_1, \dots, c_{D-1}$), i.e., the space of Möbius conjugacy classes of degree D rational maps (resp. polynomials) whose critical points are marked. Let Λ be an algebraic subfamily of rat_D^{cm} (resp.

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poly_D^{cm}) with the property that a (possibly empty) subset of the critical points are persistently attracted by a parabolic periodic point (this property is invariant by the action of Möbius transformations and therefore well-defined on the quotient). We call such Λ a *parabolic subfamily* of rat_D^{cm} (resp. poly_D^{cm}). For every $\lambda \in \Lambda$, we denote by f_λ a rational map (resp. polynomial) in the corresponding conjugacy class. Up to renumbering, we denote by $c_1, \dots, c_m$ the critical points which are active on Λ. Recall that a critical point c is *passive* on an open subset $\Lambda_0 \subseteq \Lambda$ if the sequence of holomorphic functions $\{\lambda \mapsto f_\lambda^n(c(\lambda))\}_{n\geq 1}$ is a normal family on Λ_0. Otherwise, the critical point is *active*.

Let Ω be a stable component in Λ; that is, all critical points are passive on Ω, which in this case is equivalent to asking that none of the possibly active critical points $c_1, \dots, c_m$ is active on Ω. We say that a stable component Ω is Λ-*hyperbolic* if, for every $\lambda \in \Omega$, all the critical points $c_1(\lambda), \dots, c_m(\lambda)$ are contained in the basin of some attracting cycle of f_λ. Λ-hyperbolic components are the natural generalization of the hyperbolic components of rat_D^{cm} or poly_D^{cm} (which correspond to the case where $m = 2D-2$ and $m = D-1$, respectively). Bounded Λ-hyperbolic components (i.e., those satisfying $\Omega \Subset \Lambda$) in poly_D^{cm} are the analogue of hyperbolic components for which all critical points are attracted to attracting cycles in $\mathbb{C}$, which are the hyperbolic components in the connectedness locus.

We first show that the construction of the positive semi-definite symmetric bilinear form $\langle \cdot, \cdot \rangle_G$ in [HN23a] can be extended to any Λ-hyperbolic stable component Ω, where Λ is a parabolic subfamily of poly_D^{cm} or rat_D^{cm}. Moreover, $\langle \cdot, \cdot \rangle_G$ is conformal equivalent to the pressure (pseudo) metric, which is constructed on Ω in a similar way as McMullen [McM08]; see Section 1.2 for more details.

A priori, the 2-form $\langle \cdot, \cdot \rangle_G$ is only positive semi-definite. In [HN23a], the authors gave a condition on a hyperbolic component in the moduli space of degree D rational maps under which the 2-form is non-degenerate. To obtain this, they used a deep result of Oh-Winter [OW17, Theorem 1.1] on the asymptotic distribution of repelling multipliers for hyperbolic rational maps. Since our stable Λ-hyperbolic components are more general than hyperbolic components, this result is unavailable in our setting. However, we show that $\langle \cdot, \cdot \rangle_G$ still defines a metric on every bounded Λ-hyperbolic component Ω of a parabolic subfamily Λ of poly_D^{cm}. The following theorem is our main result.

Theorem 1.1. *Let Λ be a parabolic subfamily of poly_D^{cm} and $\Omega \Subset \Lambda$ a bounded Λ-hyperbolic component. Then the 2-form $\langle \cdot, \cdot \rangle_G$ defines a metric on Ω.*

Before we move on to discuss the ideas of the proofs, we mention related works on pressure metrics in various contexts in geometry and dynamics. In Teichmüller theory, pressure metrics have been studied for quasi-Fuchsian spaces of closed surfaces [Bri10; BT08], Teichmüller spaces and quasi-Fuchsian spaces of punctured surfaces [BCK23; Kao20], and Teichmüller spaces of bordered surfaces [Xu19]. In higher Teichmüller theory, pressure metrics have been studied for deformation spaces of Anosov representations [Bri+15] and cusped Hitchin components [Bra+23]. Pressure metrics have also been defined on the moduli space of metric graphs [Kao17; PS14] and on Culler-Vogtmann outer spaces [ACR23]. Ivrii [Ivr14] studied the metric completion of $\mathcal{B}_2$ with respect to McMullen's metric. Lee, Park, and the second author [HLP24] studied the pressure metric on the space of degree $D \geq 2$ quasi-Blaschke products.

Pressure metrics are not inherently non-degenerate, i.e., they are not necessarily *Riemannian* metrics. Indeed, the construction of pressure metrics ensures only the positive semi-definiteness of the associated 2-form. Although pressure metrics are known to be positive definite in most cases, degeneracy loci are known to exist is several settings; see [Bri10; HLP24].

1.2. **Strategy of the proof.** In [HN23a], inspired by the works of Bridgeman [Bri10] and McMullen [McM08], the authors constructed a symmetric bilinear form $\langle \cdot, \cdot \rangle_G$ on any hyperbolic component in the moduli space of degree $D \geq 2$ rational maps.

Our first goal in the present paper is to show that their construction, with suitable modifications, extends to any Λ-hyperbolic component of a parabolic subfamily Λ of rat_D^{cm} (or poly_D^{cm}). Let Ω be a Λ-hyperbolic component and $\widetilde{\Omega}$ a lift of Ω in the parameter space Rat_D^{cm}. Given $\lambda_0 \in \widetilde{\Omega}$, denote by J_{λ_0} the Julia set of f_{λ_0}. As Ω is a stable component, there exists a holomorphic motion for the Julia sets as the parameter moves in $\widetilde{\Omega}$. In particular, every $\lambda_0 \in \widetilde{\Omega}$ admits a neighborhood $U(\lambda_0)$ such that a Hölder-continuous conjugacy $\Psi_\lambda : J_{\lambda_0} \to J_\lambda$ is well-defined for every $\lambda \in U(\lambda_0)$. We also observe that $\log |f'_\lambda| : J_\lambda \to \mathbb{R}$ is Hölder continuous for every $\lambda \in U(\lambda_0)$.

Fix $\eta \in (0, \log D)$. It is not difficult to see that there exists a unique number $\delta_\eta(\lambda) = \delta_\eta(f_\lambda)$ such that the *pressure* $\mathcal{P}$ of the Hölder continuous function $-\delta_\eta(\lambda) \log |f'_\lambda| : J_\lambda \to \mathbb{R}$ satisfies

$$\mathcal{P}(-\delta_\eta(\lambda) \log |f'_\lambda|) = \eta.$$

We call $\delta_\eta(\lambda)$ the *Bowen number*[1] as a celebrated result of Bowen [Bow79] states that, for a hyperbolic rational map f, the Hausdorff dimension $\delta(f)$ of $J(f)$ satisfies $\mathcal{P}(-\delta(f) \log |f'|) = 0$. We first prove (see Proposition 3.1) that the function $\delta_\eta : \widetilde{\Omega} \to \mathbb{R}$, sending λ to $\delta_\eta(\lambda)$, is real-analytic, which generalizes [Rue82] to our setting. To this end, we adapt the methods described in [SU10; UZ04] in the context of hyperbolic semi-groups of rational maps and of exponential maps respectively. The spectral gap property for the transfer operators associated to the weights $\delta_\eta(\lambda) \log |f'_\lambda|$ and their perturbations, as recently established in [BD23; BD24], allows us to deal with the non-uniform hyperbolicity due to the presence of parabolic points and to adapt those arguments in our context. Observe that the condition on η is required to apply these results, as the transfer operator does not have a unique isolated eigenvalue of multiplicity 1 for $\eta = 0$ as soon as a map has a parabolic periodic point.

Once the analyticity of the map $\delta_\eta : \widetilde{\Omega} \to \mathbb{R}$ is established, we consider the map $G_{\lambda_0} : U(\lambda_0) \to \mathbb{R}$ given by

$$G_{\lambda_0}(\lambda) := \delta_\eta(\lambda) \mathrm{Ly}_{\lambda_0}(\lambda),$$

where $\mathrm{Ly}_{\lambda_0} : U(\lambda_0) \to \mathbb{R}$ is given by $\mathrm{Ly}_{\lambda_0}(\lambda) := \int_{J_{\lambda_0}} \log |(f_\lambda \circ \Psi_\lambda)'(z)| d\nu(z)$ and ν is the unique equilibrium state of the Hölder continuous function $-\delta_\eta(\lambda_0) \log |f'_{\lambda_0}| : J_{\lambda_0} \to \mathbb{R}$. As the function Ly_{λ_0} is real-analytic by standard arguments, so is G_{λ_0}. Moreover, we show that G_{λ_0} attains a minimum at λ_0; see Section 4.1. It follows that the Hessian $G''_{\lambda_0}(\lambda_0) : T_{\lambda_0}\widetilde{\Omega} \times T_{\lambda_0}\widetilde{\Omega} \to \mathbb{R}$ of G_{λ_0} at λ_0 gives a positive semi-definite symmetric bilinear form on the tangent space $T_{\lambda_0}\widetilde{\Omega}$, that we denote by $\langle \cdot, \cdot \rangle_G$ and call the *Hessian form*. This form descends to Ω; see Section 4.2. We use the same notation for the induced form, which is still positive semi-definite.

We prove in Section 4.4 that $\langle \cdot, \cdot \rangle_G$ is conformal equivalent to the *pressure (pseudo) metric* on Ω, see Section 4.3 for the definition and details. The pressure metric was first considered by [Bri10; McM08] in the context of (quasi-)Fuchsian spaces and the moduli space of Blaschke products. Thanks to this equivalence, we can show that a key necessary condition for the degeneration of the form $\langle \cdot, \cdot \rangle_G$ at a given $[\lambda_0] \in \Omega$ and $\vec{v} \in T_{[\lambda_0]}\Omega$, proved in [HN23a] in the hyperbolic case, still holds in our context; see Lemma 4.14. Namely, if $[\lambda_0] \in \Omega$ and $\vec{v} \in T_{[\lambda_0]}\Omega$ are such that $\langle \vec{v}, \vec{v} \rangle_G = 0$, and we assume for simplicity that Ω is an open set of some $\mathbb{C}^N$, we must have

$$\frac{d}{dt}\Big|_{t=0} S_n\big(\log |f'_{\lambda_0 + t\vec{v}} \circ \Psi_t(x)|\big) = K \cdot S_n\big(\log |f'_{\lambda_0}(x)|\big) \tag{1}$$

for all n-periodic points x in the Julia set of f_{λ_0} and some constant K independent of n and x. Here, $S_n(\cdot)$ denotes the n-th Birkhoff sum and Ψ_t is the conjugation between the Julia sets of f_{λ_0} and $f_{\lambda_0 + t\vec{v}}$ induced by the holomorphic motion. Establishing (1) relies on the equivalence

[1] We avoid the name *Bowen parameter* to avoid confusion, since, in this paper, *parameters* will mostly refer to elements in the space of the maps.

between the Hessian and the pressure forms mentioned above and on the fact that any coboundary is automatically continuous. Both these ingredients rely on the spectral gap from [BD23; BD24]. The condition (1) is shown to be too rigid to hold in the hyperbolic case (up to some special exceptions) in [HN23a], by exploiting the above mentioned result by Oh-Winter [OW17]. Therefore, in that case, the form is positive definite, and hence it defines a Riemannian metric on all but finitely many hyperbolic components in rat_D^{cm} or poly_D^{cm}.

In the present case, in order to prove Theorem 1.1, we directly show that the function $d_G : \Omega \times \Omega \to \mathbb{R}$ given by

$$d_G(x,y) := \inf_{\gamma} \ell(\gamma), \qquad \text{where} \qquad \ell(\gamma) := \int_0^1 \sqrt{\langle \gamma'(t), \gamma'(t) \rangle_G}\, dt,$$

is a distance function, which implies that $\langle \cdot, \cdot \rangle_G$ defines a path metric on Ω. Here the infimum is taken over all the C^1-paths γ connecting x to y in Ω. A priori, as $\langle \cdot, \cdot \rangle_G$ is only positive semi-definite, the function d_G is only a pseudo-metric. To show that $d_G(x,y) > 0$ for any $x \neq y$ in Ω, we first show that $\langle \cdot, \cdot \rangle_G$ is real-analytic on the unit tangent bundle $UT\Omega$ of Ω. In particular, we prove that the pressure function associated to a 4-parameter family of potentials is jointly real analytic in all the parameters; see Proposition 5.1. The proof shares a similar framework as that of Proposition 3.1 and again relies on the spectral gap property of the transfer operators and their perturbations. Thanks to the analyticity of $\langle \cdot, \cdot \rangle_G$, an analysis of the possible singular sets for this form shows that, as soon as $d_G(x,y) = 0$, there exists a C^1 path γ joining them with $\ell(\gamma) = 0$; see Section 5.4.

We then proceed to show that any C^1 path $\gamma : [0,1] \to \Omega$ in Ω has strictly positive length. This is the only point in the paper where we use that Λ is a polynomial family and Ω is a *bounded* Λ-hyperbolic component. Namely, we combine the fact that the Lyapunov exponent of the measure of maximal entropy (which describes the asymptotic value of the multipliers of the periodic points, thanks to their equidistribution with respect to the measure of maximal entropy [Lyu82; Lyu83a]) is constant on Ω with the fact that (1) should hold true at every point of a possible path with zero length, taking $\vec{v}$ to be the tangent vector to the path. This forces the absolute values of the multipliers of all repelling periodic points to have a common level set, something that leads to a contradiction.

1.3. **Organization of the paper.** The paper is organized as follows. In Section 2 we present the preliminary results that we will need, in particular those related to the thermodynamic formalism and the spectral properties of the transfer operators. In Section 3, we prove the analyticity of the Bowen function δ_η. In Section 4, we construct the symmetric bilinear form $\langle \cdot, \cdot \rangle_G$ on Ω and show that it is conformal equivalent to the pressure form. We prove Theorem 1.1 in Section 5.

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2. Preliminaries

In this section, we collect basic definitions in Section 2.1. In Section 2.2 we adapt the main results of [BD23; BD24] about the spectral properties of transfer operators and their perturbations to our setting. In Section 2.3, we deduce basic properties of the geometric potential. Finally, we discuss extensions of analytic functions in Section 2.4.

2.1. **Notations and definitions.** Let $D \geq 2$ be an integer. Let Rat_D^{cm} (resp. Poly_D^{cm}) be the space of degree D rational maps (resp. polynomials) with marked critical points $c_1, \ldots, c_{2D-2}$ (resp. $c_1, \ldots, c_{D-1}$) and rat_D^{cm} (resp. poly_D^{cm}) the moduli space obtained from Rat_D^{cm} (resp. Poly_D^{cm}) by taking the quotient by all Möbius (resp. affine) conjugacies. For every $\lambda \in \mathrm{Rat}_D^{cm}$ (resp. Poly_D^{cm}), we denote by f_λ the corresponding rational map (resp. polynomial), by μ_λ its unique measure of maximal entropy [FLM83; Lyu82; Lyu83a], and by $L(\lambda) = \int \log |f'_\lambda| \mu_\lambda$ the Lyapunov exponent of μ_λ.

Let Λ be an algebraic subfamily of Rat_D^{cm} (resp. Poly_D^{cm}) with the property that a (possibly empty) subset of the critical points are persistently attracted by a parabolic periodic point. We say that such Λ is a *parabolic subfamily* of Rat_D^{cm} (resp. Poly_D^{cm}). Examples of these subfamilies are the families $\mathrm{Per}_n(\eta)$ for $n \in \mathbb{N}$ and η root of unity, which are defined as

$$\mathrm{Per}_n(\eta) := \{\lambda \colon \exists x \colon f_\lambda^n(x) = x; f'_\lambda(x) = \eta\}.$$

We refer to [Ber13; Mil93; Sil07] for the description and the geometry of these algebraic submanifolds, and to [BB09; BB11; Duj14; Gau16] for their distribution in the parameter space and their role in the dynamical understanding of the bifurcation phenomena. Every parabolic subfamily is then a finite intersection of such families (where Rat_D^{cm} or Poly_D^{cm} is thought of as the empty intersection). It is clear that the quotient by Möbius or affine conjugacies is well-defined on parabolic subfamilies, and we will use the same name for the images of these families by the quotient map. By a slight abuse of notation, we can think of parabolic subfamilies of Poly_D^{cm} (resp. poly_D^{cm}) also as parabolic subfamilies of Rat_D^{cm} (resp. rat_D^{cm}), where we allow (and require) the further critical relation $f_\lambda^{-1}(\{\infty\}) = \{\infty\}$.

Recall that a critical point c is called *passive* on an open subset $\Lambda_0 \subseteq \Lambda$ if the sequence of holomorphic functions $\{\lambda \mapsto f_\lambda^n(c(\lambda))\}_{n \geq 1}$ is a normal family on Λ_0. Otherwise, the critical point is called *active* on Λ_0. It follows from the definition and the fact that every parabolic basin contains at least a critical point that, in any parabolic subfamily Λ given by k parabolic relations, there are at least k passive critical points on Λ.

Recall [Lyu83b; MSS83] that an open subset $\Omega \subset \Lambda$ is in the *stability locus* of Λ if all critical points are passive on Ω. We also say that the family is stable on Ω. Ω is a *stable component* if it is a connected component of the stability locus. We say that a stable component $\Omega \Subset \Lambda$ is Λ-*hyperbolic* if, for every $\lambda \in \Omega$, every critical point $c_j(\lambda)$ such that c_j is active on Λ is contained in the basin of some attracting cycle for f_λ. We say that Ω is a bounded Λ-hyperbolic component if we have $\Omega \Subset \Lambda$ for the topology induced by Λ.

In the sequel, we will need the following simple facts about polynomial bounded Λ-hyperbolic components. We refer to [Zha22a; Zha22b] for some topological properties of these components in the case of cubic polynomials.

Lemma 2.1. *Let Λ be a parabolic subfamily of poly_D^{cm} and $\Omega \Subset \Lambda$ a bounded Λ-hyperbolic component. Then*

(1) all the critical points which are active on Λ are contained in the basin of some attracting cycle in $\mathbb{C}$ for every $\lambda \in \Omega$;
(2) $L(\lambda) \equiv \log D$.

Proof. If some critical point is in the basin of infinity for some $\lambda_0 \in \Omega$, the same must be true for all $\lambda \in \Omega$. It is a standard fact that the map f_{λ_0} can be deformed, staying inside Ω, to make the modulus of this critical point to become as large as desired. This contradicts the assumption $\Omega \Subset \Lambda$ and proves (1).

It follows from (1) that every critical point must have bounded orbit for every $\lambda \in \Omega$ (as it can either be attracted to a parabolic cycle or an attracting cycle in $\mathbb{C}$). Hence, the Green function at every critical point is equal to 0, and (2) follows from the classical Przyticki formula [Prz85]. □

2.2. **Transfer operators.** We fix in this section a rational map f of degree $D \geq 2$ and two real numbers $q > 2$ and $0 < \beta \leq 2$. We denote by $J(f)$ the Julia set of f. By [BD23; BD24], there exists a norm $\|\cdot\|_\diamond$ for real-valued functions on $\mathbb{P}^1$ satisfying the following properties for every $g, h\colon \mathbb{P}^1 \to \mathbb{R}$:

- **(N1)** $\|g\|_{\log^q} \lesssim \|g\|_\diamond \lesssim \|g\|_{C^\beta}$;
- **(N2)** $\|gh\|_\diamond \lesssim \|g\|_\diamond \|h\|_\diamond$;
- **(N3)** $\|f_* g\|_\diamond \lesssim \|g\|_\diamond$,

where the implicit constants are independent of g and h. We denote by $\|\cdot\|_{C^\beta}$ the β-Hölder norm, and by $\|\cdot\|_{\log^q}$ the q-log-Hölder norm, which is defined as

$$\|g\|_{\log^q} := \|g\|_{L^\infty} + \sup_{x \in \mathbb{P}^1, r>0} (1 + |\log r|)^q \cdot \big(\sup_{B(x,r)} g - \inf_{B(x,r)} g \big)$$

for every $g\colon \mathbb{P}^1 \to \mathbb{R}$. When working on $\mathbb{P}^1$, we will always use the distance $\mathrm{dist}_{\mathbb{P}^1}$ induced by the spherical metric. Recall also that the push-forward operator f_* is defined by

$$(f_* g)(y) := \sum_{f(x)=y} g(x)$$

for every $y \in \mathbb{P}^1$, where the D preimages of y are counted with multiplicity.

By defining $\|g\|_\diamond := \|\Re g\|_\diamond + \|\Im g\|_\diamond$, we see that a norm with the same properties exists also for complex-valued functions on $\mathbb{P}^1$ (see also [BD24, Section 5.3]). We set

$$B_\diamond := \{g : \mathbb{P}^1 \to \mathbb{C} : \|g\|_\diamond < \infty\}.$$

By **(N1)**, the space $(B_\diamond, \|\cdot\|_\diamond)$ is a Banach space and contains all β-Hölder continuous functions. We denote by $L(B_\diamond)$ the space of bounded linear operators on $B_\diamond$. Given a *potential* (or *weight*) $\phi \in B_\diamond$, the (Ruelle-Perron-Frobenius) *transfer operator* $\mathcal{L}_\phi : B_\diamond \to B_\diamond$ is defined by

$$\mathcal{L}_\phi(g)(y) := \sum_{f(x)=y} e^{\phi(x)} g(x).$$

By [BD24], the norm $\|\cdot\|_\diamond$ can be chosen so that $\mathcal{L}_\phi \in L(B_\diamond)$ for every $\phi \in B_\diamond$ with $\max \phi - \min \phi < \log D$. More precisely,

- **(N4)** for any such $\phi \in B_\diamond$ with $\max \phi - \min \phi < \log D$ and every $\psi \in B_\diamond$, the function $t \mapsto \mathcal{L}_{\phi + t\psi}$ is analytic as operators in $L(B_\diamond)$ for $|t|$ sufficiently small.

Recall that the analyticity of the map $t \mapsto \mathcal{L}_{\phi+t\psi}$ above means that, for every t_0 sufficiently small, there exist operators $L_{t_0,j} \in L(B_\diamond)$ such that $\mathcal{L}_{\phi+t\psi} = \sum_{j\geq 0} (t-t_0)^j L_{t_0,j}/j!$ for every t in a neighbourhood of t_0. It is not difficult to see that, for every t_0 such that $\max(\phi + t_0\psi) - \min(\phi + t_0\psi) < \log D$, we can take $L_{j,t_0}(\cdot) := \mathcal{L}_{\phi+t_0\psi}(\psi^j \cdot)$, and that we have

$$\|L_{j,t_0}\|_\diamond \leq \|\mathcal{L}_{\phi+t_0\psi}\|_\diamond \cdot \|\psi^j\|_\diamond \leq c^j \|\mathcal{L}_{\phi+t_0\psi}\|_\diamond \cdot \|\psi\|_\diamond^j,$$

where c is the implicit constant in **(N2)** and $\|\mathcal{L}_{\phi+t_0\psi}\|_\diamond$ is bounded by the assumption on t_0.

Recall that, for every continuous function $\phi\colon \mathbb{P}^1 \to \mathbb{R}$, the *topological pressure* $\mathcal{P}(\phi)$ of ϕ is defined by

$$\mathcal{P}(\phi) := \sup_{m \in \mathcal{M}_f} \left(h_m(f) + \int \phi \, dm \right)$$

where $\mathcal{M}_f$ is the set of f-invariant probability measures on $\mathbb{P}^1$ and $h_m(f)$ is the measure-theoretic entropy of f with respect to the measure m. A measure $m = m(\phi) \in \mathcal{M}_\sigma$ is called an *equilibrium state* of ϕ if $\mathcal{P}(\phi) = h_m(\sigma) + \int \phi\, dm$.

We say that a bounded linear operator $\mathcal{L} \in L(B_\diamond)$ has a *spectral gap* if $\mathcal{L}$ has an isolated real eigenvalue κ of multiplicity 1 and the rest of the spectrum of $\mathcal{L}$ is contained in a disk of radius strictly smaller than κ. By [BD24], the norms $\|\cdot\|_\diamond$ can be taken to also satisfy this property. Namely,

(N5) for every $\phi \in B_\diamond$ with $\max\phi - \min\phi < \log D$, the transfer operator $\mathcal{L}_\phi$ has a spectral gap with respect to the norm $\|\cdot\|_\diamond$, with the largest eigenvalue equal to $e^{\mathcal{P}(\phi)}$. Moreover, there exists a unique equilibrium state μ_ϕ associated with ϕ.

In this paper, we will need to consider (families of) weights ϕ which are Hölder continuous on $J(f)$, but not on $\mathbb{P}^1$. The definitions of pressure and equilibrium states can also be stated in this setting. On the other hand, in order to apply the results above, we will need to modify (or directly define) them outside of $J(f)$. This will be achieved by means of the following elementary lemma, which also gives a uniform control on the extension, under suitable assumptions. We give the proof for the reader's convenience.

Lemma 2.2. *Let $W_1 \Subset W_2 \subset \mathbb{R}^\ell$ be open sets and $\{\phi_s\colon J(f) \to \mathbb{R}\}_{s\in W_2}$ a family of β-Hölder continuous functions with the property that $s \mapsto \phi_s$ is continuous with respect to the β-Hölder norm. Assume also that there exists a constant $M < +\infty$ such that $\max\phi_s < M$ for all $s \in W_2$.*

Then, there exists a family $\{\widetilde{\phi}_s\colon \mathbb{P}^1 \to \mathbb{R}\}_{s\in W_1}$ of β-Hölder continuous functions with the property that $s \mapsto \widetilde{\phi}_s$ is continuous with respect to the $(\beta/2)$-Hölder norm and we have $\max\widetilde{\phi}_s < M$ and $\widetilde{\phi}_s = \phi_s$ on $J(f)$ for all $s \in W_1$.

Proof. For a single β-Hölder continuous map $\phi\colon J(f) \to \mathbb{R}$, the classical McShane-Whitney theorem [McS34; Whi34] gives a β-Hölder continuous extension $\tilde{\phi}\colon \mathbb{P}^1 \to \mathbb{R}$ with $\tilde{\phi} \le \max\phi$. Recall that such extension is given by the explicit formula

$$\tilde{\phi}(z) := \max_{x\in J(f)} \left(\phi(x) - C\mathrm{dist}_{\mathbb{P}^1}(x,z)^\beta\right),$$

where C is a bound for the β-Hölder semi-norm of ϕ. Observe that the modulus of continuity of $\tilde{\phi}$ is the same as that of ϕ, i.e., by construction, the β-Hölder semi-norm of $\tilde{\phi}$ is equal to the β-Hölder semi-norm of ϕ. In the following, for simplicity, we denote by $\|\cdot\|'_{C^\beta}$ the β-Hölder semi-norm, and observe that we have $\|\cdot\|_{C^\beta} = \|\cdot\|_{L^\infty} + \|\cdot\|'_{C^\beta}$.

Let us now consider a family ϕ_s as in the statement. Consider the compact metric space $\overline{W_1} \times \mathbb{P}^1$, endowed with the product metric of $\overline{W_1}$ (with the standard Euclidean distance dist_E) and $\mathbb{P}^1$ (with the spherical distance), and its compact subset $\overline{W_1} \times J(f)$. Define the function $\Phi\colon \overline{W_1} \times J(f)$ by $\Phi(s,z) = \phi_s(z)$. Observe that Φ is β-Hölder continuous, and let C be a bound for its β-Hölder semi-norm. Consider the McShane-Whitney extension of Φ to $\overline{W_1} \times \mathbb{P}^1$, which is given by

$$\tilde{\Phi}(s,z) = \sup_{(t,x)\in\overline{W_1}\times J(f)} \phi_s(x) - C\big((\mathrm{dist}_E(s,t) + \mathrm{dist}_{\mathbb{P}^1}(z,x)\big)^\beta.$$

For $s \in W_1$, set $\tilde{\phi}_s(\cdot) := \tilde{\Phi}(s,\cdot)$. The upper bound for these functions as in the statement follows from the inequality $\max\tilde{\Phi} \le \max\Phi$.

We now show the continuity of the family $\{\tilde{\phi}_s\}$ with respect to the $(\beta/2)$-Hölder norm. It follows from their definition that the $\tilde{\phi}_s$'s satisfy

$$\|\tilde{\phi}_{s_1} - \tilde{\phi}_{s_2}\|_{L^\infty} \to 0 \quad \text{as} \quad \mathrm{dist}_E(s_1,s_2) \to 0. \tag{2}$$

Given $s_1, s_2 \in W_1$ and $z_1, z_2 \in \mathbb{P}^1$, we will show the inequality

$$|\phi_{s_1}(z_1) - \phi_{s_2}(z_1) - \phi_{s_1}(z_2) - \phi_{s_2}(z_2)| \leq 2C(\mathrm{dist}_{\mathbb{P}^1}(z_1, z_2))^{\beta/2}(\mathrm{dist}_E(s_1, s_2))^{\beta/2}. \tag{3}$$

Together with (2), this inequality implies that we have

$$\|\phi_{s_1} - \phi_{s_2}\|_{C^{\beta/2}} \to 0 \quad \text{as} \quad \mathrm{dist}_E(s_1, s_2) \to 0,$$

and hence the desired continuity with respect to the $(\beta/2)$-Hölder norm.

Observe first that, as $\|\tilde{\Phi}\|'_{C^\beta} \leq C$, we have $\|\tilde{\phi}_s\|'_{C^\beta} = \|\tilde{\Phi}(s, \cdot)\|'_{C^\beta} \leq C$ for all $s \in W_1$ and $\|\tilde{\Phi}(\cdot, z)\|'_{C^\beta} \leq C$ for all $z \in \mathbb{P}^1$. We need to consider the following two cases.

Case 1: $\mathrm{dist}_E(s_1, s_2) \geq \mathrm{dist}_{\mathbb{P}^1}(z_1, z_2)$**.**

In this case, we can bound the left-hand side of (3) as

$$\begin{aligned}|\phi_{s_1}(z_1) - \phi_{s_2}(z_1) - \phi_{s_1}(z_2) - \phi_{s_2}(z_2)| &\leq (\|\tilde{\Phi}(s_1, \cdot)\|'_{C^\beta} + \|\tilde{\Phi}(s_2, \cdot)\|'_{C^\beta})(\mathrm{dist}_{\mathbb{P}^1}(z_1, z_2))^\beta \\ &\leq 2C(\mathrm{dist}_{\mathbb{P}^1}(z_1, z_2))^\beta \\ &\leq 2C(\mathrm{dist}_E(s_1, s_2))^{\beta/2}(\mathrm{dist}_{\mathbb{P}^1}(z_1, z_2))^{\beta/2}.\end{aligned}$$

Case 2: $\mathrm{dist}_E(s_1, s_2) \leq \mathrm{dist}_{\mathbb{P}^1}(z_1, z_2)$**.**

Reversing the roles of the s_j's and z_j's, we now bound the left-hand side of (3) as

$$\begin{aligned}|\phi_{s_1}(z_1) - \phi_{s_2}(z_1) - \phi_{s_1}(z_2) - \phi_{s_2}(z_2)| &\leq (\|\tilde{\Phi}(\cdot, z_1)\|'_{C^\beta} + \|\tilde{\Phi}(\cdot, z_2)\|'_{C^\beta})(\mathrm{dist}_E(s_1, s_2))^\beta \\ &\leq 2C(\mathrm{dist}_E(s_1, s_2))^\beta \\ &\leq 2C(\mathrm{dist}_E(s_1, s_2))^{\beta/2}(\mathrm{dist}_{\mathbb{P}^1}(z_1, z_2))^{\beta/2}.\end{aligned}$$

Hence, (3) holds in either case. The proof is complete. □

Lemma 2.3. *Let $\phi\colon J(f) \to \mathbb{R}$ and $\tilde{\phi}\colon \mathbb{P}^1 \to \mathbb{R}$ be continuous functions with $\tilde{\phi}_{|J(f)} = \phi$ and $\max \tilde{\phi} = \max \phi < \mathcal{P}(f_{|J(f)}, \phi)$. Then ϕ and $\tilde{\phi}$ have the same equilibrium states, which are supported on $J(f)$. In particular, we have $\mathcal{P}(f_{|J(f)}, \phi) = \mathcal{P}(f, \tilde{\phi})$.*

Proof. It follows from the definitions that any equilibrium state for $\tilde{\phi}$ supported on $J(f)$ is also an equilibrium state for ϕ. Hence, it is enough to show that we have

$$h_\nu(f) + \int \tilde{\phi} d\nu < \mathcal{P}(f, \tilde{\phi})$$

for every invariant probability measure ν whose support is outside $J(f)$. Any such invariant measure necessarily satisfies $h_\nu(f) = 0$. It then follows from the assumptions that we have

$$\int \tilde{\phi} d\nu \leq \max \tilde{\phi} = \max \phi < \mathcal{P}(f_{|J(f)}, \phi) \leq \mathcal{P}(f, \tilde{\phi}).$$

The assertion follows. □

Combining [BD24] with [IR12, Main Theorem], one can see that the condition $\max_{z \in \mathbb{P}^1} \phi - \min_{z \in \mathbb{P}^1} \phi < \log D$ can be weakened to $\max_{z \in \mathbb{P}^1} \phi < \mathcal{P}(\phi)$ in both **(N4)** and **(N5)** (and actually to $\max_{z \in J(f)} \phi < \mathcal{P}(\phi)$). More precisely, we have the following lemma.

Lemma 2.4. *Let f be a rational map of degree $D \geq 2$ and $\phi\colon J(f) \to \mathbb{R}$ a β-Hölder continuous function. If the Lyapunov exponent of each equilibrium state of ϕ is strictly positive, then there exists a β-Hölder continuous extension $\tilde{\phi}$ of ϕ to $\mathbb{P}^1$ such that the transfer operator $\mathcal{L}_{\tilde{\phi}}\colon B_\diamond \to B_\diamond$ has a spectral gap, and the function $t \mapsto \mathcal{L}_{\tilde{\phi}+t\psi}$ is analytic as operators in $L(B_\diamond)$ near $t = 0$ for every $\psi \in B_\diamond$. Here $\|\cdot\|_\diamond$ is chosen so that $\|\cdot\|_\diamond \lesssim \|\cdot\|_{C^\beta}$ in* **(N1)**.

Proof. Suppose that the Lyapunov exponent of each equilibrium state of ϕ is strictly positive. Then, by [IR12, Main Theorem], the potential ϕ is *hyperbolic*, namely, there exists an integer $n \geq 1$ such that $\max_{z\in J(f)} S_n\phi < \mathcal{P}(f^n, S_n\phi)$, where $S_n\phi$ denotes the n-th Birkhoff sum $\phi + \phi\circ f + \cdots + \phi \circ f^{n-1}$ of ϕ. Up to replacing f by f^n (which does not change the statistical properties in the statement), we can assume that $n = 1$, i.e., that we have $\max_{z\in J(f)} \phi < \mathcal{P}(f, \phi)$. Let $\tilde\phi$ be a β-Hölder continuous extension of ϕ as in Lemma 2.2. In particular, we have $\tilde\phi \leq \max_{J(f)} \phi$, which gives $\tilde\phi \leq \mathcal{P}(f, \phi) = \mathcal{P}(f, \tilde\phi)$, where the last equality follows from Lemma 2.3.

We now observe that, in [BD23] and [BD24], the assumption $\max \tilde\phi - \min \tilde\phi < \log D$ can be replaced by $\max \tilde\phi < \mathcal{P}(f, \tilde\phi)$ as soon as the last quantity has already been defined (this is not the case in [BD23; BD24], hence the need for an assumption not directly involving the pressure). In particular, the same proof as in [BD24] gives the spectral gap for the action of $\mathcal{L}_{\tilde\phi}$ with respect to the norm $\|\cdot\|_\diamond$ chosen as in the statement, and the other properties. The assertion follows. ☐

In particular, the above lemmas allow us to apply the machinery of [BD23; BD24] to functions which are a priori defined only on $J(f)$ (or which are Hölder continuous only on $J(f)$). More precisely, for every $g\colon J(f) \to \mathbb{R}$, define

$$\|g\|_\diamond := \inf\{\|\tilde g\|_\diamond \colon \tilde g\colon \mathbb{P}^1 \to \mathbb{R}, \quad \tilde g_{|J(f)} = g\}$$

and

$$B_\diamond(J) := \{g\colon J(f) \to \mathbb{R} : \|g\|_\diamond < \infty\}. \tag{4}$$

By Lemma 2.2, every β-Hölder continuous function on $J(f)$ belongs to $B_\diamond(J)$. By **(N1)**, every element of $B_\diamond(J)$ is q-log-Hölder-continuous for some $q > 2$ depending on the choice of the norm $\|\cdot\|_\diamond$. Given $\phi, g\colon J(f) \to \mathbb{R}$, it is clear that we have $\|\mathcal{L}_\phi g\|_\diamond \leq \|\mathcal{L}_{\tilde\phi}\tilde g\|_\diamond$ for every extensions $\tilde\phi, \tilde g$ of ϕ and g. In particular, for every $\phi \in B_\diamond(J)$ with $\max\phi < \mathcal{P}(f, \phi)$, the operator $\mathcal{L}_\phi$ is bounded and has a spectral gap on $B_\diamond(J)$, with the largest eigenvalue equal to $e^{\mathcal{P}(f_{|J(f)},\phi)} = e^{\mathcal{P}(f,\tilde\phi)}$.

Definition 2.5. Given a continuous function $\psi : J(f) \to \mathbb{R}$ and an invariant probability measure m supported on $J(f)$, we define formally

$$\mathrm{Var}(\psi, m) := \lim_{n\to\infty} \frac{1}{n} \int_{J(f)} \left| \sum_{i=0}^{n-1} \psi \circ f^i(x) \right|^2 dm \in [0, +\infty].$$

The following proposition is a direct consequence of **(N4)** and **(N5)**, together with the above extension lemmas; see for instance [McM08; PP90].

Proposition 2.6. *Let $\{\phi_t\}_{t\in(-1,1)}$ be a smooth path in $C^\alpha(J(f), \mathbb{R})$ for some $\alpha \in (0, 1)$. Suppose that ϕ_0 admits a unique equilibrium state $m = m(\phi_0)$ and that m has a strictly positive Lyapunov exponent. Set $\dot\phi_0 = d\phi_t/dt|_{t=0}$. Then we have*

$$\left.\frac{d\mathcal{P}(\phi_t)}{dt}\right|_{t=0} = \int_{J(f)} \dot\phi_0 dm. \tag{5}$$

If the expressions in (5) *are equal to zero, then we have*

$$\left.\frac{d^2\mathcal{P}(\phi_t)}{dt^2}\right|_{t=0} = \mathrm{Var}(\dot\phi_0, m) + \int \ddot\phi_0 dm.$$

In the statement above and in the rest of the paper, the notation $\dot\phi_0 = d\phi_t/dt|_{t=0}$ stands for the *function* which is defined pointwise as $\dot\phi_0(z) = d\phi_t(z)/dt|_{t=0}$ for every $z \in J(f)$.

Lemma 2.7. *Fix $\phi, \psi \in C^\alpha(J(f))$. Suppose that ϕ admits a unique equilibrium state $m = m(\phi)$ and that m has a strictly positive Lyapunov exponent. Then we have $\mathrm{Var}(\psi, m(\phi)) = 0$ if and only if ψ is C^0-cohomologous to 0, i.e., there exists a continuous function $h : J(f) \to \mathbb{R}$ such that $\psi = h - h \circ f$.*

Proof. It is a standard fact that we have $\mathrm{Var}(\psi, m) = 0$ if and only if ψ is a $L^2(m)$-coboundary, i.e., if there exists $h \in L^2(m)$ such that $\psi = h - h \circ f$. In order to conclude the proof, it is enough to show that a L^2-coboundary is a C^0-coboundary. The proof is a standard application of the spectral gap for the norm $\| \cdot \|_\diamond$, see for instance [FMT03, Lemma 3.4 and Corollary 3.5] and [BD24, Proposition 5.9]. □

Finally, let W be an open subset of $\mathbb{C}^\ell$ for some $\ell \geq 1$. We say that a map $W \ni w \mapsto \mathcal{L}_w \in L(B_\diamond)$ is *holomorphic* if for every $w \in W$, there exists $\mathcal{L}'_w \in L(B_\diamond)$ such that $||h^{-1}(\mathcal{L}_{(w+h)} - \mathcal{L}_w) - \mathcal{L}'_w||_\diamond \to 0$ as $h \to 0$. The following proposition gives a simple condition to ensure that this is the case for the transfer operators associated to some family of functions ζ_λ, giving an improvement of **(N4)** when the parameter is complex and the dependence of the potential on the parameter is sufficiently regular.

Proposition 2.8. *Let $\{\zeta_w\}_{w \in W}$ be a family of functions in $B_\diamond$ such that*

(1) there exists $w_0 \in W$ such that Lyapunov exponent of each equilibrium state of ζ_{w_0} is strictly positive,
(2) the function $w \mapsto \zeta_w$ is continuous with respect to $\| \cdot \|_\diamond$;
(3) for every $z \in J(f)$, the function $W \ni w \mapsto \zeta_w(z)$ is holomorphic.

Then, the map $W \ni w \mapsto \mathcal{L}_{\zeta_w|_{J(f)}} \in L(B_\diamond(J))$ is holomorphic.

Proof. By the first assumption and Lemma 2.4, we have $\mathcal{L}_{\zeta_{w_0}} \in L(B_\diamond)$. It is enough to show that the map $W \ni w \mapsto \mathcal{L}_{\zeta_w} \in L(B_\diamond)$ is continuous as this, together with the third assumption, implies the holomorphicity of $\mathcal{L}_w$; see for instance [UZ04, Lemma 7.1] or [SU10, Lemma 5.1].

In order to show the continuity of $w \mapsto \mathcal{L}_w$, by the property **(N3)** of the norm $\| \cdot \|_\diamond$, for every $w, w' \in W$ we have

$$\|(\mathcal{L}_w - \mathcal{L}_{w'})g\|_\diamond = \|f_*\big((e^{\zeta_w} - e^{\zeta_{w'}})g\big)\|_\diamond \lesssim \|(e^{\zeta_w} - e^{\zeta_{w'}})g\|_\diamond,$$

where the implicit constant is independent of w, w'. It follows from the above estimates and **(N2)** that we have

$$\|(\mathcal{L}_w - \mathcal{L}_{w'})g\|_\diamond \lesssim \|e^{\zeta_w} - e^{\zeta_{w'}}\|_\diamond \|g\|_\diamond \lesssim \|e^{\zeta_w}\|_\diamond \|1 - e^{\zeta_{w'} - \zeta_w}\|_\diamond \|g\|_\diamond \lesssim e^{\|\zeta_w\|_\diamond} \|1 - e^{\zeta_{w'} - \zeta_w}\|_\diamond \|g\|_\diamond,$$

where again the implicit constants are independent of w, w'. The assertion follows from the fact that $e^t - 1 \lesssim t$ for small $t > 0$, **(N2)**, and the assumption on the continuity of ζ_w with respect to $\| \cdot \|_\diamond$. □

2.3. **Applications to $\phi = -\theta \log |f'|$.** When f only has attracting or parabolic periodic points, the function $\phi = -\log |f'|$ is smooth on $J(f)$. Up to redefining it outside of some neighbourhood of $J(f)$, the results of the previous section apply, up to multiplying it by a sufficiently small constant (in order to verify the requirement $\max \phi < \mathcal{P}(f, \phi)$). We then have the following lemmas, which we will refer to several times in the sequel. They are both well-known (see for example [DU91]), but we give the proofs to show how they can be deduced from the formalism of the previous section.

Lemma 2.9. *Let $\theta > 0$ be such that $\mathcal{P}(-\theta \log |f'|) > 0$. Then every equilibrium state of the function $-\theta \log |f'| : J(f) \to \mathbb{R}$ has strictly positive Lyapunov exponent.*

Proof. We have

$$0 < \mathcal{P}(-\theta \log |f'|) = h_\nu - \theta L_\nu, \qquad (6)$$

where ν is an equilibrium state of $-\theta \log |f'|$ and h_ν and L_ν are the measure-theoretic entropy and the Lyapunov exponent of ν, respectively. Since ν is supported on the Julia set J, its Lyapunov exponent is non-negative, i.e., we have $L_\nu \geq 0$ [Prz93]. Hence, as $\theta > 0$, (6) implies that $h_\nu > 0$. Since the Hausdorff dimension of ν is non-negative and satisfies $\text{H.dim}(\nu) \cdot L_\nu = h_\nu$, this gives $L_\nu > 0$. □

Lemma 2.10. *The function $\theta \mapsto \mathcal{P}(-\theta \log |f'|)$ is strictly decreasing as θ increases from 0 to the first zero of the function $\mathcal{P}(-\theta \log |f'|)$.*

Proof. Take $\theta_0 \in [0, T)$, where $T > 0$ is the first zero of the function $\theta \mapsto \mathcal{P}(-\theta \log |f'|)$. We apply Proposition 2.6 with $\phi_s = -(\theta_0 + s) \log |f'|$, for s in a sufficiently small neighbrouhood of 0. Observe that $\phi_0 = -\theta_0 \log |f'|$. Since $\mathcal{P}(\phi_0) > 0$, by Lemmas 2.9 and 2.4 ϕ_0 has a unique equilibrium state $\nu = \nu(\phi_0)$, whose Lyapunov exponent L_ν is strictly positive.

By (5), we have

$$\frac{d\mathcal{P}(-\theta \log |f'|)}{dt}\bigg|_{\theta=\theta_0} = \frac{d\mathcal{P}(\phi_s)}{ds}\bigg|_{s=0} = -\int_{J(f)} \log |f'| d\nu = -L_\nu < 0.$$

The conclusion follows. □

By [SU03], the pressure function $t \mapsto \mathcal{P}(-t \log |f'|)$ is actually analytic on the interval between 0 and its first zero. We will reprove this fact in Section 3, where we will show a stronger joint analyticity property for the pressure function in both t and f.

2.4. **Extension of analytic functions.** In later sections, we will often need to extend a family of transfer operators, depending analytically on some real parameters, to a *holomorphic* family of transfer operators, in order to apply the perturbation theory on the spectra of the operators. To achieve this, we will use Lemma 2.11 below. We also refer to [Rug08] for a different method of extension.

For every integer $d \geq 1$, consider the embedding $\iota_d \colon \mathbb{C}^d \to \mathbb{C}^{2d}$ given by

$$(x_1 + iy_1, \ldots, x_d + iy_d) \mapsto (x_1, y_1, \ldots, x_d, y_d). \tag{7}$$

Observe, in particular, that $\mathbb{C}^d$ is embedded by ι_d in $\mathbb{C}^{2d}$ as the set of points of real coordinates (which, in turn, is parametrized by $\mathbb{C}^d$ by means of ι_d). For every $z \in \mathbb{C}^\ell$ and every $r > 0$, denote by $D_\ell(z, r)$ the ℓ-dimensional polydisk in $\mathbb{C}^\ell$ centered at z and with radius r. Observe that, with the above identification, we have $\iota_d(D_d(0, r)) \subset D_{2d}(0, r)$ and, more generally, $\iota_d(D_d(z, r)) \subset D_{2d}(\iota_d(z), r)$ for every $z \in \mathbb{C}^d$.

Lemma 2.11 ([SU10, Lemma 6.4])**.** *For every $M \geq 0$, $R > 0$, $\lambda_0 \in \mathbb{C}^d$, and every complex analytic function $\phi : D_d(\lambda_0, R) \to \mathbb{C}$ which is bounded in modulus by M, there exists a complex analytic function $\tilde{\phi} : D_{2d}(\iota_d(\lambda_0), R/4) \to \mathbb{C}$ that is bounded in modulus by $4^d M$ and such that the restriction of $\tilde{\phi} \circ \iota_d$ to $D_d(\lambda_0, R/4)$ coincides with the real part $\Re(\phi)$ of ϕ.*

3. Analyticity of the Bowen function

We fix in this section a parabolic subfamily Λ of rat_D^{cm}. Recall that this implies that a (possibly empty) subset of the critical points is persistently contained in parabolic basins. In particular, these critical points are passive on all of Λ. We also fix a Λ-hyperbolic component Ω in Λ. Recall that we denote by J_λ the Julia set of f_λ. Observe also that, for every $\lambda \in \Omega$, the map $\log |f'_\lambda|$ is Hölder continuous in a neighbourhood of J_λ.

Fix $\eta \in (0, \log D)$. Given $\lambda \in \Omega$, define $\delta_\eta(\lambda)$ to be the unique real number such that

$$\mathcal{P}(-\delta_\eta(\lambda) \log |f'_\lambda|) = \eta.$$

Note that the number $\delta_\eta(\lambda)$ exists and is unique as the pressure function $t \mapsto \mathcal{P}(-t\log|f'_\lambda|)$ is strictly decreasing as t increases from 0 up to the first zero of $\mathcal{P}$ by Lemma 2.10. Moreover, we have $0 < \delta_\eta(\lambda) \leq 2$ for every $0 < \eta < \log D$ and $\lambda \in \Omega$. If Λ and Ω are such that f_λ is hyperbolic for every $\lambda \in \Omega$ and we take $\eta = 0$, the number $\delta_0(\lambda)$ is called the *Bowen parameter*, as in this case, Bowen proved that $\delta_0(\lambda)$ equals the Hausdorff dimension of the Julia set of f_λ; see [Bow79]. In our case, abusing notation, we call $\delta_\eta(\lambda)$ the Bowen number of f_λ. The main result of this section is the following proposition, which generalizes Ruelle's result [Rue82] to every Λ-hyperbolic component.

Proposition 3.1. *The Bowen function $\delta_\eta : \Omega \to \mathbb{R}$ given by $\lambda \mapsto \delta_\eta(\lambda)$ is real-analytic.*

Fix $\lambda_0 \in \Omega$ and denote $\ell := \dim_{\mathbb{C}}\Omega$. In the following, we will work in a chart of Ω centred at λ_0. In particular, we can assume that $\lambda_0 = 0$ and $\lambda \in D_\ell(0,1)$. Let $R_0 < 1$ be a real number such that the map $\Psi_\lambda : J_0 \to J_\lambda$ conjugating the dynamical systems (J_0, f_0) and (J_λ, f_λ) is well-defined for all $\lambda \in D_\ell(0, R_0)$. For $\lambda \in D_\ell(0,R_0)$ and $\theta \in \mathbb{R}$, consider the potential function $\phi_{(\theta,\lambda)} : J_0 \to \mathbb{R}$ given by

$$\phi_{(\theta,\lambda)}(z) := -\theta \log\left|f'_\lambda \circ \Psi_\lambda(z)\right|.$$

Proposition 3.1 follows from the following proposition. Indeed, once we establish that the function $(\theta,\lambda) \mapsto \mathcal{P}(\phi_{(\theta,\lambda)})$ is real-analytic in a neighbourhood of $(\delta_\eta(0), 0)$, since $\{\eta\}$ is a closed set in $\mathbb{R}$ and we have $\frac{d}{d\theta}\big|_{\theta=\delta_\eta(\lambda)}\mathcal{P}(-\theta\log|f'_\lambda|) < 0$ for every λ by Lemma 2.10, it follows from the analytic implicit function theorem that the function $\lambda \mapsto \delta_\eta(\lambda)$ is analytic on Ω.

Proposition 3.2. *There exists $R > 0$ such that the function $(\theta,\lambda) \mapsto \mathcal{P}(\phi_{(\theta,\lambda)})$ is real-analytic for $(\theta,\lambda) \in (\delta_\eta(0) - R, \delta_\eta(0) + R) \times D_\ell(0,R)$.*

The rest of this section is devoted to the proof of Proposition 3.2. Our proof follows the same general outline as that of [SU10, Proof of Theorem A] or [UZ04, Proof of Theorem 9.3]. The spectral gap property for the transfer operators $\mathcal{L}_{\phi(\theta,\lambda)}$ with respect to the norm $\|\cdot\|_\diamond$ as in Section 2.2 will allow us to adapt those arguments in our context.

For every $z \in J_0$, consider the map

$$\psi_z(\lambda) := \frac{f'_\lambda \circ \Psi_\lambda(z)}{f'_0(z)}.$$

It follows from the absolute continuity of the holomorphic motion of the Julia sets that, shrinking R_0 if necessary, for all $z \in J_0$ and $\lambda \in D_\ell(0,R_0)$, we have

$$|\psi_z(\lambda) - 1| < 1/5.$$

Then, for every $z \in J_0$, there exists a branch of $\log\psi_z$ sending 0 to 0 and whose modulus is bounded by 1/4. Applying Lemma 2.11 to the complex analytic function $\log\psi_z$, we see that the real analytic function $\Re\log\psi_z : D_\ell(0,R_0) \to \mathbb{R}$ has an analytic extension $\widetilde{\Re\log\psi_z} : D_{2\ell}(0,R') \to \mathbb{R}$ for some $R' \in (0,R_0)$. Recall that $D_\ell(0,R_0)$ is seen as a subset of the points of $\mathbb{C}^{2\ell}$ with real coordinates by means of the immersion ι_ℓ as in (7), and that we have $\iota_\ell(0) = 0 \in \mathbb{C}^{2\ell}$.

For $(\theta,\lambda) \in \mathbb{C} \times D_{2\ell}(0,R')$, define $\zeta_{(\theta,\lambda)} : J_0 \to \mathbb{C}$ by

$$\zeta_{(\theta,\lambda)}(z) := -\theta\widetilde{\Re\log\psi_z}(\lambda) + \theta\log|f'_0(z)|.$$

Note that $\zeta_{(\theta,\lambda)} = \phi_{(\theta,\lambda)}$ for every $(\theta,\lambda) \in \mathbb{R} \times D_\ell(0,R')$.

Let β be such that the conjugacy map $\Psi_\lambda : J_0 \to J_\lambda$ is β-Hölder continuous for all $\lambda \in D_\ell(0,R')$. We first note that the map $(\theta,\lambda) \mapsto \zeta_{(\theta,\lambda)}$ is continuous with respect to the β-Hölder norm (see for instance [SU10, Lemma 6.6] for a similar computation). Applying Lemma 2.2 to the families $\Re\zeta_{(\theta,\lambda)}$ and $\Im\zeta_{(\theta,\lambda)}$, and shrinking R', gives an extended family $\mathbb{C} \times D_{2\ell}(0,R') \ni (\theta,\lambda) \mapsto \widetilde{\zeta}_{(\theta,\lambda)} : \mathbb{P}^1 \to \mathbb{C}$

which is continuous with respect to the $\beta/2$-Hölder norm and satisfies $\widetilde{\zeta}_{(\theta,\lambda)}(z) = \zeta_{(\theta,\lambda)}(z)$ for every $(\theta, \lambda) \in \mathbb{C} \times \iota_1^{-1} D_{2\ell}(0, R')$ and $z \in J_0$.

Fix a norm $||\cdot||_\diamond$ as in Section 2.2 such that $||\cdot||_\diamond \lesssim ||\cdot||_{C^{\beta/2}}$ and recall that we denote by $B_\diamond(J_0)$ the Banach space defined by the norm $||\cdot||_\diamond$ as in (4), and by $L(B_\diamond(J_0))$ the space of the bounded linear operators on $(B_\diamond(J_0), \|\cdot\|_\diamond)$. For $(\theta, \lambda) \in \mathbb{C} \times D_{2\ell}(0, R')$, consider the complex transfer operator $\mathcal{L}_{(\theta,\lambda)} := \mathcal{L}_{\widetilde{\zeta}_{(\theta,\lambda)}}$.

Lemma 3.3. *There exists $R'' \in (0, R')$ such that the map $D_\ell(\delta_\eta(0), R'') \times D_{2\ell}(0, R'') \ni (\theta, \lambda) \mapsto \mathcal{L}_{(\theta,\lambda)} \in L(B_\diamond(J))$ is holomorphic.*

In the proof of the above lemma, we will use the fact that we have $\mathcal{P}(\phi_{(\delta_\eta(\lambda),\lambda)}) \equiv \eta > 0$ to verify the assumptions of the lemmas in Section 2.3, and hence of Proposition 2.8. This is the reason why we take $\eta \in (0, \log D)$, rather than $\eta = 0$ as in Bowen's paper [Bow79].

Proof. We verify that the three assumptions of Proposition 2.8 are satisfied when considering the family of weights $\widetilde{\zeta}_{(\theta,\lambda)}$ instead of ζ_w.

For (1), since we have $\mathcal{P}(\phi_{(\delta_\eta(0),0)}) = \eta > 0$, Lemma 2.9 ensures that the desired property holds at the parameter $w_0 = (\delta_\eta(0), 0)$.

For (2), we observe that, up to choosing $R'' \in (0, R')$ sufficiently small, the family $(\theta, \lambda) \mapsto \widetilde{\zeta}_{(\theta,\lambda)}$ is continuous with respect to the $\beta/2$-Hölder norm by Lemma 2.2. The desired condition then holds thanks to the bound $||\cdot||_\diamond \lesssim ||\cdot||_{C^{\beta/2}}$ which we required in the choice of $\|\cdot\|_\diamond$.

For (3), for each fixed $z \in J_0$, the map $(\theta, \lambda) \mapsto \zeta_{(\theta,\lambda)}(z)$ is holomorphic by definition.

As all the assumptions of Proposition 2.8 are satisfied, the assertion follows from Proposition 2.8. □

We can now conclude the proof of Proposition 3.2, which also concludes the proof of Proposition 3.1.

End of the proof of Proposition 3.2. We let R'' be as in Lemma 3.3 and denote by $\mathcal{L}_{\zeta_{(\theta,\lambda)}} \in L(B_\diamond(J))$ the transfer operator associated to $(\theta, \lambda) \in D_\ell(\delta_\eta(0), R'') \times D_\ell(0, R'')$. For every $(\theta, \lambda) \in \mathbb{R} \times D_\ell(\lambda_0, R'')$, the value $e^{P(\zeta_{(\theta,\lambda)})}$ is a simple isolated eigenvalue of the transfer operator $\mathcal{L}_{\zeta_{(\theta,\lambda)}}$. It follows from Lemma 3.3 and the Kato-Rellich perturbation theorem [Kat95] that there exists $0 < R < R''$ and an analytic function $\mathcal{A} : D_\ell(\delta_\eta(0), R) \times D_{2\ell}(0, R) \to \mathbb{C}$ such that $\mathcal{A}(\theta, \lambda)$ is a simple isolated eigenvalue for $\mathcal{L}_{(\theta,\lambda)}$ for every $(\theta, \lambda) \in D_\ell(\delta_\eta(0), R) \times D_{2\ell}(0, R)$ and $e^{P(\zeta_{(\theta,\lambda)})} = \mathcal{A}(\theta, \lambda)$ for every $(\theta, \lambda) \in (\delta_\eta(0) - R, \delta_\eta(0) + R) \times D_{2\ell}(0, R)$. The assertion follows by possibly reducing R, taking the logarithm of the function $\mathcal{A}$, and recalling that $D_\ell(0, R)$ is identified to a subset of $\mathbb{C}^{2\ell}$ by means of the map ι_ℓ. The proof is complete. □

4. The Hessian form $\langle\cdot,\cdot\rangle_G$ and the pressure form $\langle\cdot,\cdot\rangle_{\mathcal{P}}$

We continue to use the notations from the previous section. We fix a Λ-hyperbolic component Ω of a parabolic family $\Lambda \subset \mathrm{rat}_D^{cm}$. In this section, we show that the construction in [HN23a] of a positive semi-definite symmetric bilinear form $\langle\cdot,\cdot\rangle_G$ for hyperbolic components in the moduli space of rational maps can be extended to Ω. To this end, we first construct a positive semi-definite symmetric bilinear form $\langle\cdot,\cdot\rangle_G$ on $\widetilde{\Omega}$, which is a lift of Ω in the parameter space Rat_D^{cm} and show that it descends to a positive semi-definite symmetric bilinear form on Ω; see Sections 4.1 and 4.2. We construct the pressure form $\langle\cdot,\cdot\rangle_{\mathcal{P}}$ on Ω in Section 4.3 and prove the conformal equivalence of $\langle\cdot,\cdot\rangle_G$ and $\langle\cdot,\cdot\rangle_{\mathcal{P}}$ in Section 4.4.

4.1. **The Hessian 2-form $\langle\cdot,\cdot\rangle_G$ on $\widetilde{\Omega}$.** Recall that J_λ denotes the Julia set of f_λ, for $\lambda \in \widetilde{\Omega}$. Fix $\lambda_0 \in \widetilde{\Omega}$. Let $U(\lambda_0)$ be a neighborhood of λ_0 such that the conjugacy $\Psi_\lambda : J_{\lambda_0} \to J_\lambda$ is well-defined for any $f_\lambda \in U(\lambda_0)$. Note that we can take $U(\lambda_0) = \widetilde{\Omega}$ if $\widetilde{\Omega}$ is simply connected. Fix $\eta \in (0, \log D)$. For simplicity, we denote by $\nu := \nu_{\eta,\lambda_0}$ the unique equilibrium state on J_{λ_0} for the potential $-\delta_\eta(\lambda_0)\log|f'_{\lambda_0}|$. We note that $-\delta_\eta(\lambda_0)\log|f'_{\lambda_0}|$ has a unique equilibrium state by the fact that $\mathcal{P}(-\delta_\eta(\lambda_0)\log|f'_{\lambda_0}|) = \eta > 0$, see Lemmas 2.9 and 2.4.

Definition 4.1. The function $\mathrm{Ly}_{\lambda_0} : U(\lambda_0) \to \mathbb{R}$ is defined as

$$\mathrm{Ly}_{\lambda_0}(\lambda) := \int_{J_\lambda} \log|f'_\lambda| d\left((\Psi_\lambda)_*\nu\right) = \int_{J_{\lambda_0}} \log|f'_\lambda \circ \Psi_\lambda| d\nu.$$

Lemma 4.2. *The function $\mathrm{Ly}_{\lambda_0} : U(\lambda_0) \to \mathbb{R}$ is harmonic. In particular, it is real-analytic.*

Proof. We show that $dd^c_\lambda \mathrm{Ly}_{\lambda_0}(\lambda) \equiv 0$, where dd^c_λ denotes the complex Laplacian. Denote by π_Ω and $\pi_{\mathbb{P}^1}$ the natural projection of $\Omega \times \mathbb{P}^1$ onto its factors. With these notations, we see that, formally, we have

$$dd^c_\lambda \mathrm{Ly}_{\lambda_0}(\lambda) = (\pi_\Omega)_*(dd^c_{\lambda,z}\log|f'_\lambda \circ \Psi_\lambda| \wedge (\pi_{\mathbb{P}^1})^*\nu). \tag{8}$$

For every $z_0 \in J_{\lambda_0}$, we have

$$dd^c_{\lambda,z}\log|f'_\lambda \circ \Psi_\lambda(z_0)| = 0$$

since the map $\lambda \mapsto f'_\lambda \circ \Psi_\lambda(z_0)$ is a non-vanishing holomorphic function on Ω for every $z_0 \in J_{\lambda_0}$. The assertion follows from Fubini's theorem, which also justifies that the currents and the equality in (8) are well-defined. □

Definition 4.3. The function $G_{\lambda_0} : U(\lambda_0) \to \mathbb{R}$ is defined as

$$G_{\lambda_0}(\lambda) := \delta_\eta(\lambda)\mathrm{Ly}_{\lambda_0}(\lambda) = \delta_\eta(\lambda)\int_{J_{\lambda_0}} \log|f'_\lambda \circ \Psi_\lambda| d\nu.$$

Lemma 4.4. *The function $G_{\lambda_0} : U(\lambda_0) \to \mathbb{R}$ is real-analytic.*

Proof. The statement follows from Proposition 3.1 and Lemma 4.2. □

Let $G : X \to \mathbb{R}$ be a smooth real-valued function on an ℓ-dimensional smooth manifold X. The *Hessian* of G at $x \in X$ with respect to a chart $u : U(x) \to \mathbb{R}^\ell$ is the map $G''_{(u)}(x) : T_xX \times T_xX \to \mathbb{R}$ represented by the 2-form

$$G''_{(u)}(x) = \sum_{i,j=1}^{\ell} \frac{\partial^2 G_{(u)}}{\partial u_i \partial u_j}(x) du_i \otimes du_j.$$

We note that if $u : U(x) \to \mathbb{R}^\ell$ and $v : V(x) \to \mathbb{R}^\ell$ are two different charts, we have $G''_{(u)}(x) = G''_{(v)}(x)$ as soon as $DG(x)$ vanishes, where $DG(x)$ is the differential $DG(x) : T_xX \to T_{G(x)}\mathbb{R}$ (which a priori depends on the coordinate charts, but the condition of its vanishing does not); see [BT08, Section 7]. If $x \in X$ is such that $DG(x) = 0$, then $G''(x) := G''_{(u)}(x)$ (with respect to any chart u) is a well-defined symmetric bilinear form $G''(x) : T_xX \times T_xX \to \mathbb{R}$ on T_xX. Moreover, if x is a local minimum for G, then $G''(x)$ is positive semi-definite.

In what follows, we will show that G_{λ_0} has a minimum at λ_0. Therefore we have $DG_{\lambda_0}(\lambda_0) = 0$ and the Hessian $G''_{\lambda_0}(\lambda_0) : T_{\lambda_0}\widetilde{\Omega} \times T_{\lambda_0}\widetilde{\Omega} \to \mathbb{R}$ is well-defined at λ_0 and is positive semi-definite.

Proposition 4.5. *Fix $\lambda_0 \in \widetilde{\Omega}$. We have $G_{\lambda_0}(\lambda_0) \leq G_{\lambda_0}(\lambda)$ for all $\lambda \in U(\lambda_0)$.*

Proof. Recall that we denote by ν the unique equilibrium state on J_{λ_0} for the weight $-\delta_\eta(\lambda_0)\log|f'_{\lambda_0}|$. To ease notation, we also set $\nu_\lambda := (\Psi_\lambda)_*\nu$ for every $\lambda \in U(\lambda_0)$. Then we have

$$\mathrm{Ly}(\nu, f_\lambda) = \int_{J_\lambda} \log|f'_\lambda| d\nu_\lambda \quad \text{for all } \lambda \in U(\lambda_0).$$

Since the weight $-\delta_\eta(\lambda_0)\log|f'_{\lambda_0}|$ has pressure η with respect to f_{λ_0} and $\nu = \nu_{\lambda_0}$ is its (unique) equilibrium state, by the definition of pressure we have

$$h_{\nu_{\lambda_0}}(f_{\lambda_0}) = \eta + \delta_\eta(\lambda_0)\int_{J_{\lambda_0}} \log|f'_{\lambda_0}| d\nu_{\lambda_0}. \tag{9}$$

Since $\nu = \nu_{\lambda_0}$ is f_{λ_0}-invariant, ν_λ is f_λ-invariant for every $\lambda \in U(\lambda_0)$. Since the measure-theoretic entropy is invariant under topological conjugacy, it follows that we have

$$h_{\nu_{\lambda_0}}(f_{\lambda_0}) = h_{(\Psi_\lambda)_*\nu_{\lambda_0}}(f_\lambda) = h_{\nu_\lambda}(f_\lambda). \tag{10}$$

Again by the definition of pressure and the identity $P(-\delta_\eta(\lambda)\log|f'_\lambda|) = \eta$, we have

$$h_{\nu_\lambda}(f_\lambda) \le \eta + \delta_\eta(\lambda)\int_{J_\lambda} \log|f'_\lambda| d\nu_\lambda \quad \text{for all } \lambda \in U(\lambda_0). \tag{11}$$

Combining (9), (10), and (11), we obtain the inequality $\delta_\eta(\lambda_0)\mathrm{Ly}(\nu, f_{\lambda_0}) \le \delta_\eta(\lambda)\mathrm{Ly}(\nu, f_\lambda)$. The assertion follows. □

Therefore, the Hessian of G_{λ_0} at λ_0 defines a positive semi-definite symmetric bilinear form $\langle\cdot,\cdot\rangle_G$ on the tangent space $T_{\lambda_0}\widetilde{\Omega}$ as follows.

Definition 4.6. For every $\vec{u}, \vec{v} \in T_{\lambda_0}\widetilde{\Omega}$, we define

$$\langle \vec{u}, \vec{v}\rangle_G := (G''_{\lambda_0}(\lambda_0))(\vec{u}, \vec{v}).$$

For every $\vec{v} \in T_{\lambda_0}\widetilde{\Omega}$, we will also denote $\|\vec{v}\|_G := \sqrt{\langle \vec{v}, \vec{v}\rangle_G}$.

Lemma 4.7. *Let $\gamma(t), t \in (-1,1)$ be a smooth path in $U(\lambda_0)$ with $\gamma(0) = \lambda_0$ and $\gamma'(0) = \vec{v} \in T_{\lambda_0}\widetilde{\Omega}$. Then*

$$||\vec{w}||_G^2 = \frac{d^2}{dt^2}\Big|_{t=0} G_{\lambda_0}(\gamma(t)).$$

Proof. For every γ as in the statement, we have

$$\frac{d^2}{dt^2}\Big|_{t=0} G_{\lambda_0}(\gamma(t)) = G''_{\lambda_0}(\gamma(0))(\gamma'(0), \gamma'(0)) + DG_{\lambda_0}(\gamma(0)) \cdot \gamma''(0) = G''_{\lambda_0}(\lambda_0)(\vec{v}, \vec{v}).$$

The second equality follows from the identity $DG_{\lambda_0}(\gamma(0)) = DG_{\lambda_0}(\lambda_0) = 0$ given by Proposition 4.5. The assertion follows. □

4.2. **The Hessian 2-form $\langle\cdot,\cdot\rangle_G$ on Ω.** We now show that the Hessian form $\langle\cdot,\cdot\rangle_G$ on (the tangent bundle of) $\widetilde{\Omega}$ descends to a symmetric bilinear form on (the tangent bundle of) Ω. We first need a preliminary lemma.

Lemma 4.8. *Fix $\lambda_0 \in \widetilde{\Omega}$. If $f_1, f_2 \in \widetilde{\Omega}$ are Möbius conjugate, then $\log|f'_1 \circ \Psi_{f_1}|$ and $\log|f'_2 \circ \Psi_{f_2}|$ are C^0-cohomologous. In particular, we have $\delta_\eta(\lambda_{f_1}) = \delta_\eta(\lambda_{f_2})$. Here $\Psi_{f_i} : J_{\lambda_0} \to J(f_i), i = 1, 2$ is the conjugacy map.*

Proof. Let g be a Möbius transformation such that $f_1 = g \circ f_2 \circ g^{-1}$ and observe that $g = \Psi_{f_1} \circ \Psi_{f_2}^{-1}$. Then, for every $x \in J_{\lambda_0}$, we have

$$\begin{aligned}\log|f_1' \circ \Psi_{f_1}(x)| &= \log|f_1' \circ g \circ \Psi_{f_2}(x)| = \log|(g \circ f_2 \circ g^{-1})'(g \circ \Psi_{f_2}(x))| \\ &= \log|g'(f_2 \circ \Psi_{f_2}(x))| + \log|f_2'(\Psi_{f_2}(x))| + \log|(g^{-1})'(g \circ \Psi_{f_2}(x))| \\ &= \log|g'(f_2 \circ \Psi_{f_2}(x))| + \log|f_2'(\Psi_{f_2}(x))| - \log|g'(\Psi_{f_2}(x))| \\ &= \log|g'(\Psi_{f_2} \circ f_{\lambda_0}(x))| + \log|f_2'(\Psi_{f_2}(x))| - \log|g'(\Psi_{f_2}(x))|.\end{aligned}$$

The first assertion follows by setting $h := \log|g'(\Psi_{f_2}(x))|$ and observing that h is continuous as g has no critical points. As the pressure function only depends on cohomology classes [PP90], we have $\mathcal{P}(-t\log|f_1' \circ \Psi_{f_1}|) = \mathcal{P}(-t\log|f_2' \circ \Psi_{f_2}|)$ for all $t \in \mathbb{R}$. Therefore, we have $\delta_\eta(f_1) = \delta_\eta(f_2)$ and the proof is complete. ☐

Lemma 4.9. *Fix $[\lambda] \in \Omega$ and $\vec{v} \in T_{[\lambda]}\Omega$. Let $\gamma(t), t \in (-1,1)$ be a smooth curve in Ω with $\gamma(0) = [\lambda]$ and $\gamma'(0) = \vec{v}$. If $\widetilde{\gamma}_1(t)$ and $\widetilde{\gamma}_2(t)$ are two smooth lifts of $\gamma(t)$ to $\widetilde{\Omega}$, then we have $G_{\widetilde{\gamma}_1(0)}(\widetilde{\gamma}_1(t)) = G_{\widetilde{\gamma}_2(0)}(\widetilde{\gamma}_2(t))$ for every $t \in (-1,1)$.*

Proof. Let $\widetilde{\gamma}_1(t)$ and $\widetilde{\gamma}_2(t)$ be two lifts of $\gamma(t)$ to $\widetilde{\Omega}$ as in the statement. Since the definitions are local, we may assume that we have $\widetilde{\gamma}_1(t) \subset U(\widetilde{\gamma}_1(0))$ and $\widetilde{\gamma}_2(t) \subset U(\widetilde{\gamma}_2(0))$ for all $t \in (-1,1)$, where the neighbourhood $U(\lambda)$ of $\lambda \in \widetilde{\Omega}$ is as in the previous section. Since $\widetilde{\gamma}_1(t)$ and $\widetilde{\gamma}_2(t)$ are Möbius conjugate, we have $\delta_\eta(\widetilde{\gamma}_1(t)) = \delta_\eta(\widetilde{\gamma}_2(t))$ for every $t \in (-1,1)$ by Lemma 4.8. Let $M : J_{\widetilde{\gamma}_2(0)} \to J_{\widetilde{\gamma}_1(0)}$ be the Möbius conjugacy map, and ν_i be the equilibrium state of $-\delta_\eta(\widetilde{\gamma}_i(0))\log|f'_{\widetilde{\gamma}_i(0)}|$ for $i = 1, 2$.

We first claim that $M_*\nu_2 = \nu_1$. We compute

$$\begin{aligned}h_{M_*\nu_2}(f_{\widetilde{\gamma}_1(0)}) + \int_{J_{\widetilde{\gamma}_1(0)}} \log|f'_{\widetilde{\gamma}_1(t)}|d(M_*\nu_2) &= h_{\nu_2}(f_{\widetilde{\gamma}_2(0)}) + \int_{M^{-1}J_{\widetilde{\gamma}_1(0)}} \log|f'_{\widetilde{\gamma}_1(t)} \circ M|d\nu_2 \\ &= h_{\nu_2}(f_{\widetilde{\gamma}_2(0)}) + \int_{J_{\widetilde{\gamma}_2(0)}} \log|f'_{\widetilde{\gamma}_2(t)}|d\nu_2 \\ &= \mathcal{P}(-\delta_\eta(\widetilde{\gamma}_2(0))\log|f'_{\widetilde{\gamma}_2(0)}|) \text{ (as } \nu_2 \text{ is an equilibrium state)} \\ &= \mathcal{P}(-\delta_\eta(\widetilde{\gamma}_1(0))\log|f'_{\widetilde{\gamma}_1(0)}|) \text{ (by Lemma 4.8).}\end{aligned}$$

Therefore, $M_*\nu_2$ is an equilibrium state for $-\delta_\eta(\widetilde{\gamma}_1(0))\log|f'_{\widetilde{\gamma}_1(0)}|$. Since $-\delta_\eta(\widetilde{\gamma}_1(0))\log|f'_{\widetilde{\gamma}_1(0)}|$ has a unique equilibrium state, this gives the desired equality $M_*\nu_2 = \nu_1$.

It follows from the above that we have

$$\begin{aligned}G_{\widetilde{\gamma}_1(0)}(\widetilde{\gamma}_1(t)) = \delta_\eta(\widetilde{\gamma}_1(t)) \int_{J_{\widetilde{\gamma}_1(0)}} \log|f'_{\widetilde{\gamma}_1(t)}|d\nu_1 &= \delta_\eta(\widetilde{\gamma}_2(t)) \int_{M(J_{\widetilde{\gamma}_2(0)})} \log|f'_{\widetilde{\gamma}_1(t)}|d(M_*\nu_2) \\ = \delta_\eta(\widetilde{\gamma}_2(t)) \int_{J_{\widetilde{\gamma}_2(0)}} \log|f'_{\widetilde{\gamma}_1(t)} \circ M|d\nu_2 &= G_{\widetilde{\gamma}_2(0)}(\widetilde{\gamma}_2(t))\end{aligned}$$

for every $t \in (-1,1)$. The assertion follows. ☐

The above lemma implies that the following definition is well-posed.

Definition 4.10. For every $[\lambda] \in \Omega$ and $\vec{v} \in T_{[\lambda]}\Omega$, we define

$$||\vec{v}||_G := \|\widetilde{\vec{v}}\|_G = ||\widetilde{\gamma}'(0)||_G$$

where $\tilde{\lambda}$ is any representative of $[\lambda]$, $\widetilde{\vec{v}}$ is a lift of the tangent vector $\vec{v}$ to $T_{\tilde{\lambda}}\tilde{\Omega}$, and $\widetilde{\gamma}(t)$, $t \in (-1,1)$ is any smooth real 1-dimensional curve in $\widetilde{\Omega}$ with $\widetilde{\gamma}(0) = \widetilde{\lambda}$ and $\gamma'(0) = \widetilde{\vec{v}}$.

4.3. **The pressure 2-form $\langle\cdot,\cdot\rangle_{\mathcal{P}}$ on Ω.** From now on, we will use λ (instead of $[\lambda]$ as before) to denote an element of Ω. By abuse of notation, we will refer to f_λ and J_λ as the objects corresponding to λ, even if a priori λ should stand for an equivalence class. This identification is meaningful by the lemmas in the previous section, as we can choose a lift of representatives of the elements of Ω in $\widetilde{\Omega}$.

In this section, we construct the pressure form $\langle\cdot,\cdot\rangle_{\mathcal{P}}$ on Ω. Fix $\lambda_0 \in \Omega$. Since, for every $\lambda \in \Omega$, the dynamical system (J_λ, f_λ) is topologically conjugate to $(J_{\lambda_0}, f_{\lambda_0})$ by means of the holomorphic motion, we can think of $(J_{\lambda_0}, f_{\lambda_0})$ as a *model dynamics* for every f_λ with $\lambda \in \Omega$. Recall that we denote by $\Psi_\lambda : J_{\lambda_0} \to J_\lambda$ the conjugacy map.

For a fixed $\eta \in (0, \log D)$, recall that $\delta_\eta(\lambda)$ is the unique real number satisfying

$$\mathcal{P}(-\delta_\eta(\lambda) \log |f'_\lambda \circ \Psi_\lambda|) = \eta.$$

Let $\mathcal{C}_\eta(J_{\lambda_0})$ be the set of cohomology classes of Hölder continuous functions with pressure η with respect to f_{λ_0}, that is,

$$\mathcal{C}_\eta(J_{\lambda_0}) := \{\phi : \phi \in C^\alpha(J_{\lambda_0}, \mathbb{R}) \text{ for some } \alpha > 0, \mathcal{P}(\phi) = \eta\}/\sim$$

where $\phi_1 \sim \phi_2$ if ϕ_1 and ϕ_2 are C^0-cohomologous on J_{λ_0}.

Definition 4.11. The *thermodynamic mapping* $\mathscr{E} : \Omega \to \mathcal{C}_\eta(J_{\lambda_0})$ is defined by

$$\mathscr{E}(\lambda) := [-\delta_\eta(\lambda) \log |f'_\lambda \circ \Psi_\lambda|].$$

The thermodynamic mapping is well-defined by Lemma 4.8.

Fix $\lambda \in \Omega$ and $\mathscr{E}(\lambda) \in \mathcal{C}_\eta(J_{\lambda_0})$. Let ν be the equilibrium state for a representative ϕ of $\mathscr{E}(\lambda)$, whose existence and uniqueness are guaranteed by the fact that $\mathcal{P}(-\delta_\eta(\lambda) \log |f'_\lambda \circ \Psi_\lambda|) = \eta > 0$ and Lemmas 2.9 and 2.4. By Proposition 2.6, the tangent space of $\mathcal{C}_\eta(J_{\lambda_0})$ at $\mathscr{E}(\lambda)$ can be identified with

$$T_{\mathscr{E}(\lambda)}\mathcal{C}_\eta(J_{\lambda_0}) = \left\{\psi \colon \psi \in C^\alpha(J_{\lambda_0}, \mathbb{R}) \text{ for some } \alpha > 0, \int_{J_{\lambda_0}} \psi d\nu = 0\right\}/\sim,$$

where we used the fact that, by definition, the pressure is constant on $\mathcal{C}_\eta(J_{\lambda_0})$. Following [McM08, p. 375], we define the *pressure form* $||\cdot||_{pm}$ *on* $\mathscr{E}(\Omega) \subset \mathcal{C}_\eta(J_{\lambda_0})$ as follows. Given $\psi \in T_{\mathscr{E}(\lambda)}\mathcal{C}_\eta(J_{\lambda_0})$, we define

$$||[\psi]||^2_{pm} := \frac{\mathrm{Var}(\psi, \nu)}{\eta - \int_{J_{\lambda_0}} \phi d\nu}.$$

Given $\vec{w} \in T_\lambda\Omega$, let $\gamma(t) = [f_t]$, $t \in (-1, 1)$ be a smooth path in Ω with $\gamma(0) = \lambda$ and $\gamma'(0) = \vec{w}$. Letting ϕ_t be a representative of the class $\mathscr{E}(\gamma(t))$, we define

$$\|\vec{w}\|_{\mathcal{P}} := ||\dot{\phi}_0||_{pm} = \frac{\mathrm{Var}(\dot{\phi}_0, \nu)}{\eta - \int_{J_{\lambda_0}} \phi_0 d\nu}. \tag{12}$$

We call $\|\cdot\|_{\mathcal{P}}$ the *pressure form on* Ω. Observe that $||\cdot||_{\mathcal{P}}$ is positive semi-definite on $T_\lambda\Omega$ since we have $\mathrm{Var}(\dot{\phi}_0, \nu) \geq 0$ and $\eta - \int_{J_{\lambda_0}} \phi_0 d\nu > 0$.

4.4. **Conformal equivalence of $\langle\cdot,\cdot\rangle_G$ and $\langle\cdot,\cdot\rangle_{\mathcal{P}}$ on Ω.** We continue to denote by λ the elements of Ω, as explained at the beginning of the previous section.

Fix $\lambda_0 \in \Omega$ and $\vec{v} \in T_{\lambda_0}\Omega$. Let $\gamma(t)$, $t \in (-1, 1)$ be a smooth path in Ω with $\gamma(0) = \lambda_0$ and $\vec{v} = \gamma'(0)$. Let $\widetilde{\gamma}(t)$ be a lift of $\gamma(t)$ in $\widetilde{\Omega}$, and denote by f_t a map corresponding to $\widetilde{\gamma}(t)$.

For every $x \in J_{\lambda_0}$ and $t \in (-1, 1)$, set

$$g(t, x) := -\delta_\eta(\widetilde{\gamma}(t)) \log |f'_t \circ \Psi_{\widetilde{\gamma}(t)}(x)|.$$

Denote by $\dot{g}(0,\cdot)$ and $\ddot{g}(0,\cdot)$ the real-valued functions on J_{λ_0} given by

$$\dot{g}(0,x) = \frac{d}{dt}\Big|_{t=0} g(t,x) \quad \text{and} \quad \ddot{g}(0,x) = \frac{d^2}{dt^2}\Big|_{t=0} g(t,x) \quad \text{for all } x \in J_{\lambda_0}$$

and denote by ν the (unique) equilibrium state of $g(0,x) = -\delta_\eta(\lambda_0)\log|f_0'|$.

Proposition 4.12. *We have*

$$||\vec{v}||^2_{\mathcal{P}} = \frac{||\vec{v}||^2_G}{\eta - \int_{J_{\lambda_0}} g(0,x)d\nu(x)}.$$

In particular, the Hessian form $||\cdot||_G$ is conformal equivalent to the pressure form $||\cdot||_{\mathcal{P}}$.

Proof. By the definition of $g(t,x)$, $\vec{v}$ can be identified with $\dot{g}(0,\cdot)$. By (12) and Proposition 2.6 (2), we have

$$||\vec{v}||^2_{\mathcal{P}} = \frac{\operatorname{Var}(\dot{g}(0,x),\nu)}{\eta - \int_{J_{\lambda_0}} g(0,x)d\nu(x)} = \frac{-\int_{J_{\lambda_0}} \ddot{g}(0,x)d\nu(x)}{\eta - \int_{J_{\lambda_0}} g(0,x)d\nu(x)}.$$

Hence, it is enough to show that we have

$$||\vec{v}||^2_G = -\int_{J_{\lambda_0}} \ddot{g}(0,x)d\nu(x).$$

By the Definition 4.3 of $G_{\tilde{\gamma}(0)}(\tilde{\gamma}(t))$, we have

$$G_{\tilde{\gamma}(0)}(\tilde{\gamma}(t)) = \delta_\eta(\tilde{\gamma}(t)) \int_{J_{\lambda_0}} \log|f_t' \circ \Psi_{\tilde{\gamma}(t)}|d\nu = -\int_{J_{\lambda_0}} g(t,x)d\nu(x).$$

The assertion follows from the Definition 4.10 of $||\cdot||_G$, after taking two derivatives in t in the last expression. □

Corollary 4.13. *The following assertions are equivalent:*

(1) $||\vec{v}||_G = 0$;
(2) $||\vec{v}||_{\mathcal{P}} = 0$;
(3) $\operatorname{Var}(\dot{g}(0,x),\nu) = 0$;
(4) $\dot{g}(0,x)$ *is a C^0-coboundary, i.e., it is C^0-cohomologous to zero.*

Proof. The equivalence between the first three assertions immediately follows from Proposition 4.12. The equivalence between (3) and (4) follows from Lemma 2.7. □

We conclude this section with the following lemma, which we will need to prove that the forms introduced so far induce a path metric on Ω.

Lemma 4.14. *If $||\vec{v}||_G = 0$, then there exists a constant $K \in \mathbb{R}$ such that, for every $n \in \mathbb{N}$, we have*

$$\frac{d}{dt}\Big|_{t=0} S_n\big(\log|f_t' \circ \Psi_{\tilde{\gamma}(t)}(x)|\big) = K \cdot S_n\big(\log|f' \circ \Psi_{\tilde{\gamma}(0)}(x)|\big)$$

for all n-periodic points x of f in J_{λ_0}. Here $S_n\phi$ denotes the Birkhoff sum of ϕ.

Proof. By the assumption on $||\vec{v}||_G$ and Corollary 4.13, the derivative $\dot{g}(0,x)$ of the map $g(t,x) = -\delta(f_t)\log|f_t' \circ \Psi_{f_t}(x)|$ is a C^0-coboundary. Hence, there exists a continuous function $h : J_{f_0} \to \mathbb{R}$ such that $\dot{g}(0,x) = h(x) - h(f_0(x))$ for every $x \in J_{f_0}$. Let $x \in J_{f_0}$ be a n-periodic point of f_0. Then, we have

$$\begin{aligned}0 = h(x) - h(f_0^n(x)) &= h(x) - h(f_0(x)) + h(f_0(x)) - \cdots - h(f_0^n(x))\\ &= \frac{d}{dt}\Big|_{t=0} g(t,x) + \frac{d}{dt}\Big|_{t=0} g(t, f_0(x)) + \cdots + \frac{d}{dt}\Big|_{t=0} g(t, f_0^{n-1}(x))\\ &= -\frac{d}{dt}\Big|_{t=0} \delta_\eta(f_t) S_n\big(\log|f_t' \circ \Psi_{f_t}(x)|\big).\end{aligned}$$

Applying the chain rule and using the inequality $\delta_\eta(f_0) > 0$, we obtain

$$\frac{d}{dt}\Big|_{t=0} S_n\big(\log|f_t' \circ \Psi_{f_t}(x)|\big) = -\frac{\frac{d}{dt}\big|_{t=0}\delta_\eta(f_t)}{\delta_\eta(f_0)} \cdot S_n\big(\log|f' \circ \Psi_f(x)|\big).$$

Therefore, the assertion follows choosing $K := \frac{d}{dt}\big|_{t=0}\delta_\eta(f_t)/\delta_\eta(f_0)$. □

5. The Hessian form defines a path metric

We prove our main result, Theorem 1.1, in this section. We first prove two preliminary results in Sections 5.1–5.3 stating that $\langle\cdot,\cdot\rangle_G$ is real-analytic on the unit tangent bundle $UT\Omega$ of Ω, and that any C^1-path in Ω has strictly positive length with respect to $\langle\cdot,\cdot\rangle_G$. We conclude the proof of Theorem 1.1 in Section 5.4.

5.1. **Analyticity of $\langle\cdot,\cdot\rangle_G$ on the unit tangent bundle.** We fix in this section a Λ-hyperbolic component Ω in a parabolic family Λ in rat_D^{cm} and we show that the bilinear form $\langle\cdot,\cdot\rangle_G$ is analytic on the unit tangent bundle $UT\Omega$ of Ω. Recall that this form depends on a parameter $\eta \in (0, \log D)$, which will be fixed throughout this section.

As in Section 4.3, we will think of Ω and Λ as subfamilies of Rat_D^{cm}, by means of suitable lifts. We fix λ_0 and only work in a sufficiently small neighbourhood of λ_0 in Ω. By means of suitable charts, we can then assume that $\lambda_0 = 0$ and that $D_\ell(0, R_0) \subset \Omega$, where ℓ is the complex dimension of Ω and Λ.

Let $\{\vec{v}_s\}_{s\in D_1(0,R_0)}$ be a holomorphic family of elements of $\mathbb{C}^\ell \setminus \{\vec{0}\}$, i.e., we assume that the map $s \mapsto \vec{v}_s \in \mathbb{C}^\ell \setminus \{\vec{0}\}$ is holomorphic. Observe also that we can identify $T_\lambda\Omega$ with $\mathbb{C}^\ell$ for every $\lambda \in D_\ell(0, R_0)$. Consider the map $\gamma\colon D_\ell(0, R_0) \times D_1(0, R_0) \times D_1(0, R_0) \to \mathbb{C}^\ell$ given by

$$\gamma(\lambda, t, s) = \lambda + t\vec{v}_s.$$

Up to shrinking R_0, we can assume that the image of γ is contained in Ω. Moreover, it is clear from the definition that γ satisfies

$$\gamma(\lambda, 0, s) = \lambda \quad \text{and} \quad \frac{d}{dt}\Big|_{t=0}\gamma(\lambda, t, s) = \vec{v}_s \in T_\lambda\Omega \quad \text{for all } \lambda \in D_\ell(0, R_0) \text{ and } s \in D_1(0, R_0).$$

For every $(\theta, \lambda, t, s) \in \mathbb{R} \times D_\ell(0, R_0) \times D_1(0, R_0) \times D_1(0, R_0)$, we also define

$$\phi_{(\theta,\lambda,t,s)} := -\delta_\eta(\lambda)\log|f_\lambda'| + \theta\log|f'_{\gamma(\lambda,t,s)} \circ \Psi_{\lambda,t,s}| : J_\lambda \to \mathbb{R}$$

where we denote by $\Psi_{\lambda,t,s} : J_\lambda \to J_{\lambda,t,s}$ the conjugacy map induced by the holomorphic motion on Ω.

Proposition 5.1. *There exists $0 < R < R_0$ such that the map $(\delta_\eta(\lambda_0) - R, \delta_\eta(\lambda_0) + R) \times D_\ell(0, R) \times D_1(0, R) \times D_1(0, R) \ni (\theta, \lambda, t, s) \mapsto \mathcal{P}(\phi_{(\theta,\lambda,t,s)})$ is real-analytic.*

We will show Proposition 5.1 in Section 5.2. We first deduce the following corollary, giving the analyticity of the metric $||\cdot||_G$ on the tangent bundle of Ω. For every $\lambda \in \Omega$, we denote by ν_λ the unique equilibrium state of $-\delta_\eta(\lambda)\log|f_\lambda'| : J_\lambda \to \mathbb{R}$.

Corollary 5.2. *There exists $R \in (0, R_0)$ such that the map*

$$D_\ell(0,R)\times D_1(0,R)\times D_1(0,R) \ni (\lambda,t,s) \mapsto G_0(\gamma(\lambda,t,s)) = \delta_\eta(\gamma(\lambda,t,s)) \int_{J_\lambda} \log|f'_{\gamma(\lambda,t,s)} \circ \Psi_{\lambda,t,s}| d\nu_\lambda$$

is real-analytic. Moreover, the map $D_\ell(0,R) \times D_1(0,R) \ni (\lambda, s) \mapsto (G''_\lambda(\lambda))(\vec{v}_s, \vec{v}_s)$ is real-analytic.

Proof. By Propositions 2.6 and 5.1, taking the first derivative with respect to θ of the pressure function $\mathcal{P}(\phi_{(\theta,\lambda,t,s)})$, for every $(\lambda, t, s) \in D_\ell(0,R) \times D_1(0,R) \times D_1(0,R)$ we have

$$\frac{d}{d\theta}\Big|_{\theta=0} \mathcal{P}(-\delta_\eta(\lambda)\log|f'_\lambda| + \theta \log|f'_{\gamma(\lambda,t,s)} \circ \Psi_{\lambda,t,s}|) = \int_{J_\lambda} \log|f'_{\gamma(\lambda,t,s)} \circ \Psi_{\lambda,t,s}| d\nu_\lambda.$$

Hence, again by Proposition 5.1, the map

$$D_\ell(0,R) \times D_1(0,R) \times D_1(0,R) \ni (\lambda,t,s) \mapsto \int_{J_\lambda} \log|f'_{\gamma(\lambda,t,s)} \circ \Psi_{\lambda,t,s}| d\nu_\lambda$$

is real-analytic. By Proposition 3.1 and the definition of $\gamma(\lambda, t, s)$, the map

$$D_\ell(0,R)\times D_1(0,R)\times D_1(0,R) \ni (\lambda,t,s) \mapsto G_0(\gamma(\lambda,t,s)) = \delta_\eta(\gamma(\lambda,t,s)) \int_{J_\lambda} \log|f'_{\gamma(\lambda,t,s)} \circ \Psi_{\lambda,t,s}| d\nu_\lambda$$

is real-analytic. This gives the first assertion.

Taking two derivatives in t of the function $G_0(\gamma(\lambda, t, s))$ and evaluating at $t = 0$, we see that the map

$$D_\ell(0,R) \times D_1(0,R) \ni (\lambda,s) \mapsto \frac{d^2}{dt^2}\Big|_{t=0} G_0(\gamma(\lambda,t,s)) = (G''_\lambda(\lambda))(\vec{v}_s, \vec{v}_s)$$

is real-analytic. This completes the proof. □

5.2. **Proof of Proposition 5.1.** We again follow the general strategy as the proof of [SU10, Theorem A] or [UZ04, Theorem 9.3]. As the arguments are similar to those of the proof of Proposition 3.2, we will just sketch them.

For every $z \in J_0$ and $\lambda \in \Omega$, we denote $z_\lambda := \Psi_\lambda(z)$. For every $z \in J_0$ and $(\lambda, t, s) \in D_\ell(0, R_0) \times D_1(0, R_0) \times D_1(0, R_0)$, consider the map

$$\psi_z(\lambda,t,s) := \frac{f'_{\gamma(\lambda,t,s)} \circ \Psi_{\lambda,t,s}(z_\lambda)}{f'_\lambda(z_\lambda)},$$

where we recall that $\Psi_{\lambda,t,s} : J_\lambda \to J_{\lambda,t,s}$ is the conjugacy map. Up to shrinking R_0 if necessary, for all $z \in J_0$ and $(\lambda, t, s) \in D_\ell(0, R_0) \times D_1(0, R_0) \times D_1(0, R_0)$, we have $|\psi_z(\lambda, t, s) - 1| < 1/5$. Then, for every $z \in J_0$, there exists a branch of $\log \psi_z$ sending 0 to 0 and whose modulus is bounded by 1/4. By Lemma 2.11, up to further shrinking R_0, the analytic map $\Re \log \psi_z : D_\ell(0, R_0) \times D_1(0, R_0) \times D_1(0, R_0) \to \mathbb{R}$ has an analytic extension $\widetilde{\Re \log \psi_z} : D_{2\ell}(0, R_0) \times D_2(0, R_0) \times D_2(0, R_0) \to \mathbb{C}$. Recall that $D_\ell(0, R_0)$ (resp. $D_\ell(0, R_0)$) is seen as a subset of the points of $\mathbb{C}^{2\ell}$ (resp. $\mathbb{C}$) with real coordinates by means of the immersion ι_ℓ (resp. ι_1) as in (7), and that we have $\iota_\ell(0) = 0 \in \mathbb{C}^{2\ell}$ and $\iota_1(0) = 0 \in \mathbb{C}^2$.

For $(\theta, \lambda, t, s) \in \mathbb{C} \times D_{2\ell}(0, R_0) \times D_2(0, R_0) \times D_2(0, R_0)$, consider the map $\zeta_{(\theta,\lambda,t,s)} : J_0 \to \mathbb{C}$ given by

$$\begin{aligned}\zeta_{(\theta,\lambda,t,s)}(z) &:= -\delta_\eta(\lambda)\log|f'_\lambda(z_\lambda)| - \theta \widetilde{\Re\log\psi_z}(\lambda,t,s) + \theta\log|f'_\lambda(z_\lambda)| \\ &= -\theta\widetilde{\Re\log\psi_z}(\lambda,t,s) + (\theta - \delta_\eta(\lambda))\log|f'_\lambda(z_\lambda)|.\end{aligned}$$

Let β be such that the conjugacy maps $\Psi_\lambda : J_0 \to J_\lambda$ and $\Psi_{\lambda,t,s} : J_\lambda \to J_{\lambda,t,s}$ are β-Hölder continuous for all $\lambda \in D_\ell(0, R_0)$ and all $(\lambda, t, s) \in D_\ell(0, R_0) \times D_1(0, R_0) \times D_1(0, R_0)$, respectively.

The map $(\theta, \lambda, t, s) \mapsto \zeta_{(\theta,\lambda,t,s)}$ is then continuous with respect to the β-Hölder norm. Applying Lemma 2.2 to the families $\Re\zeta_{(\theta,\lambda,t,s)}$ and $\Im\zeta_{(\theta,\lambda,t,s)}$, up to shrinking R_0, gives an extended family $\mathbb{C} \times D_{2\ell}(0, R_0) \times D_2(0, R_0) \times D_2(0, R_0) \ni (\theta, \lambda, t, s) \mapsto \widetilde{\zeta}_{(\theta,\lambda,t,s)} : \mathbb{P}^1 \to \mathbb{C}$ which is continuous with respect to the $\beta/2$-Hölder norm and satisfies $\widetilde{\zeta}_{(\theta,\lambda,t,s)}(z) = \zeta_{(\theta,\lambda,t,s)}(z)$ for every $(\theta, \lambda, t, s) \in \mathbb{C} \times \iota_{\ell+2}^{-1}(D_{2\ell}(0, R_0) \times D_2(0, R_0) \times D_2(0, R_0))$ and $z \in J_0$.

Fix a norm $||\cdot||_\diamond$ as in Section 2.2 such that $||\cdot||_\diamond \lesssim ||\cdot||_{C^{\beta/2}}$ and, for $(\theta, \lambda, t, s) \in \mathbb{C} \times D_{2\ell}(0, R_0) \times D_2(0, R_0) \times D_2(0, R_0)$, consider the complex transfer operator $\mathcal{L}_{(\theta,\lambda,t,s)} := \mathcal{L}_{\widetilde{\zeta}_{(\theta,\lambda,t,s)}}$. As in Lemma 3.3, there exists $0 < R_1 < R_0$ such that the map $D_1(\delta_\eta(0), R_1) \times D_{2\ell}(0, R_0) \times D_2(0, R_0) \times D_2(0, R_0) \ni (\theta, \lambda, t, s) \mapsto \mathcal{L}_{\widetilde{\zeta}_{(\theta,\lambda,t,s)}} \in L(B_\diamond(J_0))$ is holomorphic. As for Proposition 3.2, Proposition 5.1 then follows from Kato-Rellich perturbation theorem.

5.3. **Non-degeneracy along C^1 paths.** We fix in this section a *bounded* Λ-hyperbolic component Ω of a parabolic subfamily Λ in poly_D^{cm}. Using the metric $||\cdot||_G$, given a C^1 path $\gamma\colon (0,1) \to \Omega$, we define the *length* $\ell_G(\gamma)$ of γ as

$$\ell_G(\gamma) := \int_0^1 \|\gamma'(t)\|_G dt.$$

The main result of this section is the following proposition, which states that the metric assigns a positive length to any (non-trivial) C^1 path in Ω. The assumptions on the polynomial family and on the boundedness of Ω in our main theorem will be used in the proof of this result.

Proposition 5.3. *We have $\ell_G(\gamma) > 0$ for any non-trivial C^1 path $\gamma\colon (0,1) \to \Omega$.*

Observe that, if $\ell_G(\gamma) = 0$, we must have $\|\gamma'(t)\|_G = 0$ for almost every $t \in (0,1)$ (and hence, by continuity, for all $t \in (0,1)$). For simplicity, we will say that the metric is *degenerate along* γ if this happens. The following lemma, which follows immediately from Lemma 4.14, characterizes when such a situation can occur.

Lemma 5.4. *Let $\gamma\colon (0,1) \to \Omega$ be a C^1 path. Then the metric $||\cdot||_G$ is degenerate along γ if and only if for every $t \in (0,1)$ and for every $n \in \mathbb{N}$, we have*

$$\frac{d}{dt}\Big|_{t=0} S_n\big(\log|f'_{\gamma(t)}(x(\gamma(t)))|\big) = K(\gamma(t)) S_n\big(\log|f'_{\gamma(t)}(x(\gamma(t)))|\big)$$

for every repelling n-periodic point $x(\gamma(t))$, where $K(\gamma(t)) := \delta_\eta(\gamma(0))^{-1} \frac{d}{dt}\big|_{t=0} \delta_\eta(\gamma(t))$.

The following corollary is an immediate consequence of the previous lemma.

Corollary 5.5. *Let $\gamma\colon (0,1) \to \Omega$ be a C^1 path along which the metric $||\cdot||_G$ degenerates. Then the following assertions hold:*

(1) for every repelling n-periodic point x and $t_1, t_2 \in (0,1)$, we have

$$S_n\big(\log|f'_{\gamma(t_2)}(x(\gamma((t_2)))|\big) = S_n\big(\log|f'_{\gamma(t_1)}(x(\gamma(t_1)))|\big) \cdot e^{\widetilde{K}(t_2,t_1)},$$

where $\widetilde{K}(t_2, t_1) = \int_{t_1}^{t_2} K(\gamma(t))dt$;

(2) for every pair of motions of repelling n-periodic points x_i, x_j, there exists a positive constant $a_{i,j}$ such that

$$S_n\big(\log|f'_{\gamma(t)}(x_i(\gamma(t)))|\big) = a_{i,j} S_n\big(\log|f'_{\gamma(t)}(x_j(\gamma(t)))|\big)$$

for every t in $(0,1)$.

We can now prove Proposition 5.3.

Proof of Proposition 5.3. Suppose by contradiction that we have $\ell_G(\gamma) = 0$. Then, the metric must degenerate along γ. We denote by $\{x_i(\lambda)\}_{i\geq 1}$ the set of maps parametrizing the repelling periodic points on Ω, and let n_i be the period of the corresponding cycle.

Fix $s_0 \in (0,1)$ and an index i_0. It follows from Corollary 5.5 that, for every $i \geq 1$ and every $s \in (0,1)$, we have

$$S_{n_{i_0} n_i}\big(\log |f'_{\gamma(s)}(x_i(\gamma(s)))|\big) = a_{i,i_0} e^{\widetilde{K}(s,s_0)} S_{n_{i_0} n_i}\big(\log |f'_{\gamma(s_0)}(x_{i_0}(\gamma(s_0)))|\big)$$

for some strictly positive (as both $x_i(\gamma(s))$ and $x_{i_0}(\gamma(s_0))$ are repelling) constants a_{i,i_0}.

As Ω is bounded, we have $L(\gamma(s)) \equiv \log D$ on Ω by Lemma 2.1. Hence, it follows from the equidistribution of periodic points with respect to the measure of maximal entropy [Lyu82; Lyu83a] that, for every $s \in (0,1)$, we have

$$\begin{aligned}
\log D &= \lim_{n\to\infty} \frac{1}{n_{i_0} n} \frac{1}{D^{n_{i_0} n}} \sum_{x_i\colon n_i = n_{i_0} n} a_{i,i_0} e^{\widetilde{K}(s,s_0)} S_{n_{i_0} n}\big(\log |f'_{\gamma(s_0)}(x_{i_0}(\gamma(s_0)))|\big)\\
&= e^{\widetilde{K}(s,s_0)} \cdot \lim_{n\to\infty} \left(\frac{1}{D^{n_{i_0} n}} \sum_{x_i\colon n_i = n} a_{i,i_0} \cdot \frac{1}{n_{i_0} n} S_{n_{i_0} n}\big(\log |f'_{\gamma(s_0)}(x_{i_0}(\gamma(s_0)))|\big)\right)\\
&= e^{\widetilde{K}(s,s_0)} \cdot \left(\lim_{n\to\infty} \frac{1}{D^{n_{i_0} n}} \sum_{x_i\colon n_i = n} a_{i,i_0} \right) \cdot \frac{1}{n_{i_0}} S_{n_{i_0}}\big(\log |f'_{\gamma(s_0)}(x_{i_0}(\gamma(s_0)))|\big),
\end{aligned}$$

where in the last step we used the identity

$$S_{n_{i_0} n}\big(\log |f'_{\gamma(s_0)}(x_{i_0}(\gamma(s_0)))|\big) = n S_{n_{i_0}}\big(\log |f'_{\gamma(s_0)}(x_{i_0}(\gamma(s_0)))|\big).$$

We deduce that the function $\widetilde{K}(s,s_0)$ is independent of s. By Corollary 5.5 (1), this shows that the absolute values of all the multipliers of the $x_i(\gamma(s))$'s are constant along γ. However, this is impossible as, by [JX23, Theorem 8.25], there are only finitely many conjugacy classes of rational maps, not in the locus of (conjugacy classes of) flexible Lattès maps, having the same set of absolute values of repelling multipliers. Therefore $\ell_G(\gamma) = 0$ and the proof is complete. □

Remark 5.6. The conclusion of the proof of Proposition 5.3 can be achieved also without making use of [JX23]. Indeed, assume as above that the absolute values of all the multipliers of the $x_i(\gamma(s))$'s are constant along γ. For every i, the absolute value of the multiplier of $x_i(\lambda)$ is an harmonic function on Ω. This gives a family of harmonic functions with a non-trivial common level set (possibly corresponding to a different real value, larger than 1, for each function) which, by the above, must contain the image of γ. Up to reparametrization, as Λ is algebraic, we can also assume that $\Lambda = \mathbb{C}^\ell$, where $\ell = \dim_{\mathbb{C}}\Lambda$. It follows from the maximum principle that this common level set cannot be bounded in Λ (as, otherwise, all the harmonic functions would be constant in the region bounded by the level set, hence constant there, hence on Λ, which contradicts the choice of Ω as before). Hence, all of the repelling points stay repelling, with constant modulus of their multiplier, along some path going to infinity in $\Lambda = \mathbb{C}^\ell$. This implies that, for every λ belonging to this path, we have $L(\lambda) = \log D$. This contradicts the fact that, outside of a compact subset of Λ, all critical points escape to infinity, which gives $L(\lambda) > \log D$ by the Przyticki formula.

5.4. **Proof of Theorem 1.1.** In this section we conclude the proof of our main theorem. We will use the following theorem by Mityagin [Mit15], describing the zero set of a non-trivial real analytic function.

Theorem 5.7. *Let $\mathcal{O} \subset \mathbb{R}^l$ be an open set and $\psi\colon \mathcal{O} \to \mathbb{R}$ an analytic function not identically vanishing. Then, the set $\{\psi = 0\}$ is covered by a countable union of (not necessarily closed) analytic submanifolds of $\mathcal{O}$.*

In [Mit15], the author only states that, under the assumptions of the theorem, the set $\{\psi = 0\}$ has zero Lebesgue measure. However, the proof is constructive and, as already noted in [Kuc16, Remark 5.23], gives a decomposition of this set as a countable union of smooth submanifolds of real co-dimension at least 1. As each of these submanifolds (obtained by means of the implicit function theorem) is in the common zero locus of analytic functions (namely ψ and some of its derivatives), one can see that these sets are indeed analytic submanifolds.

We now conclude the proof of our main result; that is, we need to show that the function $d_G : \Omega \times \Omega \to \mathbb{R}$ given by

$$d_G(x, y) := \inf_{\gamma} \int_0^1 \|\gamma'(t)\|_G dt$$

is a distance function. Recall that the infimum is taken over all the C^1-paths γ connecting x to y in Ω.

Proof of Theorem 1.1. Let Λ and Ω be as in the statement. Observe that d_G is a pseudo-metric; namely, we have $d_G(x, x) = 0$ and $d_G(x, y) = d_G(y, x)$ for every $x, y \in \Omega$, and d_G satisfies the triangle inequality. We need to show $d_G(x, y) > 0$ for $x \neq y \in \Omega$.

By Corollary 5.2, the pseudo-metric d_G induced by the Hessian form is described by a collection of positive semi-definite bilinear forms A^1_λ on $T_\lambda\Omega$, depending analytically on the point $\lambda \in \Omega$. Denote by $\mathscr{D}_1(\lambda)$ the determinant of the matrix representing A^1_λ. By Corollary 5.2, $\mathscr{D}_1 : \Omega \to \mathbb{R}$ is real-analytic. It is clear that the pseudo-metric d_G is indeed a metric outside of the zero locus S_1 of the analytic map $\mathscr{D}_1 : \Omega \to \mathbb{R}$. Hence, it is enough to prove that $d_G(x, y) > 0$ for every $x, y \in S_1$.

We first show that we cannot have $S_1 = \Omega$. Observe that $S_1 = \Omega$ means that at every $\lambda \in \Omega$ there is at least one degenerate direction for the metric. For $j = 1, \ldots, 2\ell$ where $\ell := \dim_{\mathbb{C}}\Omega$, set

$$U_j := \{\lambda \in \Omega : \exists \vec{v}_1, \ldots, \vec{v}_j \in T_\lambda\Omega \text{ linearly independent with } \|\vec{v}_1\|_G = \ldots = \|\vec{v}_j\|_G = 0\}.$$

It is straightforward to see that $S_1 = U_1 \supseteq U_2 \supseteq \cdots \supseteq U_{2\ell}$. There are two cases to consider.

Case 1: $U_{j+1} \neq U_j$ for some $j \in \{1, \ldots, 2\ell - 1\}$. Let $j^\star$ be the minimum j satisfying this property. Then $U_{j^\star} \setminus U_{j^\star+1}$ contains an open set $\mathcal{A}$ of Ω. By the definition of $\mathcal{A}$ and the analyticity of the metric, there exists a $j^\star$-dimensional subbundle $V \subset T\Omega$ on $\mathcal{A}$ such that, for every $\lambda \in \mathcal{A}$, $\|\cdot\|_G$ is degenerate on the fiber V_λ. Consider a C^1 path $\gamma\colon (0, 1) \to \mathcal{A}$ whose tangent $\gamma'(t)$ is contained in $V_{\gamma(t)}$ for every $t \in (0, 1)$. By construction, we have $\ell_G(\gamma) = 0$, contradicting Proposition 5.3.

Case 2: If $U_{j+1} = U_j$ for all $j = 1, \ldots, 2\ell - 1$, then $U_{2\ell} = U_1 = \Omega$, meaning that we have $\|\cdot\|_G \equiv 0$ on the tangent bundle $T\Omega$. Then any C^1-path in Ω has length 0, contradicting Proposition 5.3.

Therefore, as $S_1 \neq \Omega$, the function $\mathscr{D}_1 : \Omega \to \mathbb{R}$ is not identically zero on Ω and, up to working locally, we can apply Theorem 5.7 to $\mathscr{D}_1 : \Omega \to \mathbb{R}$. It follows that there is a countable collection $\{S^j_1\}_{j\geq 1}$ of connected analytic submanifolds of Ω of real co-dimension at least 1 covering S_1. It is enough to show that $d_G(x, y) > 0$ for any two distinct points x and y belonging to the same real co-dimension one submanifold in the collection $\{S^j_1\}_{j\geq 1}$, say S^1_1.

If $\dim_{\mathbb{R}}\Omega = 2$, the proof is complete by Proposition 5.3. Indeed, as the submanifold S^1_1 is smooth and one-dimensional, it itself gives a smooth path joining x and y in S^1_1. As the length of this path is strictly positive by Proposition 5.3, the proof in this case is complete.

We now treat the general case where $\dim_{\mathbb{R}}\Omega > 2$. By the above argument, S^1_1 is an analytic submanifold, and we need to show that $d_G(x, y) > 0$ for any pair of distinct points $x, y \in S^1_1$. Observe that any path between x and y not contained in S^1_1 must necessarily have a positive length. Hence, we can restrict ourselves to paths with are contained in S^1_1, whose length can be computed by considering the restrictions of the pseudo-metric represented by A^1_λ to the (real) tangent spaces

$T_\lambda S_1^1$. Let $\mathscr{D}_2(\lambda)$ be the determinant of the associated matrix of A_λ^2 on the tangent space $T_\lambda S_1^1$. Then $\mathscr{D}_2 : S_1^1 \to \mathbb{R}$ is an analytic function. We argue as above that the zero locus S_2 of $\mathscr{D}_2 : S_1^1 \to \mathbb{R}$ is not equal to S_1^1, and is covered by a countable collection $\{S_2^j\}_{j\geq 1}$ of connected analytic submanifolds of S_1^1. As before, we can then assume that x and y belong to the same component S_2^1 of S_2, and that any continuous path of trivial length joining x and y must be contained in S_2^1, which is an analytic submanifold of Ω of real dimension $\dim_{\mathbb{R}}\Omega - 2$.

Working by induction, we see that any continuous path of trivial length between x and y must be contained in an analytic real one-dimensional submanifold $S^1_{\dim_{\mathbb{R}}\Omega-1}$, where S^1_{j+1} is a component of the singular locus of the restriction of the pseudo-metric to S^1_j. The conclusion now follows from Proposition 5.3, as in the case where $\dim_{\mathbb{R}}\Omega = 2$. The proof is complete. □

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Dipartimento di Matematica, Università di Pisa, Largo Bruno Pontecorvo 5, 56127 Pisa, Italy
Email address: fabrizio.bianchi@unipi.it

Department of Mathematics, University of Oklahoma, Norman, OK 73019
Email address: he@ou.edu