ANALYTICITY OF THE HAUSDORFF DIMENSION AND METRIC STRUCTURES ON MISIUREWICZ FAMILIES OF POLYNOMIALS

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Abstract. Consider a holomorphic family $(f_\lambda)_{\lambda\in\Lambda}$ of polynomial maps on $\mathbb{C}$ with the property that a critical point of f_λ is persistently preperiodic to a repelling periodic point of f_λ. Let Ω be a bounded stable component of Λ with the property that, for all $\lambda \in \Omega$, all the other critical points of f_λ belong to attracting basins. In this paper, we introduce a dynamically meaningful geometry on Ω by constructing a natural path metric on Ω coming from a 2-form $\langle\cdot,\cdot\rangle_G$. Our construction uses thermodynamic formalism. A key ingredient is the spectral gap of adapted transfer operators on suitable Banach spaces, which also implies the analyticity of $\langle\cdot,\cdot\rangle_G$ on the unit tangent bundle of Ω. As part of our construction, we recover a result of Skorulski and Urbański stating that the Hausdorff dimension of the Julia set of f_λ varies analytically over Ω.

1. Introduction

Let S be a closed surface of genus $g \geq 2$. The Teichmüller space $T(S)$ of S, which parametrizes the hyperbolic structures on S, plays a fundamental role in modern mathematics. The topology and geometry of Teichmüller spaces have been investigated from numerous viewpoints. Ahlfors [Ahl54] proved that $T(S)$ is homeomorphic to $\mathbb{R}^{6g-6}$. On the other hand, $T(S)$ carries a number of natural metrics defined from different perspectives, e.g., the Teichmüller metric, the Weil-Petersson metric, and the Thurston metric; see for instance [Hub06; IT92].

From the perspective of Sullivan's dictionary [Sul85; McM94; DSU17], stable components (in the sense of [Lyu83b; MSS83]) of moduli spaces of rational maps (seen as holomorphic dynamical systems on the Riemann sphere) are the natural counterparts in complex dynamics of Teichmüller spaces of closed surfaces. This correspondence provides both the motivation and the tools to examine both the topology and the geometry of such stable components, in parallel to the theories of Teichmüller spaces.

Ever since McMullen's [McM08] construction of the Weil-Petersson metric on the space of degree $d \geq 2$ Blaschke products in complex dynamics, there has been an extensive study of Weil-Petersson metrics on stable components of moduli spaces. Ivrii [Ivr14] studied the completeness properties of McMullen's metric for degree 2 Blaschke products. Nie and the second author [HN23] constructed Weil-Petersson metrics on general hyperbolic components in moduli spaces of rational maps. Lee, Park, and the second author [HLP24; HLP25] studied the degeneracy loci of the Weil-Petersson metric on spaces of quasi-Blaschke products. In our earlier paper [BH24], we studied the Weil-Petersson metric on stable components of polynomials families with a persistent parabolic point.

In this paper, we extend the theory of Weil-Petersson metrics in complex dynamics to stable components of a *Misiurewicz family* of polynomials (i.e., a family where some critical point is persistently preperiodic to some repelling periodic point). The construction of the Weil-Petersson metric requires a deep analysis of the spectral properties of adapted transfer operators to deal with the presence of critical points in the Julia sets and the lack of uniform hyperbolicity.

1.1. **Statement of results.** Denote by Poly_D^{cm} (resp. Rat_D^{cm}) the space of critically marked degree $D \geq 2$ polynomials (resp. rational maps). A not necessarily closed subfamily Λ of Poly_D^{cm} (resp.

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Rat_D^{cm}) is a *Misiurewicz family* if it is the open subset of a family given by a finite number of critical relations of the form

$$f^{n_i}(c_i(\lambda)) = f^{n_i+m_i}(c_i(\lambda)) = r_i(\lambda),$$

where $c_i(\lambda)$ is the i-th marked critical point of f_λ, n_i, m_i are positive integers, and each $r_i(\lambda)$ is a repelling periodic point for every $f_\lambda \in \Lambda$. The dynamics of a Misiurewicz polynomial $f_\lambda \in \Lambda$ on its Julia set is not uniformly hyperbolic, due to the presence of critical points. We say that a stable component $\Omega \Subset \Lambda$ is Λ-*hyperbolic* if, for every $\lambda \in \Omega$, every critical point $c_j(\lambda)$ such that c_j is active on Λ is contained in the basin of some attracting cycle for f_λ.

Our main goal in this paper is to define a Weil-Petersson path-metric on a bounded Λ-hyperbolic component Ω of a Misiurewicz subfamily Λ of poly_D^{cm}. To this end, we construct a positive semi-definite symmetric bilinear form $\langle \cdot, \cdot \rangle_G$ on each tangent space $T_\lambda \Omega$, which will turn out to be equivalent to the pressure 2-form. The construction of $\langle \cdot, \cdot \rangle_G$ is valid on any Λ-hyperbolic component of a Misiurewicz subfamily Λ of rat_D^{cm}.

As the first step of our construction of the 2-form $\langle \cdot, \cdot \rangle_G$, we show that the Hausdorff dimension function is real-analytic.

Theorem 1.1. *Let Ω be a Λ-hyperbolic component of a Misiurewicz subfamily Λ of rat_D^{cm}. The Hausdorff dimension function $\delta\colon \Omega \to (0,2)$ sending λ to the Hausdorff dimension of the Julia set of f_λ is real-analytic.*

In fact, Theorem 1.1 is obtained as a corollary (see Corollary 3.2) of a stronger analyticity result; see Theorem 3.1. We also remark that Theorem 1.1 is already known; see for example Skorulski-Urbanski [SU14]. Our proof of Theorem 3.1, and hence of Theorem 1.1, is very different from Skorulski-Urbanski's proof [SU14]. The framework of our proof will be crucial in later parts of the paper. More details of the proof strategies will be given in Section 1.2.

We then construct the Weil-Petersson metric on each tangent space $T_\lambda \Omega$. The construction follows the general framework of [HN23; BH24].

By construction, the 2-form $\langle \cdot, \cdot \rangle_G$ may not be non-degenerate; namely, there may exist a non-zero tangent vector $\vec{v} \in T_\lambda \Omega$ such that $\langle \vec{v}, \vec{v} \rangle_G = 0$. For example, in [HLP24; HLP25], Lee, Park and the second author studied the degeneracy loci of the Weil-Petersson metric on spaces of quasi-Blaschke products. On the other hand, in [HN23], Nie and the second author gave a sharp condition under which the Weil-Petersson metric is non-degenerate on (uniformly) hyperbolic components of rational maps. This result deeply exploited the uniform hyperbolicity and in particular a result by Oh-Winter [OW17] on the distribution of the multiplier of the periodic points. Even though we cannot rely on that result in our Misiurewicz setting, we can still prove that $\langle \cdot, \cdot \rangle_G$ defines a path-metric on Ω.

Theorem 1.2. *Let Ω be a bounded Λ-hyperbolic component of a Misiurewicz subfamily Λ of poly_D^{cm}. Then the function $d_G : \Omega \times \Omega \to \mathbb{R}$ given by*

$$d_G(x, y) := \inf_\gamma \int_0^1 \|\gamma'(t)\|_G dt$$

is a distance function. Here the infimum is taken over all the C^1-paths γ connecting x to y in Ω.

The above theorem parallels the result we obtained in [BH24] for the case of parabolic families of polynomials. We note that the analysis in [BH24] does not apply to the case of Misiurewicz maps. More specifically, for parabolic maps, the geometric potential is Hölder continuous, hence we could apply the machinery of [BD23; BD24] to the transfer orerator associated to such weights and their perturbations. For Misiurewicz maps, we will instead consider suitable transfer operators on an enlarged tower space, where the dynamics becomes expanding.

1.2. **Strategies of the proofs.** The main technical difficulty of the paper lies in the proof of Theorem 3.1, which states that a certain pressure function $(t_1, \lambda_1, t_2, \lambda_2) \mapsto P(t_1, \lambda_1, t_2, \lambda_2)$ related to two geometric potentials is real-analytic. As consequences of Theorem 3.1, we obtain Theorem 1.1 as well as the analyticity of the two-form $\langle \cdot, \cdot \rangle_G$ on the unit tangent bundle of Ω.

To prove Theorem 3.1, we first consider a suitable extension $T\colon \mathcal{T} \to \mathcal{T}$ of the dynamics to an enlarged *tower space*, see Section 3.2, which provides a way to deal with critical points in the Julia set for a single Misiurewicz map. Since we will work with a stable component of Misiurewicz maps, we generalize the work of Makarov-Smirnov [MS03] to a holomorphic family of tower maps $\{T_\lambda \colon \mathcal{T}_\lambda \to \mathcal{T}_\lambda\}_{\lambda \in \Omega}$. A key observation is that the Hölder continuous conjugation between Julia sets in the same Λ-hyperbolic component extends to a Hölder continous conjugation between the corresponding tower dynamics; see Lemma 3.9.

For $t_i \in \mathbb{R}, \lambda_i \in \Omega$, we construct a 4-parameter family of transfer operators $\mathcal{L}_{t_1,\lambda_1,t_2,\lambda_2} \colon C^\kappa(\mathcal{T}) \to C^\kappa(\mathcal{T})$ acting on the Banach space of κ-Hölder continuous functions of $\mathcal{T}$. The four parameters of the transfer operators come from the potential functions; namely, we consider a 4-parameter family of potentials $\zeta_{(t_1,\lambda_1,t_2,\lambda_2)}$ to define $\mathcal{L}_{t_1,\lambda_1,t_2,\lambda_2} = \mathcal{L}_{\zeta_{(t_1,\lambda_1,t_2,\lambda_2)}}$. In fact, our potential $\zeta_{(t_1,\lambda_1,t_2,\lambda_2)}$ is of the form $\zeta_{(t_1,\lambda_1)} + \zeta_{(t_2,\lambda_2)}$, where each $\zeta_{(t_i,\lambda_i)}$ is a modified version of the geometric potential $-t_i \log |f'_{\lambda_i}|$.

Then an important step is to prove that such transfer operators have a *spectral gap* for certain ranges of the parameters; see Proposition 3.12. In particular, the spectral gap property follows from a Lasota-Yorke type of estimate, which is locally uniform in all the parameters $t_1, \lambda_1, t_2, \lambda_2$.

To show that the pressure function is indeed analytic, and not only separately analytic in its variables, we will employ an extension argument, see for instance [SU10; UZ04]. More specifically, we show that we can extend the 4-parameter family of potential functions $\zeta_{t_1,\lambda_1,t_2,\lambda_2}$ to complex parameters $t_1, \lambda_1, t_2, \lambda_2$, in such a way that the 4-parameter family of transfer operators $\mathcal{L}_{t_1,\lambda_1,t_2,\lambda_2}$ becomes a *holomorphic* family of operators; see Proposition 3.14 and Lemma 3.17. Then it follows from the perturbation theory that the eigenvalue, and therefore the pressure, varies real-analytically with the parameters.

1.3. **Organization of the paper.** The paper is organized as follows. Section 2 collects basic facts about Misiurewicz maps and Λ-hyperbolic components. We prove Theorem 3.1 and therefore Theorem 1.1 in Section 3. We construct the 2-form $\langle \cdot, \cdot \rangle_G$ and the pressure form in Section 4. Finally, we prove Theorem 1.2 in Section 5.

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2. Misiurewicz families and Λ-hyperbolic components

Let $D \geq 2$ be an integer. Let Rat_D^{cm} (resp. Poly_D^{cm}) be the space of degree D rational maps (resp. polynomials) with marked critical points $c_1, \ldots, c_{2D-2}$ (resp. $c_1, \ldots, c_{D-1}$) and rat_D^{cm} (resp. poly_D^{cm}) the moduli space obtained from Rat_D^{cm} (resp. Poly_D^{cm}) by taking the quotient by all Möbius (resp. affine) conjugacies. For every $\lambda \in \mathrm{Rat}_D^{cm}$ (resp. Poly_D^{cm}), we denote by f_λ the corresponding rational map (resp. polynomial), by μ_λ its unique measure of maximal entropy [FLM83; Lyu82; Lyu83a], and by $L(\lambda) = \int \log |f'_\lambda| \mu_\lambda$ the Lyapunov exponent of μ_λ. The following lemma collects the properties of the function $L(\lambda)$ that we shall need in the sequel. Observe that, as $\log |f'_\lambda|$ is not necessarily continuous on the support of μ_λ, the first assertion is not a direct consequence of the equidistribution of the periodic points with respect to μ_λ [Lyu82], but requires a more quantitive equidistribution, see for instance [BDM08; BD19]. The second assertion is the classical Przytycki formula [Prz85] for polynomials.

Lemma 2.1. *For every* $\lambda \in \mathrm{Rat}_D^{cm}$*, we have*

(1) $L(\lambda) = \lim_{n\to\infty} D^{-n} \sum_{x\in \mathrm{Pern}(\lambda)} \log |f'_\lambda(x)|$,

where $\mathrm{Pern}(\lambda)$ *is the set of periodic points of* f_λ *of (exact or not exact) period* n*. Moreover, for every* $\lambda \in \mathrm{Poly}_D^{cm}$*, we also have*

(2) $L(\lambda) = \log D + \sum_{j=1}^{D-1} G_\lambda(c_j(\lambda))$,

where $G_\lambda(z) := \lim_{n\to\infty} D^{-n} \log \max\{1, |f_\lambda^n(z)|\}$ *is the* Green function *of* f_λ*.*

Let now Λ_0 be an algebraic subfamily of Rat_D^{cm} (resp. Poly_D^{cm}).

Definition 2.2. We say that Λ_0 is given by *critical relations* if it is given by a finite number of non-trivial equations of the form

$$f_\lambda^{n_1}(c_{i_1}(\lambda)) = f_\lambda^{n_2}(c_{i_2}(\lambda)).$$

We say that Λ is a *Misiurewicz family* if it is the open subset of a family given by a finite number of critical relations of the form

$$f^{n_i}(c_i(\lambda)) = f^{n_i+m_i}(c_i(\lambda)) = r_i(\lambda),$$

where each $r_i(\lambda)$ is a repelling periodic point for every $f_\lambda \in \Lambda$ and each m_i is strictly positive.

We refer to [BB09; BB11; DeM01; Duj14; DF08; GOV19; Oku14] for the description and distribution in the parameter space of these submanifolds and to [Ber13; BBD18; Bia19; Duj14; Lev81] for their role in the understanding of the bifurcation phenomena and related problems.

It is clear that the quotient by Möbius or affine conjugacies is well-defined on Misiurewicz families, and we will use the same name for the images of these families by the quotient map. By a slight abuse of notation, we can think of Misiurewicz subfamilies of Poly_D^{cm} (resp. poly_D^{cm}) also as Misiurewicz subfamilies of Rat_D^{cm} (resp. rat_D^{cm}), where we allow (and require) the further critical relation $f_\lambda^{-1}(\{\infty\}) = \{\infty\}$.

Let now Λ be a Misiurewicz family and Λ_0 the algebraic family given by the associated critical relations. Recall that a critical point c is called *passive* on an open subset Λ' of Λ (or Λ_0) if the sequence of holomorphic functions $\{\lambda \mapsto f_\lambda^n(c(\lambda))\}_{n\geq 1}$ is a normal family on Λ'. Otherwise, the critical point is called *active* on Λ'. It follows from the definition that the critical points c_i which are preperiodic to the periodic points r_i are passive on Λ (this is true also on Λ_0, although it may be needed to pass to a finite cover to be able to follow the periodic points r_i on all Λ_0).

Recall [Lyu83b; MSS83] that an open subset Ω of Λ or Λ_0 is in the *stability locus* of the family if all critical points are passive on Ω. We also say that the family is stable on Ω. We say that Ω is a *stable component* if it is a connected component of the stability locus.

Definition 2.3. We say that a stable component $\Omega \Subset \Lambda$ is Λ-*hyperbolic* if, for every $\lambda \in \Omega$, every critical point $c_j(\lambda)$ such that c_j is active on Λ is contained in the basin of some attracting cycle for f_λ.

We say that Ω is a *bounded* Λ-hyperbolic component if we have $\Omega \Subset \Lambda$ for the topology induced by Λ. In this case, if $\Lambda \subseteq \mathrm{Poly}_D^{cm}$, all the critical points which are active on Λ are contained in the basin of some attracting cycle in $\mathbb{C}$ (rather than just $\mathbb{P}^1(\mathbb{C})$) for every $\lambda \in \Omega$. Observe that in this case, as all critical points have bounded orbit, by Lemma 2.1 (2) we also have

$$L(\lambda) \equiv \log D \text{ on } \Omega. \tag{1}$$

3. Analyticity of pressure functions

The main goal of this section is to prove Theorem 1.1. As we will see, we in fact prove a stronger result (see Theorem 3.1) and obtain Theorem 1.1 as a corollary (see Corollary 3.2). To prove Theorem 3.1, we first generalize the spectral study carried out by Makarov-Smirnov [MS03]. Our generalization is twofold – we extend the study in [MS03] (which was done for a single Misiurewicz map) to a stable holomorphic family of Misiurewicz maps, and we consider spectral properties for a more general 4-parameter family of transfer operators $\mathcal{L}_{(t_1,\lambda_1,t_2,\lambda_2)}$ which encode both the geometric potential at a given parameter and its perturbation in the direction of another parameter. Then we combine it with a general method which allows us to deduce analyticity of the pressure function; see for instance [SU10].

Let Ω be a Λ-hyperbolic component of a Misiurewicz family Λ. We assume for simplicity that the family is defined by a single critical relation $f_\lambda(c(\lambda)) = r(\lambda)$, as the arguments are essentially the same in the general case.

Fix $\lambda_0 \in \Omega$ and consider a neighborhood $N(\lambda_0) \Subset \Omega$ of λ_0. For each $\lambda \in N(\lambda_0)$, we denote by f_λ a representative of the conjugacy class λ and by J_λ the Julia set of f_λ. Recall that, by [MSS83; Lyu83b], there exists a *holomorphic motion* of the sets J_λ over $N(\lambda_0)$, i.e., a family of conjugations $h_\lambda \colon J_{\lambda_0} \to J_\lambda$ which depend holomorphically on λ. The family of the maps $(h_\lambda)_{\lambda\in\overline{N(\lambda_0)}}$ is compact in the space of γ-Hölder continuous maps from J_{λ_0} to $\mathbb{P}^1$, for some γ depending on $N(\lambda_0)$. We can take $\gamma \to 1$ as $N(\lambda_0)$ shrinks to λ_0. In particular, we will always assume in what follows that $N(\lambda_0)$ is sufficiently small so that the associated γ satisfies $\gamma > \gamma_0$ for some $\gamma_0 < 1$ whose precise value will be chosen when needed. Recall also that all these conjugations can be extended to quasiconformal homeomorphisms from $\mathbb{P}^1(\mathbb{C})$ to itself, and so in particular to Hölder continuous maps.

For every $t \in \mathbb{R}$ and $\lambda \in \Omega$, the *pressure function* is defined as

$$p(t,\lambda) := \limsup_{n\to\infty} \frac{1}{n} \log \sum_{z\in f_\lambda^{-n}(c(\lambda))} |(f_\lambda^n)'(z)|^{-t}. \tag{2}$$

We refer to [PRS04] for a number of equivalent definitions and characterizations of the pressure. For any fixed λ, the function $t \mapsto p(t,\lambda)$ is strictly decreasing. By [McM00], for every $\lambda \in \Omega$, the unique zero of the map $t \mapsto p(t,\lambda)$ is equal to the Hausdorff dimension of the Julia set of f_λ. We denote this number by $\delta(\lambda)$.

In the following, it will be useful to observe that, thanks to the conjugating maps h_λ, we also have

$$p(t,\lambda) = \limsup_{n\to\infty} \frac{1}{n} \log \sum_{z\in f_{\lambda_0}^{-n}(c(\lambda_0))} |(f_\lambda^n)' \circ h_\lambda(z)|^{-t}$$

for every $t \in \mathbb{R}$ and $\lambda \in N(\lambda_0)$. More generally, given $t_1, t_2 \in \mathbb{R}$ and $\lambda_1, \lambda_2 \in \Omega$, we define a *joint pressure function*

$$P(t_1,\lambda_1,t_2,\lambda_2) := \limsup_{n\to\infty} \frac{1}{n} \log \sum_{z\in f_{\lambda_0}^{-n}(c(\lambda_0))} |(f_{\lambda_1}^n)'(h_{\lambda_1}(z))|^{-t_1} |(f_{\lambda_2}^n)'(h_{\lambda_2}(z))|^{-t_2}. \tag{3}$$

Observe that we have $P(t_1,\lambda_1,0,\lambda_2) = p(t_1,\lambda_1)$ for every $t_1, \lambda_1, \lambda_2$.

The following is the main result of this section. We denote by $B(\lambda,R)$ the ball of radius R centered at λ.

Theorem 3.1. *For every $\lambda \in \Omega$ there exists $R > 0$ such that the function $P(t_1,\lambda_1,t_2,\lambda_2)$ is real analytic on $(\delta(\lambda)-R, \delta(\lambda)+R) \times B(\lambda,R) \times (-R,R) \times B(\lambda,R)$.*

The proof of Theorem 3.1 consists of two steps. The first step is to prove spectral properties of a 4-parameter family of transfer operators on a tower dynamical system; see Proposition 3.12. Sections 3.1 and 3.2 contain the setup and preliminary estimates for the tower dynamics, adapted to a holomorphic family of Misurewicz maps. The study of transfer operators is carried out in Section 3.3. The second step is to extend the potential function to complex parameters, so that the 4-parameter family of transfer operators can be seen as a *holomorphic* family of transfer operators. This step, carried out in Sections 3.4 and 3.5, uses a general framework of Sumi-Urbanski [SU10] and allows us to apply the Kato-Rellich perturbation theory.

Corollary 3.2. *For every $\lambda \in \Omega$ there exists $R > 0$ such that the function $p(t, \lambda)$ is real analytic on $(\delta(\lambda) - R, \delta(\lambda) + R) \times B(\lambda, R)$. In particular, the function $\lambda \mapsto \delta(\lambda)$ is analytic on Ω.*

Proof. The first assertion is a consequence of Theorem 3.1, applied with $t_2 = 0$. The second assertion follows from the first one and the implicit function theorem. □

Remark 3.3. The potential functions appearing in (3) are in fact the same used to define and study the so-called *Manhattan curves* [Bur93; Sha98]. In [BH25], among other things, we study the Manhattan curves associated to two hyperbolic rational maps, and more generally holomorphic endomorphisms of $\mathbb{CP}^k$. Theorem 3.1 in particular also shows that the Manhattan curve associated to two maps in the same Λ-hyperbolic component of a Misiurewicz family is analytic.

3.1. **Branched covers and generalized conformal Cantor sets.** We first recall the general setup of [MS03], which describes a model dynamics $F: P_1 \to U_0$ for every map in Ω. As in [MS03], for simplicity, we will only consider the case of *generalized conformal Cantor sets*, which corresponds to the case where all but one critical points of F escape to infinity. However, the results and the arguments hold without this assumption. To further simplify things, we will only consider the case of our interest, i.e., we will assume that the non-escaping critical point c is preperiodic to a repelling cycle (while c is only assumed to be non-recurrent in [MS03]). Without loss of generality, up to taking an iterate of the map, we assume that the image of c is a fixed point r, whose multiplier has modulus equal to $\chi > 1$.

Let U_0 be an open Jordan domain in $\mathbb{C}$ and P_1 a collection of a finite number of open topological disks whose closures are disjoint and contained in U_0. Let $F: P_1 \to U_0$ be a proper analytic function. We assume that F is a branched cover of order 2 on a component U_1 of P_1 and a biholomorphism on every other component. We assume that the number of components of P_1 is equal to $D - 1 \geq 1$, so that $F: P_1 \to U_0$ is a branched cover of degree D.

Observe that the (unique) critical point c of F belongs to U_1. For every $n \geq 2$, we set $P_n := f^{-n}(U_0)$. The *Julia set* $J(F)$ is then given by the intersection $\cap_n P_n$, and by assumption we have $c \in J(F)$. For every $n \geq 0$ and $x \in J(F)$, we denote by $P_n(x)$ the component of P_n containing x. For simplicity, we will also set $U_n := P_n(c)$. By assumption, we have $F(c) = r \notin U_1$, and this point is a fixed repelling point for F. Hence, we can also denote $V_n := P_n(r) \neq U_n$. Observe that V_{n+1} is mapped univalently by F to V_n and that $F: U_{n+1} \to V_n$ is a branched cover of degree 2. For every $n \geq 0$, we also define the *critical annulus* $D_n := U_n \setminus \overline{U_{n+1}}$. Hence, $\{D_n\}_{n \geq 0}$ is a collection of annuli around the critical point c. For every $n \geq 2$, F^n is a cover of degree 2 from D_n to $U_0 \setminus V_n$. In particular, there exist $2(D-2)$ components of P_n in D_n, and, for each of the $D - 2$ components of P_1 different from U_0, two of them are mapped univalently to it.

Example 3.4. Assume that the degree of F is equal to 3. In this case, P_1 is given by $U_1 \cup V_1$, where $U_1 = P_1(c)$ and $V_1 = P_1(r)$. The set P_2 is given by the following five sets:

- $U_2 = P_2(c)$, which is mapped by F to V_1 with degree 2,
- $V_2 := P_2(r)$, which is mapped univalently by F to V_1;
- a topological disc $W_2 \subset V_1 \setminus V_2$, which is mapped univalently by F to U_1, and

- two topological disks $Y_{2,1}, Y_{2,2} \subset U_1 \setminus U_2$, each mapped univalently by F to U_1.

More generally, for every $n \geq 2$, there exist two components $Y_{n,1}, Y_{n,2} \subset D_{n-1}$ and a component $W_n \subset V_{n-1} \setminus V_n$ of P_n which are all mapped univalently by F^n to U_1.

Consider now the family $(f_\lambda)_{\lambda \in N(\lambda_0)}$. The map F is a model for the dynamics of every f_λ (to see this, it is enough to take as U_0 any connected open neighborhood of the Julia set $J(f_\lambda)$). To simplify the notations, we will assume that $\lambda_0 = 0$ and, to avoid an extra conjugation in the steps below, we will directly think of F as the map f_0 and U_0 as a connected neighborhood of $J(f_0)$. In particular, we have $f_\lambda \circ h_\lambda = h_\lambda \circ f_0 = h_\lambda \circ F$ for every $z \in J(f_0) = J(F)$ and $\lambda \in N(0)$. Observe that we restrict to $N(0)$ in order to have uniform estimates for the maps h_λ in the sequel. The critical point $c(\lambda)$ and the repelling point $r(\lambda)$ whose relation $f_\lambda(c(\lambda)) = r(\lambda)$ defines Ω satisfy $c(0) = c$ and $r(0) = r$.

For every $\lambda \in N(0)$, we let χ_λ be the multiplier of the repelling fixed point $r(\lambda)$, and we also set $\hat{\chi} := \inf_{\lambda \in N(0)} |\chi_\lambda| > 1$. The following lemma is a direct consequence of [MS03, Lemma 1] applied to every f_λ with $\lambda \in N(0)$.

Lemma 3.5. *For every $k \geq 0$, $z \in D_k$, and $\lambda \in N(0)$, we have*

$$\operatorname{diam}(h_\lambda(U_k)) \sim \chi_\lambda^{-k/2}, \quad |(f_\lambda^k)'(h_\lambda(z))| \sim \chi_\lambda^{k/2}, \quad \textit{and} \quad |(f_\lambda^k)''(h_\lambda(z))| \lesssim \chi_\lambda^k,$$

where the implicit constants are independent of k, z, and λ.

3.2. **Tower dynamics.** We now recall the construction of the *tower extension* for the model dynamics $F\colon P_1 \to U_0$. For dynamical systems exhibiting some form of hyperbolicity, the tower method has been extensively used to make the dynamics uniformly expanding on some auxiliary tower space, see for instance [Hof80; You98].

We first consider the product $U_0 \times \mathbb{N}$ and its subset

$$\mathcal{T} := \sqcup_{k \geq 1} \mathcal{T}_k,$$

where $\mathcal{T}_1 := (P_1, 1)$ and $\mathcal{T}_k := (U_k, k)$ for every $k \geq 2$ are the *floors* of the tower. Then, consider the map $T = T_0\colon \mathcal{T} \to \mathcal{T} \cup (U_0, 1)$ given by

$$T\colon (z,k) \mapsto \begin{cases} (z, k+1) & z \in U_{k+1} \\ (F^k(z), 1) & z \notin U_{k+1}. \end{cases} \tag{4}$$

Example 3.6. Consider again the case where $d = 3$ as in Example 3.4. We see that every point not on the first floor of $\mathcal{T}$ has exactly one preimage under T (given by the corresponding point on the floor just below it). On the other hand, on the first floor:

- every point in V_1 has exactly one preimage in V_2;
- every point in U_1 has infinitely many preimages: one in W_2, and then, for every $n \geq 2$, one in each $Y_{n,1}$ and $Y_{n,2}$.

Remark 3.7. Observe that all the points of the form (z, k) with z on the boundary of U_k are mapped (to the first floor and) to the boundary of V_1. In particular, their images are not in $\mathcal{T}$.

Fix now a constant $1 < \chi_* < \sqrt{\hat{\chi}}$. Consider the following metric on $\mathcal{T}$: by definition, points in different floors have infinite distance, and the distance inside $\mathcal{T}_k$ is the usual Euclidean distance multiplied by the factor $(\chi_*)^{k-1}$. We will denote by $d_{\mathcal{T}}$ the induced distance function on $\mathcal{T}$. Observe, in particular, that the map F is expanding with respect to this distance. More precisely, for every $(z, k) \in \mathcal{T}$, by Lemma 3.5 we have

$$T'(z,k) = \begin{cases} \chi_* & z \in U_{k+1} \\ \chi_*^{1-k} (F^k)'(z) \gtrsim (\sqrt{\hat{\chi}}/\chi_*)^k & z \notin \overline{U_{k+1}}. \end{cases}$$

Observe that T' is differentiable at (z,k), unless z belongs to the boundary of U_k. Observe also that the diameter of $\mathcal{T}_k$ goes to zero exponentially by the choice of $\chi_* < \sqrt{\hat{\chi}}$ and Lemma 3.5.

For every $\lambda \in N(0)$, we can similarly construct a tower dynamics $T_\lambda \colon \mathcal{T}_\lambda \to \mathcal{T}_\lambda \cup (h_\lambda(U_0), 1)$, where $\mathcal{T}_\lambda$ is the tower defined replacing P_1 and U_k with $h_\lambda(P_1)$ and $h_\lambda(U_k)$, respectively, and T_λ is obtained replacing F^k with f_λ^k in the definition (4) of T. By the choice of χ_*, for every λ the map T_λ is expanding on $\mathcal{T}_\lambda$. The following lemma is a consequence of [MS03, Lemma 2] applied at every λ. It essentially follows from Lemma 3.5, as [MS03, Lemma 2] follows from [MS03, Lemma 1].

Lemma 3.8. *There is a constant C independent of n and λ such that if T_λ^n is defined and differentiable at y, then*

$$\frac{|(T_\lambda^n)''(y)|}{|(T_\lambda^n)'(y)|^2} \leq C.$$

The conjugations h_λ between F and the f_λ's can be lifted to the towers. More precisely, for every λ we can consider the map H_λ given by

$$H_\lambda((z,k)) := (h_\lambda(z), k) \quad \text{ for every } (z,k) \in \mathcal{T}.$$

As the maps h_λ are γ-Hölder continuous, the same is true for the restriction of H_λ to each level of the tower $\mathcal{T}$. On the other hand, because of the dilation of the metric at every floor, the modulus of continuity of H_λ a priori gets worse and worse as $k \to \infty$. The following lemma shows that, nevertheless, the maps H_λ are still Hölder continuous on the full tower.

Lemma 3.9. *Up to taking $N(0)$ and κ sufficiently small, the family $\{H_\lambda\}_{\lambda \in \overline{N(0)}}$ is compact in the space of κ-Hölder continuous maps.*

Proof. Recall that we are assuming that $N(0)$ is so small that all the h_λ's are (uniformly) γ-Hölder continuous, for some $\gamma > \gamma_0$, where γ_0 is close to 1.

Fix $k \in \mathbb{N}$ and take two points $x, y \in \mathcal{T}_k$ with $d_{\mathcal{T}}(x,y) < r$ (recall that points on different floors have infinite distance). As mentioned above, by the first estimate in Lemma 3.5 and the choice $\chi_* < \sqrt{\hat{\chi}}$, we necessarily have $r \lesssim \chi_*^{k-1} \hat{\chi}^{-k/2} \to 0$ as $k \to \infty$. By definition, the points x and y correspond to points $a, b \in \mathbb{C}$ whose Euclidean distance is less than $\chi_*^{1-k} r$. It follows from the Hölder-continuity of h_λ that we have

$$|h_\lambda(a) - h_\lambda(b)| \leq C_\gamma (\chi_*^{1-k} r)^\gamma,$$

where the constant C_γ is independent of $\lambda \in N(0)$. This implies that we have

$$d_{\mathcal{T}_\lambda}(H_\lambda(x), H_\lambda(y)) \lesssim \chi_*^k (\chi_*^{-k} r)^\gamma = \chi_*^{k(1-\gamma)} r^\gamma,$$

where the implicit constant is independent of λ and k.

Taking γ_0 sufficiently close to 1, let $\kappa > 0$ be sufficiently close to 0 such that $\chi_* < \hat{\chi}^{\frac{\gamma_0 - \kappa}{2(1-\kappa)}}$. This is possible by the condition $\chi_* < \sqrt{\hat{\chi}}$. It follows from the above bound for r that we have

$$\chi_*^{k(1-\gamma)} r^{\gamma-\kappa} \lesssim \chi_*^{k(1-\gamma)} (\chi_*^k \hat{\chi}^{-k/2})^{\gamma-\kappa} = \chi_*^{k(1-\gamma)+k(\gamma-\kappa)} \hat{\chi}^{-k(\gamma-\kappa)/2} = \chi_*^{k(1-\kappa)} \hat{\chi}^{-k(\gamma-\kappa)/2} \lesssim 1,$$

where the implicit constant is now independent of k. Hence, we have

$$d_{\mathcal{T}_\lambda}(H_\lambda(x), H_\lambda(y)) \lesssim r^\kappa,$$

where again the implicit constant is independent of k. This completes the proof. □

3.3. **Transfer operators.** We denote by $C(\mathcal{T})$ the space of bounded continuous functions $g\colon \mathcal{T} \to \mathbb{R}$ endowed with the L^∞-norm. For every $t_1, t_2 > 0$ and $\lambda_1, \lambda_2 \in N(0)$, consider the transfer operator $\mathcal{L}_{t_1,\lambda_1,t_2,\lambda_2}\colon C(\mathcal{T}) \to C(\mathcal{T})$ defined as

$$\mathcal{L}_{t_1,\lambda_1,t_2,\lambda_2} g(x) := \sum_{T(y)=x} g(y)|R_{\lambda_1}(y)|^{-t_1}|R_{\lambda_2}(y)|^{-t_2}$$

where, for every $y \in \mathcal{T}_k$ and $\lambda \in N(0)$, the value $R_\lambda(y)$ is defined as

$$R_\lambda(y) := \begin{cases} \chi_* & y \in U_{k+1} \times \{k\} \\ \chi_*^{1-k} \cdot |(f_\lambda^k)' \circ h_\lambda| & y \notin \overline{U_{k+1}} \times \{k\}. \end{cases}$$

Observe that, thanks to the conjugating maps H_λ, $\mathcal{L}_{t_1,\lambda_1,t_2,\lambda_2}$ can be equivalently defined as

$$\mathcal{L}_{t_1,\lambda_1,t_2,\lambda_2} g(x) := \sum_{T(y)=x} g(y)|T'_{\lambda_1}(H_{\lambda_1}(y))|^{-t_1}|T'_{\lambda_2}(H_{\lambda_2}(y))|^{-t_2}.$$

The second writing has the advantage of allowing us to apply, for every $\lambda_1, \lambda_2 \in N(0)$, the uniform estimates of [MS03] (see Lemmas 3.5 and 3.8).

Observe that the operator $\mathcal{L}_{t_1,\lambda_1,t_2,\lambda_2}$ above is well defined (i.e., for every $g \in C(\mathcal{T})$, the value of $\mathcal{L}_{t_1,\lambda_1,t_2,\lambda_2} g(x)$ is well defined for every $x \in \mathcal{T}$ and $\mathcal{L}_{t_1,\lambda_1,t_2,\lambda_2} g$ is continuous). Indeed, the only points where $\mathcal{L}_{t_1,\lambda_1,t_2,\lambda_2} g$ may be not continuous (or not defined) in $\mathcal{T} \cup (U_0, 1)$ are on $(\partial V_1, 1)$, which is outside $\mathcal{T}$ (see Example 3.6 for the case of degree 3 and Remark 3.7).

In [MS03], the authors study the special case where $t_2 = 0$ and $\lambda_1 = \lambda_0$ and obtain a spectral gap for this operator on the Banach space of Lipschitz functions on $\mathcal{T}$. As we now allow the parameters λ_1, λ_2 to change in $N(0)$, we will need to take into account the distortion given by the holomorphic motion h_λ and the map H_λ (observe that this is necessary even when $t_2 = 0$). To this aim, we will instead work with the Banach space $C^\kappa(\mathcal{T})$ of κ-Hölder continuous functions endowed with the Hölder norm $\|\cdot\|_\kappa$. For $g \in C^\kappa(\mathcal{T})$, we have

$$\|g\|_\kappa := \|g\|_\infty + \|g\|'_\kappa$$

where $\|g\|'_\kappa := \sup_{a,b \in \mathcal{T}} \frac{|g(a)-g(b)|}{d_{\mathcal{T}}(a,b)^\kappa}$ is the *Hölder constant* of g.

Lemma 3.10. *For every $t_1, t_2 \in \mathbb{R}$ with $t_1 + t_2 > 0$ and $\lambda_1, \lambda_2 \in N(0)$, the transfer operator $\mathcal{L}_{t_1,\lambda_1,t_2,\lambda_2}$ is a bounded linear operator on both $C(\mathcal{T})$ and $C^\kappa(\mathcal{T})$.*

Proof. For simplicity, we will give a proof only in the case where $d = 3$ (see Examples 3.4 and 3.6), so that the combinatorics of the problem become simpler. In particular, an explicit description of the preimages of points in $\mathcal{T}$ is given in Example 3.6. The general case can be proved in exactly the same way. To simplify the notations, we will also denote $\mathcal{L}_{t_1,\lambda_1,t_2,\lambda_2}$ by $\mathcal{L}$.

As we saw above, $\mathcal{L}$ preserves $C(\mathcal{T})$. We now show that $\mathcal{L}$ is a bounded operator on $C(\mathcal{T})$. If x is on the $(k+1)$-th floor, then x has only one preimage y under T, which is on the k-th floor of $\mathcal{T}$, and y belongs to $U_{k+1} \times \{k\}$. Then we have $T'_{\lambda_1}(y) = T'_{\lambda_2}(y) = \chi_*$ and

$$|\mathcal{L}g(x)| = |g(y)| \cdot |R_{\lambda_1}(y)|^{-t_1}|R_{\lambda_2}(y)|^{-t_2} \leq \|g\|_\infty \chi_*^{-(t_1+t_2)}.$$

Let us now take x in the first floor. If $x \in V_1 \times \{1\}$, then it has one preimages (in $V_2 \times \{1\}$) and the argument is similar to the one above. If, instead, x belongs to $U_1 \times \{1\}$, then x has countable preimages, which are as described in Example 3.6. We denote by y the preimage in W_2 and, for every $k \geq 2$, by $(y_{k,1}, k)$ and $(y_{k,2}, k)$ the preimages in $Y_{k,1} \times \{k\}$ and $Y_{k,2} \times \{k\}$, respectively. We then have

$$\begin{aligned}|\mathcal{L}g(x)| \leq &\|g\|_\infty \Big(|f'_{\lambda_1}(h_{\lambda_1}(y))|^{-t_1} |f'_{\lambda_2}(h_{\lambda_2}(y))|^{-t_2} \\ &+ \sum_{k\geq 2; j=1,2} |\chi_*^{1-k} (f^k_{\lambda_1})'(h_{\lambda_1}(y_{k,j}))|^{-t_1} |\chi_*^{1-k} (f^k_{\lambda_2})'(h_{\lambda_2}(y_{k,j}))|^{-t_2} \Big) \\ \lesssim &\|g\|_\infty \cdot \big(\chi_{\lambda_1}^{-t_1} \chi_{\lambda_2}^{-t_2} + \sum_k (\chi_*/\sqrt{\chi_{\lambda_1}})^{kt_1} (\chi_*/\sqrt{\chi_{\lambda_2}})^{kt_2}\big) \lesssim \|g\|_\infty,\end{aligned}$$

where in the second step we used Lemma 3.5 and in the last one the assumptions $\chi_* < \sqrt{\hat{\chi}}$ and $t_1 + t_2 > 0$.

Therefore, we have $\|\mathcal{L}g\|_\infty \lesssim \|g\|_\infty$ for some implicit constant independent of g.

We omit the proof of the boundedness on $C^\kappa(\mathcal{T})$ as we will prove a more precise estimate in Lemma 3.13. □

Definition 3.11. For $\mathcal{B} = C(\mathcal{T})$ or $C^\kappa(\mathcal{T})$, we define

(1) $\rho(\mathcal{L}_{t_1,\lambda_1,t_2,\lambda_2}, \mathcal{B})$ to be the spectral radius of $\mathcal{L}_{t_1,\lambda_1,t_2,\lambda_2} \colon \mathcal{B} \to \mathcal{B}$;
(2) $\rho_{ess}(\mathcal{L}_{t_1,\lambda_1,t_2,\lambda_2}, \mathcal{B}) := \inf\{\rho(\mathcal{L}_{t_1,\lambda_1,t_2,\lambda_2} - K, \mathcal{B}) \colon \text{rank } K < \infty\}$;
(3) $\eta(t_1, \lambda_1, t_2, \lambda_2) := \rho(\mathcal{L}_{t_1,\lambda_1,t_2,\lambda_2}, C(\mathcal{T}))$.

The main result of this section is the following proposition which states that, under suitable conditions, the transfer operator $\mathcal{L}_{t_1,\lambda_1,t_2,\lambda_2}$ is quasicompact on $C^\kappa(\mathcal{T})$.

Proposition 3.12. *If $t_1, t_2 \in \mathbb{R}$ with $t_1 + t_2 > 0$ and $\lambda_1, \lambda_2 \in N(0)$ are such that*

$$P(t_1, \lambda_1, t_2, \lambda_2) > -(\log \hat{\chi}/2)(t_1 + t_2), \qquad i.e., \qquad e^{P(t_1,\lambda_1,t_2,\lambda_2)} > \hat{\chi}^{-(t_1+t_2)/2}, \tag{5}$$

then the following properties hold:

(1) $\eta(t_1, \lambda_1, t_2, \lambda_2) = e^{P(t_1,\lambda_1,t_2,\lambda_2)}$;
(2) $\rho_{\mathrm{ess}}(\mathcal{L}_{t_1,\lambda_1,t_2,\lambda_2}, C^\kappa(\mathcal{T})) < \rho(\mathcal{L}_{t_1,\lambda_1,t_2,\lambda_2}, C^\kappa(\mathcal{T})) = \eta(t_1, \lambda_1, t_2, \lambda_2)$;
(3) $\eta(t_1, \lambda_1, t_2, \lambda_2)$ is a simple eigenvalue of $\mathcal{L}_{t_1,\lambda_1,t_2,\lambda_2}$ in $C^\kappa(\mathcal{T})$.

The first assertion can be proved in the same way as in [MS03, Section 2.4], by using Lemma 3.5 instead of [MS03, Lemma 1]. The key point in the proof of Proposition 3.12 is the following lemma, giving a Lasota-Yorke estimate for the iterated transfer operator with respect to the semi-norm $\|\cdot\|'_\kappa$. Once Lemma 3.13 is established, the rest of the proof carries on as in [MS03] with minimal modifications. Assumption (5) is used in that part of the proof. Observe that $t_1 + t_2 > 0$ plays the role of t in [MS03], as in Lemma 3.10.

Lemma 3.13. *For all $t_1, t_2 \in \mathbb{R}$ with $t_1 + t_2 > 0$ and $\lambda_1, \lambda_2 \in N(0)$, there exist a sequence c_n with $0 < c_n \to 0$ and a constant $C > 0$ such that for every $n \geq 1$, and every $g \in C^\kappa(\mathcal{T})$, we have*

$$\|\mathcal{L}^n_{t_1,\lambda_1,t_2,\lambda_2} g\|'_\kappa \leq c_n \eta(t_1, \lambda_1, t_2, \lambda_2)^n \|g\|'_\kappa + C\|\mathcal{L}^n_{t_1,\lambda_1,t_2,\lambda_2} |g|\|_\infty.$$

Proof. As above, for ease of notation, we will only give the proof for the case where $d = 3$ and we will simply denote $\mathcal{L}_{t_1,\lambda_1,t_2,\lambda_2}$ by $\mathcal{L}$. We also only consider the restriction to $U_1 \times \{1\}$, as the estimates are simpler otherwise (since only one preimage is involved and the system is expanding).

The n-th iterate of $\mathcal{L}$ is given by

$$\mathcal{L}^n g(x) = \sum_{T^n(y)=x} g(y) \cdot \left(\prod_{j=0}^{n-1} |R_{\lambda_1}(T^j(y))|^{-t_1} |R_{\lambda_2}(T^j(y))|^{-t_2} \right) \qquad \text{for all } g \in C(\mathcal{T}).$$

Hence, we have

$$\begin{aligned}\|\mathcal{L}^n g\|'_\kappa &= \sup_{a,b\in\mathcal{T}} \frac{1}{d_\mathcal{T}(a,b)^\kappa} \cdot |\mathcal{L}^n g(a) - \mathcal{L}^n g(b)| \\ &= \sup_{a,b\in\mathcal{T}} \frac{1}{d_\mathcal{T}(a,b)^\kappa} \cdot \Big| \sum_{T^n(y_a)=a} g(y_a)\Big(\prod_{j=0}^{n-1} |R_{\lambda_1}(T^j(y_a))|^{-t_1} |R_{\lambda_2}(T^j(y_a))|^{-t_2}\Big) \\ &\quad - \sum_{T^n(y_b)=b} g(y_b)\Big(\prod_{j=0}^{n-1} |R_{\lambda_1}(T^j(y_b))|^{-t_1} |R_{\lambda_2}(T^j(y_b))|^{-t_2}\Big)\Big|.\end{aligned}$$

We now recall (see Example 3.6) that we can pair the n-preimages of a and b under F^n depending on the set of P_n to which they belong. We will then denote these preimages by $\{a_i\}$ and $\{b_i\}$, meaning that, for every i_0, a_{i_0} and b_{i_0} belong to the same component of P_n. The above expression is then bounded by $I + II + III + IV$, where

$$I := \sup_{a,b\in\mathcal{T}} \frac{1}{d_\mathcal{T}(a,b)^\kappa} \cdot \sum_i |g(a_i) - g(b_i)| \cdot \Big(\prod_{j=0}^{n-1} |R_{\lambda_1}(T^j(a_i))|^{-t_1} |R_{\lambda_2}(T^j(a_i))|^{-t_2}\Big),$$

$$II := \sup_{a,b\in\mathcal{T}} \frac{1}{d_\mathcal{T}(a,b)^\kappa} \cdot \sum_i |g(b_i)| \cdot \Big| \prod_{j=0}^{n-1} |R_{\lambda_1}(T^j(b_i))|^{-t_1} - \prod_{j=0}^{n-1} |R_{\lambda_1}(T^j(a_i))|^{-t_1}\Big| \cdot \prod_{j=0}^{n-1} |R_{\lambda_2}(T^j(b_i))|^{-t_2},$$

$$III := \sup_{a,b\in\mathcal{T}} \frac{1}{d_\mathcal{T}(a,b)^\kappa} \cdot \sum_i |g(b_i)| \cdot \prod_{j=0}^{n-1} |R_{\lambda_1}(T^j(b_i))|^{-t_1} \cdot \Big| \prod_{j=0}^{n-1} |R_{\lambda_2}(T^j(b_i))|^{-t_2} - \prod_{j=0}^{n-1} |R_{\lambda_2}(T^j(a_i))|^{-t_2}\Big|, \quad \text{and}$$

$$\begin{aligned}IV := \sup_{a,b\in\mathcal{T}} \frac{1}{d_\mathcal{T}(a,b)^\kappa} \cdot \sum_i |g(b_i)| \cdot \prod_{j=0}^{n-1} |R_{\lambda_1}(T^j(b_i))|^{-t_1} - \prod_{j=0}^{n-1} |R_{\lambda_1}(T^j(a_i))|^{-t_1}\Big| \cdot \\ \Big| \prod_{j=0}^{n-1} |R_{\lambda_2}(T^j(b_i))|^{-t_2} - \prod_{j=0}^{n-1} |R_{\lambda_2}(T^j(a_i))|^{-t_2}\Big|.\end{aligned}$$

For I, we have

$$\begin{aligned}I &\leq \|\mathcal{L}^n 1\|_\infty \cdot \sup_{a,b\in\mathcal{T}} \frac{\max_i |g(a_i) - g(b_i)|}{d_\mathcal{T}(a,b)^\kappa} \leq \|\mathcal{L}^n 1\|_\infty \cdot \sup_{a,b\in\mathcal{T}} \frac{\max_i \|g\|'_\kappa \cdot d_\mathcal{T}(a_i,b_i)^\kappa}{d_\mathcal{T}(a,b)^\kappa} \\ &\leq \|\mathcal{L}^n 1\|_\infty \cdot \|g\|'_\kappa \cdot c_n \lesssim \eta(t_1,\lambda_1,t_2,\lambda_2)^n \cdot \|g\|'_\kappa \cdot c_n\end{aligned}$$

for some c_n as in the statement. In the third inequality, we have used the fact that $\max_i d_\mathcal{T}(a_i,b_i) \leq d_\mathcal{T}(a,b) \cdot c'_n$ for some $c'_n > 0$ with $c'_n \to 0$ as $n \to \infty$, since T is expanding.

For II, observe that we have $II = 0$ if $t_1 = 0$. Otherwise, we compute

$$\begin{aligned}II &= \sup_{a,b\in\mathcal{T}} \frac{1}{d_\mathcal{T}(a,b)^\kappa} \cdot \sum_i |g(b_i)| \cdot \big| |(T^n_{\lambda_1})'(H_{\lambda_1}(a_i))|^{-t_1} - |(T^n_{\lambda_1})'(H_{\lambda_1}(b_i))|^{-t_1}\big| \cdot \big|(T^n_{\lambda_2})'(H_{\lambda_2}(b_i))\big|^{-t_2} \\ &\lesssim |t_1| \cdot \sup_{a,b\in\mathcal{T}} \frac{1}{d_\mathcal{T}(a,b)^\kappa} \cdot \sum_i |g(b_i)| \left| \frac{(T^n_{\lambda_1})''(H_{\lambda_1}(b_i))}{(T^n_{\lambda_1})'(H_{\lambda_1}(b_i))|^{t_1+2}} \right| \cdot \big|(T^n_{\lambda_2})'(H_{\lambda_2}(b_i))\big|^{-t_2} \cdot d_\mathcal{T}(H_\lambda(y_a), H_\lambda(y_b)) \\ &\lesssim |t_1| \cdot \|\mathcal{L}^n |g|\|_\infty \cdot \sup_{a,b\in\mathcal{T}} \frac{1}{d_\mathcal{T}(a,b)^\kappa} \cdot \max_i \frac{|(T^n_\lambda)''(H_\lambda(b_i))|}{|(T^n_\lambda)'(H_\lambda(b_i))|^2} \cdot d_\mathcal{T}(H_\lambda(a_i), H_\lambda(b_i)) \\ &\lesssim |t_1| \cdot \|\mathcal{L}^n |g|\|_\infty \cdot \sup_{a,b\in\mathcal{T}} \frac{1}{d_\mathcal{T}(a,b)^\kappa} \cdot d_\mathcal{T}(a,b)^\kappa = |t_1| \cdot \|\mathcal{L}^n |g|\|_\infty,\end{aligned}$$

where in the last line we used Lemmas 3.8 and 3.9. For III, we can use the same steps as for II, up to reversing the role of (t_1, λ_1) and (t_2, λ_2). The term IV is similarly bounded by $\|\mathcal{L}^n|g|\|_\infty$ using the same estimates as in II and Lemma 3.8 on both the differences of products inside the sum (in particular, we have $IV = 0$ if either $t_1 = 0$ or $t_2 = 0$). The proof is complete. □

3.4. **Holomorphicity of complex transfer operators.** To prove Theorem 3.1, we will need to establish the *joint analyticity* of the pressure function P in all its parameters. To do this, we will need to consider suitable *complex* extensions of the operators $\mathcal{L}_{t,\lambda}$. This will be done in the next section. In this section, we collect the definitions and tools that we will need concerning this extension. Observe that Proposition 3.14 will be used also in Section 5 to prove the analyticity of the pressure form on the unit tangent bundle.

Let W be an open subset of $\mathbb{C}^\ell$ for some $\ell \geq 1$. We let $\mathcal{L}_w$ be a family of operators of the form

$$\mathcal{L}_w g(x) := \sum_{T(y)=x} e^{\zeta_w(y)} g(y),$$

where $\zeta_w \colon T^{-1}(\mathcal{T}) \to \mathbb{C}$ is a complex-valued function. Assuming that, for every w, $\mathcal{L}_w$ is a bounded linear operator on the space $C^\kappa(\mathcal{T}, \mathbb{C})$ of complex valued κ-Hölder continuous functions on $\mathcal{T}$, we say that the map $W \ni w \mapsto \mathcal{L}_w$ is *holomorphic* if for every $w \in W$, there exists a bounded operator $\mathcal{L}'_w$ on $C^\kappa(\mathcal{T})$ such that $||h^{-1}(\mathcal{L}_{w+h} - \mathcal{L}_w) - \mathcal{L}'_w||_\kappa \to 0$ as $h \to 0$.

Proposition 3.14. *Let $\{\zeta_w\}_{w\in W}$ be a family of complex valued functions on $T^{-1}(\mathcal{T})$. Assume also that*

(1) there exists $w_0 \in W$ such that $\mathcal{L}_{w_0}$ is a bounded operator on $C^\kappa(\mathcal{T}, \mathbb{C})$;
(2) the family of functions $\{w \mapsto \sum_{\hat{T}} e^{\zeta_w} \circ \hat{T}\}_{w\in W}$ is uniformly continuous with respect to $\|\cdot\|_\kappa$ on $U_1 \times \{1\}$, where $\hat{T} : U_1 \times \{1\} \to T^{-1}(U_1 \times \{1\})$ ranges over all inverse branches of T on $U_1 \times \{1\}$;
(3) the family of functions $\{\zeta_w\}_{w\in W}$ is uniformly continuous with respect to $\|\cdot\|_\kappa$ on $\mathcal{T} \setminus (U_1 \times \{1\})$;
(4) for every $x \in \mathcal{T}$, the function $W \ni w \mapsto \zeta_w(x)$ is holomorphic.

Then, the map $W \ni w \mapsto \mathcal{L}_w$ is holomorphic (and in particular consists of bounded operators).

Remark 3.15. The third condition in Proposition 3.14 could be replaced by

(3') the family of functions $\{e^{\zeta_w} \circ T^{-1}\}_{w\in W}$ is uniformly continuous with respect to $\|\cdot\|_\kappa$ on $\mathcal{T} \setminus (U_1 \times \{1\})$;

which we recognize as the same condition as (2) for points in $\mathcal{T} \setminus (U_1 \times \{1\})$, which have only one preimage under T.

Proof. As in [BH24, Proposition 2.9], it is enough to show that the map $W \ni w \mapsto \mathcal{L}_w$ is continuous as a family of operators on $C^\kappa(\mathcal{T}, \mathbb{C})$. This, together with the fourth assumption, implies the holomorphicity of $\mathcal{L}_w$; see for instance [UZ04, Lemma 7.1] or [SU10, Lemma 5.1].

For every $x \in U_1 \times \{1\}$, we have

$$\begin{aligned} |(\mathcal{L}_w - \mathcal{L}_{w'})g(x)| &= \Big| \sum_{T(y)=x} \left(e^{\zeta_w(y)} - e^{\zeta_{w'}(y)}\right) g(y)\Big| \leq \|g\|_\infty \cdot \Big| \sum_{T(y)=x} (e^{\zeta_w} - e^{\zeta_{w'}})(y)\Big| \\ &= \|g\|_\infty \cdot \Big|\Big(\sum_{\hat{T}} e^{\zeta_w} \circ \hat{T} - \sum_{\hat{T}} e^{\zeta_{w'}} \circ \hat{T}\Big)(x)\Big|. \end{aligned}$$

Therefore, we have

$$\|(\mathcal{L}_w - \mathcal{L}_{w'})g\|_{L^\infty(U_1\times\{1\})} \lesssim \|g\|_\infty \cdot \Big| \sum_{\hat{T}} e^{\zeta_w} \circ \hat{T} - \sum_{\hat{T}} e^{\zeta_{w'}} \circ \hat{T}\Big|_{L^\infty(U_1\times\{1\})}.$$

The last term goes to zero as $w' \to w$ thanks to the second assumption. The same steps as above can be used to show the continuity in $L^\infty(\mathcal{T} \setminus (U_1 \times \{1\}))$, using the third assumption (where the sum over the inverse branches contains now only one element).

We then check the continuity with respect to the semi-norm $\|\cdot\|'_\kappa$. As above, we check it on $U_1 \times \{1\}$ using the second assumption, and the check on $\mathcal{T} \setminus (U_1 \times \{1\})$ is completely analogous (and simpler) using the third assumption.

For every $x, x' \in U_1 \times \{1\}$ and $w, w' \in W$, we have

$$
\begin{aligned}
|(\mathcal{L}_w - \mathcal{L}_{w'})g(x) - & (\mathcal{L}_w - \mathcal{L}_{w'})g(x')| \\
&= \Big| \sum_{T(y)=x} e^{\zeta_w(y)} g(y) - e^{\zeta_{w'}(y)} g(y) - \sum_{T(y')=x'} e^{\zeta_w(y')} g(y') - e^{\zeta_{w'}(y')} g(y') \Big| \\
&\leq \Big| \sum_{\hat{T}} \big(e^{\zeta_w(\hat{T}(x))} - e^{\zeta_{w'}(\hat{T}(x))} \big) \big(g(\hat{T}(x)) - g(\hat{T}(x')) \big) \Big| \\
&\quad + \|g\|_\infty \cdot \Big| \sum_{\hat{T}} \big(e^{\zeta_w(\hat{T}(x))} - e^{\zeta_{w'}(\hat{T}(x))} \big) - \big(e^{\zeta_w(\hat{T}(x'))} - e^{\zeta_{w'}(\hat{T}(x'))} \big) \Big|.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
\big\| \big((\mathcal{L}_w - \mathcal{L}_{w'}) g \big)|_{U_1 \times \{1\}} \big\|'_\kappa \lesssim & \Big\| \sum_{\hat{T}} e^{\zeta_w} \circ \hat{T} - \sum_{\hat{T}} e^{\zeta_{w'}} \circ \hat{T} \Big\|_{L^\infty(U_1 \times \{1\})} \cdot \|g\|'_\kappa \\
& + \|g\|_\infty \cdot \Big\| \big(\sum_{\hat{T}} e^{\zeta_w} \circ \hat{T} - \sum_{\hat{T}} e^{\zeta_{w'}} \circ \hat{T} \big)|_{U_1 \times \{1\}} \Big\|'_\kappa .
\end{aligned}
$$

As before, the last term goes to zero as $w' \to w$ thanks to the second assumption. This completes the proof. ☐

3.5. **Extension of analytic functions.** For every integer $d \geq 1$, consider the embedding $\iota_d \colon \mathbb{C}^d \to \mathbb{C}^{2d}$ given by

$$
(x_1 + iy_1, \dots, x_d + iy_d) \mapsto (x_1, y_1, \dots, x_d, y_d). \tag{6}
$$

Observe, in particular, that $\mathbb{C}^d$ is embedded by ι_d in $\mathbb{C}^{2d}$ as the set of points of real coordinates (which, in turn, is parametrized by $\mathbb{C}^d$ by means of ι_d). For every $z \in \mathbb{C}^\ell$ and every $r > 0$, denote by $D_\ell(z, r)$ the ℓ-dimensional polydisk in $\mathbb{C}^\ell$ centered at z and with radius r. Observe that, with the above identification, we have $\iota_d(D_d(0, r)) \subset D_{2d}(0, r)$ and, more generally, $\iota_d(D_d(z, r)) \subset D_{2d}(\iota_d(z), r)$ for every $z \in \mathbb{C}^d$.

Lemma 3.16 ([SU10, Lemma 6.4])**.** *For every $M \geq 0$, $R > 0$, $\lambda_0 \in \mathbb{C}^d$, and every complex analytic function $\phi : D_d(\lambda_0, R) \to \mathbb{C}$ which is bounded in modulus by M, there exists a complex analytic function $\tilde{\phi} : D_{2d}(\iota_d(\lambda_0), R/4) \to \mathbb{C}$ that is bounded in modulus by $4^d M$ and such that the restriction of $\tilde{\phi} \circ \iota_d$ to $D_d(\lambda_0, R/4)$ coincides with the real part $\Re(\phi)$ of ϕ.*

3.6. **Proof of Theorem 3.1.** In order to prove Theorem 3.1, we will follow the general framework of [SU10, Proof of Theorem A] and apply Proposition 3.14 to a suitable family of complex operators, that we now define.

First of all, observe that the map $\chi_\lambda \colon \Omega \to \mathbb{C}$ is holomorphic and takes value in the complement of $\overline{\mathbb{D}}$. Hence, $\log|\chi_\lambda|$ is harmonic on Ω, and $|\chi_\lambda|$ has no maximum in Ω. Since the assertion is local, it is enough to prove it on any $\Omega' \Subset N(0)$. We fix such Ω' and also a $\lambda_0 \in N(0)$ with $|\chi_{\lambda_0}| > \sup_{\lambda \in \Omega'} |\chi_\lambda|$. Recall that the holomorphic motion of the Julia sets is γ-Hölder continuous on $N(0)$. We allow ourself to shrink $N(0)$ during the proof, if the estimates are uniform in Ω' and λ_0.

For every $y = (z, k) \in \mathcal{T}$ and $\lambda \in N(0)$, consider the map

$$\psi_y(\lambda) := \begin{cases} 1 & z \in U_{k+1} \times \{k\} \\ \hat{\psi}_y(\lambda) := \frac{(f_\lambda^k)' \circ h_\lambda(z)}{(f_{\lambda_0}^k)' \circ h_{\lambda_0}(z)} & z \notin \overline{U_{k+1}} \times \{k\} \end{cases}$$

and observe that we have $|\psi_y(\lambda)| = |R_\lambda(y)/R_{\lambda_0}(y)|$ for every $y \in \mathcal{T}$. It follows from the choice of λ_0 and Lemma 3.5 that, shrinking $N(0)$ if necessary, for all $y \in \mathcal{T}$ and $\lambda \in \Omega'$, we have

$$|\psi_y(\lambda) - 1| < 1/5.$$

Working in chart, we can also assume that $\Omega' = D_\ell(0, R_0)$ for some $R_0 > 0$ and $\ell := \dim_{\mathbb{C}} \Omega$. Then, for every $k \in \mathbb{N}$ and $y = (z, k) \in U_k \times \{k\}$, there exists a branch of $\log \hat{\psi}_y$ on $\Omega' = D_\ell(0, R_0)$ sending 0 to 0 and whose modulus is bounded by 1/4. Applying Lemma 3.16 to the complex analytic function $\log \hat{\psi}_y$, we see that the real analytic function $\Re \log \hat{\psi}_y \colon D_\ell(0, R_0) \to \mathbb{R}$ has an analytic extension $\widetilde{\Re\log \hat{\psi}_y} \colon D_{2\ell}(0, R') \to \mathbb{R}$ for some $R' \in (0, R_0)$ which *does not depend on* k and whose modulus is bounded by $4^{\ell-1}$. Recall that $D_\ell(0, R_0)$ is seen as a subset of the points of $\mathbb{C}^{2\ell}$ with real coordinates by means of the immersion ι_ℓ as in (6), and that we have $\iota_\ell(0) = 0 \in \mathbb{C}^{2\ell}$.

For $(t, \lambda) \in \mathbb{C} \times D_{2\ell}(0, R')$, define $\zeta_{(t,\lambda)} \colon \mathcal{T} \to \mathbb{C}$ by

$$\zeta_{(t,\lambda)}(y) := \begin{cases} -t \widetilde{\log \chi_*} & z \in U_{k+1} \times \{k\} \\ -t \widetilde{\Re\log \hat{\psi}_y}(\lambda) - t \log(\chi_*^{1-k} |(f_{\lambda_0}^k)'(h_{\lambda_0}(z))|) & z \notin \overline{U_{k+1}} \times \{k\}. \end{cases}$$

Observe that we have $e^{\zeta_{(t,\lambda)}} = R_\lambda^{-t}$ for every $(t, \lambda) \in \mathbb{R} \times D_\ell(0, R')$.

We want to apply Proposition 3.14 with $w = (t_1, \lambda_1, t_2, \lambda_2)$ and to the family of operators $\mathcal{L}_{(t_1,\lambda_1,t_2,\lambda_2)} = \mathcal{L}_{\zeta_{(t_1,\lambda_2)} + \zeta_{(t_2,\lambda_2)}}$. For each $x \in \mathcal{T}$, the map $(t_1, \lambda_1, t_2, \lambda_2) \mapsto \big(\zeta_{(t_1,\lambda_1)} + \zeta_{(t_2,\lambda_2)}\big)(x)$ is holomorphic by the definition of $\zeta_{(t,\lambda)}$. Moreover, $\mathcal{L}_{(\delta(0),0,0,0)}$ is a bounded operator on $C^\kappa(\mathcal{T}, \mathbb{C})$ by Lemma 3.10. The following lemma gives the second condition in Proposition 3.14. The third condition can be verified by a similar (and simpler) computation, as only one inverse branch is involved.

Lemma 3.17. *Up to shrinking R', the family of functions*

$$D_1(\delta(0), R') \times D_{2\ell}(0, R') \times D_1(0, R') \times D_{2\ell}(0, R') \ni (t_1, \lambda_1, t_2, \lambda_2) \mapsto \sum_{\hat{T}} e^{\zeta_{(t_1,\lambda_1)} + \zeta_{(t_2,\lambda_2)}} \circ \hat{T}$$

is uniformly continuous with respect to the Hölder norm $\| \cdot \|_\kappa$ on $U_1 \times \{1\}$.

As we will only work on $U_1 \times \{1\}$, we will drop the dependence of the norm on this space. Lemma 3.17 will follow from the following two lemmas, giving the continuity with respect to the $\| \cdot \|_\infty$-norm and the $\| \cdot \|'_\kappa$-norm, respectively. In all the statements and the proofs below, whenever an inverse branch $\hat{T}$ of T is given, we will denote by $k = k(\hat{T})$ the floor which contains the image of $\hat{T}$, which is equal to the iteration exponent as in the definition of T.

Lemma 3.18. *Up to shrinking R', there exists $L_1 > 0$ such that for any $\lambda_1, \lambda_2, \lambda'_1, \lambda'_2 \in D_{2\ell}(0, R')$, $t_1, t'_1 \in D_1(\delta(0), R')$, and $t_2, t'_2 \in D_1(0, R')$ we have*

$$\Big\| \sum_{\hat{T}} e^{\zeta_{(t'_1,\lambda'_1)} + \zeta_{(t'_2,\lambda'_2)}} \circ \hat{T} - \sum_{\hat{T}} e^{\zeta_{(t_1,\lambda_1)} + \zeta_{(t_2,\lambda_2)}} \circ \hat{T} \Big\|_\infty \le L_1 \|(t'_1, \lambda'_1, t'_2, \lambda'_2) - (t_1, \lambda_1, t_2, \lambda_2)\|.$$

Proof. We will prove a uniform continuity estimate first at fixed t_1, t_2, λ_2 and then at fixed $\lambda_1, t_2, \lambda_2$. As the problem is essentially symmetric in (t_1, λ_1) and (t_2, λ_2) (apart from the ranges of t_1 and t_2, see Remark 3.19 below for this issue), this will give the desired continuity. In particular, in all this proof t_2 and λ_2 will be fixed.

First, let us fix $t_1 \in D_1(\delta(0), R')$. For every $\lambda_1, \lambda_1' \in D_{2\ell}(0, R')$, we have

$$\begin{aligned}
&\Big|\sum_{\hat{T}} e^{\zeta(t_1,\lambda_1')+\zeta(t_2,\lambda_2)} \circ \hat{T}(w) - \sum_{\hat{T}} e^{\zeta(t_1,\lambda_1)+\zeta(t_2,\lambda_2)} \circ \hat{T}(w)\Big| \\
&\le \sum_{\hat{T}} \big(e^{\zeta(t_2,\lambda_2)} \circ \hat{T}(w)\big) e^{-\Re t_1 \cdot \log(\chi_*^{1-k}|(f_{\lambda_0}^k)'(h_{\lambda_0}(\hat{T}(w)))|)} \cdot \Big| e^{-t_1 \cdot \Re \widetilde{\log \hat{\psi}_{\hat{T}(w)}}(\lambda_1')} - e^{-t_1 \cdot \Re \widetilde{\log \hat{\psi}_{\hat{T}(w)}}(\lambda_1)} \Big| \\
&\lesssim \sum_{\hat{T}} \big(e^{\zeta(t_2,\lambda_2)} \circ \hat{T}(w)\big) e^{-\Re t_1 \cdot \log(\chi_*^{1-k}|(f_{\lambda_0}^k)'(h_{\lambda_0}(\hat{T}(w)))|)} \cdot \Big| \Re \widetilde{\log \hat{\psi}_{\hat{T}(w)}}(\lambda_1') - \Re \widetilde{\log \hat{\psi}_{\hat{T}(w)}}(\lambda_1) \Big|,
\end{aligned}$$

where the implicit constant is independent of $t_1, \lambda_1, \lambda_1'$. Since $|\Re \widetilde{\log \hat{\psi}_{\hat{T}(w)}}(\lambda)| \le 4^{\ell-1}$ *for every inverse branch* $\hat{T}$ and every λ, by Cauchy's formula we have

$$|\Re \widetilde{\log \hat{\psi}_{\hat{T}(w)}}(\lambda_1) - \Re \widetilde{\log \hat{\psi}_{\hat{T}(w)}}(\lambda_1')| \lesssim |\lambda_1 - \lambda_1'|,$$

where the implicit constant is independent of λ_1, λ_1' and of the branch $\hat{T}$ (and of t_1). As $\Re t_1 > 0$, the term $e^{-\Re t_1 \cdot \log(\chi_*^{1-k}|(f_{\lambda_0}^k)'(h_{\lambda_0}(\hat{T}(w)))|)}$ is exponentially small for $k \to \infty$, by the choice of λ_0. For similar reasons, the term $e^{\zeta(t_2,\lambda_2)} \circ \hat{T}(w)$ is also uniformly bounded. Therefore, we have

$$\Big\| \sum_{\hat{T}} e^{\zeta(t_1,\lambda_1')+\zeta(t_2,\lambda_2)} \circ \hat{T} - \sum_{\hat{T}} e^{\zeta(t_1,\lambda_1)+\zeta(t_2,\lambda_2)} \circ \hat{T} \Big\|_\infty \lesssim |\lambda_1' - \lambda_1|, \tag{7}$$

where the implicit constant is independent of $t_1, \lambda_1, \lambda_1'$ (and of t_2, λ_2), as desired.

We now fix λ_1 and consider the variation in t_1. For any $\lambda_1 \in D_{2\ell}(0, R')$ and $t_1, t_1' \in D_1(\delta(0), R')$, we have (in the first step we ignore the term $e^{\zeta(t_2,\lambda_2)} \circ \hat{T}(w)$ as it is uniformly bounded, as above)

$$\begin{aligned}
&\Big|\sum_{\hat{T}} e^{\zeta(t_1,\lambda_1)+\zeta(t_2,\lambda_2)} \circ \hat{T}(z) - \sum_{\hat{T}} e^{\zeta(t_1',\lambda_1)+\zeta(t_2,\lambda_2)} \circ \hat{T}(z)\Big| \\
&\lesssim \sum_{\hat{T}} e^{-\Re t_1 \Re \widetilde{\log \hat{\psi}_{\hat{T}(z)}}(\lambda_1)} \cdot e^{-\Re t_1 \log(\chi_*^{1-k}|(f_{\lambda_0}^k)'(h_{\lambda_0}(\hat{T}(z)))|)} \\
&\qquad\qquad \cdot \big(1 - e^{(t_1 - t_1')(\Re \widetilde{\log \hat{\psi}_{\hat{T}(z)}}(\lambda_1) - \log(\chi_*^{1-k}|(f_{\lambda_0}^k)'(h_{\lambda_0}(\hat{T}(z)))|))}\big) \\
&\lesssim |t_1 - t_1'| \sum_{\hat{T}} e^{-\Re t_1 \log(\chi_*^{1-k}|(f_{\lambda_0}^k)'(h_{\lambda_0}(\hat{T}(z)))|)} \big| \Re \widetilde{\log \hat{\psi}_{\hat{T}(z)}}(\lambda_1) - \log(\chi_*^{1-k}|(f_{\lambda_0}^k)'(h_{\lambda_0}(\hat{T}(z)))|) \big|,
\end{aligned}$$

where in the last step we bounded the factor $e^{-\Re t_1 \Re \widetilde{\log \hat{\psi}_{\hat{T}(z)}}(\lambda_1)}$ as in the previous part of the proof. To conclude, denoting $a_k := \log(\chi_*^{1-k}|(f_{\lambda_0}^k)'(h_{\lambda_0}(\hat{T}(z)))|)$ and observing that $a_k \to \infty$ for $k \to \infty$, we just need to bound the expression

$$\sum_{\hat{T}} e^{-\Re t_1 a_k} \big| \Re \widetilde{\log \hat{\psi}_{\hat{T}(z)}}(\lambda_1) - a_k \big|.$$

As the term $\Re \widetilde{\log \hat{\psi}_{\hat{T}(z)}}(\lambda_1)$ is bounded and t_1 belongs to a small neighborhood of $\delta(0)$, the expression in bounded and the desired bound

$$\Big\| \sum_{\hat{T}} e^{\zeta(t_1,\lambda_1)+\zeta(t_2,\lambda_2)} \circ \hat{T} - \sum_{\hat{T}} e^{\zeta(t_1',\lambda_1)+\zeta(t_2,\lambda_2)} \circ \hat{T} \Big\|_\infty \lesssim |t_1 - t_1'|, \tag{8}$$

follows, where the implicit constant is independent of t_1, t_1', λ_1 (and of t_2, λ_2). The assertion now follows from (7) and (8). ☐

Remark 3.19. In the proof of Lemma 3.18, when working with t_2 instead of t_1, we do not have an exponential bound for $e^{-\Re t_2 \log(\chi_*^{1-k}|(f_{\lambda_0}^k)'(h_{\lambda_0}(\hat{T}(z)))|)}$ as t_2 may be equal to 0. On the other hand, the term $e^{-\Re t_1 \log(\chi_*^{1-k}|(f_{\lambda_0}^k)'(h_{\lambda_0}(\hat{T}(z)))|)}$ is still present (and exponentially small), and can be used to bound the expressions in the same way as above.

Lemma 3.20. *Up to shrinking R', there exists a constant $L_2 \geq 1$ such that for any $\lambda_1, \lambda_2, \lambda_1', \lambda_2' \in D_{2\ell}(0, R')$, $t_1, t_1' \in D_1(\delta(0), R')$, and $t_2, t_2' \in D_1(0, R')$, we have*

$$\Big\| \sum_{\hat{T}} e^{\zeta_{(t_1,\lambda_1)} + \zeta_{(t_2,\lambda_2)}} \circ \hat{T} - \sum_{\hat{T}} e^{\zeta_{(t_1',\lambda_1')} + \zeta_{(t_2',\lambda_2')}} \circ \hat{T} \Big\|_\kappa' \leq L_2 ||(t_1', \lambda_1', t_2, \lambda_2') - (t_1, \lambda_1, t_2, \lambda_2)||.$$

Proof. We first observe that, thanks to Lemma 3.18 and the fact that $e^{-\Re t_1 \cdot \log(\chi_*^{1-k}|(f_{\lambda_0}^k)'(h_{\lambda_0}(\hat{T}(w)))|)}$ is exponentially small for $k \to \infty$ and $e^{\zeta_{(t_2,\lambda_2)}} \circ \hat{T}(w)$ is uniformly bounded, it will be enough to prove a *uniform* continuity with respect to the κ-Hölder seminorm $\|\cdot\|_\kappa'$ for all terms $e^{\zeta_{(t_1,\lambda_1)}} \circ \hat{T}$. Namely, we do not need to control the constants in an exponentially small way, as the summable coefficients are provided by the bounds for $\|\cdot\|_\infty$ in Lemma 3.18. We will then prove, for every $\hat{T}$, the κ-Hölder continuity of both the terms in $\zeta_{(t_1,\lambda_1)} \circ \hat{T}$. The desired bound then follows computing the variation of the function $e^{\zeta_{(t_1,\lambda_1)} \circ \hat{T}}$, which we can naturally write as the product of two exponentials, one of which is uniformly bounded and the other exponentially small in k.

As in Lemma 3.18 we first consider the variation in t_1 at a fixed λ_1. For every inverse branch $\hat{T}$ of T on $U_1 \times \{1\}$ and every $z, w \in U_1 \times \{1\}$ we have

$$\Big| \log |(f_{\lambda_0}^k)'(h_{\lambda_0}(\hat{T}(w)))| - \log |(f_{\lambda_0}^k)'(h_{\lambda_0}(\hat{T}(z)))| \Big| \lesssim d(z, w)^\gamma, \tag{9}$$

where the implicit constant is independent of $z, w, \hat{T}$ and we recall that $\gamma > \kappa$ is the Hölder exponent of the holomorphic motion over $N(0)$. Indeed, observe that both $\hat{T}(z)$ and $\hat{T}(w)$ belong to $D_k \times \{k\}$. For every $\lambda_3 \in N(0)$, by Lemma 3.8 we then have

$$\Big| \log(f_{\lambda_3}^k)'(h_{\lambda_1}(\hat{T}(w))) - \log(f_{\lambda_3}^k)'(h_{\lambda_1}(\hat{T}(z))) \Big| \lesssim \frac{|(f_{\lambda_3}^k)''|}{|(f_{\lambda_3}^k)'|^2} C_\gamma d(z, w)^\gamma \lesssim d(z, w)^\gamma,$$

which gives (9) taking $\lambda_3 = \lambda_1$. Applying the above inequality with $\lambda_3 = \lambda_1$ and $\lambda_3 = \lambda_0$ and taking the difference, we obtain

$$| \log \hat{\psi}_{\hat{T}(w)}(\lambda_1) - \log \hat{\psi}_{\hat{T}(z)}(\lambda_1)| \lesssim d(z, w)^\gamma.$$

We deduce from Lemma 3.16 that we have

$$|\Re \widetilde{\log \hat{\psi}_{\hat{T}(w)}}(\lambda_1) - \Re \widetilde{\log \hat{\psi}_{\hat{T}(z)}}(\lambda_1)| \lesssim 4^\ell d(z, w)^\gamma, \tag{10}$$

where again the implicit constants are independent of $\hat{T}, z, w$. Together, (9) and (10) give the Hölder continuity in t_1 of the two terms in the definition of $\zeta_{(t_1,\lambda_1)} \circ \hat{T}$, as desired.

We then consider the variation in λ_1 for a fixed t_1. Observe that the second term in the definition of $\zeta_{(t_1,\lambda_1)}$ does not depend on λ_1, hence we only have to prove the continuity in λ_1 of its first term, i.e., of $\Re \widetilde{\log \hat{\psi}_{\hat{T}(z)}}(\lambda_1)$.

Since by (10) we have $|\Re\widetilde{\log \hat{\psi}}_{\hat{T}(w)}(\lambda_1) - \Re\widetilde{\log \hat{\psi}}_{\hat{T}(z)}(\lambda_1)| \leq 4^\ell \cdot d(z,w)^\kappa$, up to shrinking R', by Cauchy's formula, we have

$$|\Re\widetilde{\log \hat{\psi}}_{\hat{T}(w)}(\lambda_1) - \Re\widetilde{\log \hat{\psi}}_{\hat{T}(z)}(\lambda_1) - \Re\widetilde{\log \hat{\psi}}_{\hat{T}(w)}(\lambda_1') + \Re\widetilde{\log \hat{\psi}}_{\hat{T}(z)}(\lambda_1')| \lesssim |\lambda_1 - \lambda_1'| \cdot d(z,w)^\kappa$$

for any $\lambda_1, \lambda_1' \in D_{2\ell}(0, R')$. This gives the desired continuity and completes the proof. □

Lemma 3.17 is then a consequence of Lemmas 3.18 and 3.20. We can now complete the proof of Theorem 3.1.

Proof of Theorem 3.1. By the above, and in particular by Lemma 3.17, up to shrinking R' the family of operators $\mathcal{L}_{(t_1,\lambda_1,t_2,\lambda_2)} = \mathcal{L}_{\zeta_{(t_1,\lambda_1,t_2,\lambda_2)}}$ verifies all the assumptions of Proposition 3.14. Therefore, by that proposition, the map $D_\ell(\delta(0), R') \times D_{2\ell}(0,R') \times D_1(0,R') \times D_{2\ell}(0,R') \ni (t_1,\lambda_1,t_1,\lambda_2) \mapsto \mathcal{L}_{(t_1,\lambda_1,t_2,\lambda_2)} \in C^\kappa(\mathcal{T})$ is holomorphic. Thanks to Proposition 3.12, the assertion is now a consequence of Kato-Rellich perturbation theorem. □

4. The Hessian form $\langle\cdot,\cdot\rangle_G$ and the pressure form $\langle\cdot,\cdot\rangle_{\mathcal{P}}$

Let Ω be a Λ-hyperbolic component of a Misurewicz family $\Lambda \subset \mathrm{rat}_D^{\mathrm{cm}}$. In this section, we construct the Hessian form $\langle\cdot,\cdot\rangle_G$ and the pressure form $\langle\cdot,\cdot\rangle_{\mathcal{P}}$. Thanks to Corollary 3.2, the construction of the Hessian form $\langle\cdot,\cdot\rangle_G$ and the pressure form $\langle\cdot,\cdot\rangle_{\mathcal{P}}$ can be obtained using the same arguments as in [HN23; BH24]. Here we give their definitions and the main steps of their constructions for completeness. For simplicity, we will work with a lift of Ω to $\mathrm{Rat}_D^{\mathrm{cm}}$ and construct a metric there. It follows from standard arguments that this metric indeed descends to the quotient (and is degenerate on vectors tangent to the fibers); see [HN23].

Fix $\lambda_0 \in \Omega$. Let $U(\lambda_0)$ be a neighborhood of λ_0 such that the conjugacy $h_\lambda\colon J_{\lambda_0} \to J_\lambda$ is well-defined for any $f_\lambda \in U(\lambda_0)$. We denote by $\nu = \nu_{\lambda_0}$ the unique equilibrium state on J_{λ_0} for the potential $-\delta(\lambda_0)\log|f'_{\lambda_0}|$. We note that the existence and uniqueness of ν is guaranteed by [MS03], see also Proposition 3.12 and [PRS03].

Consider the function $\mathrm{Ly}_{\lambda_0}\colon U(\lambda_0) \to \mathbb{R}$ given by

$$\mathrm{Ly}_{\lambda_0}(\lambda) := \int_{J_\lambda} \log|f'_\lambda| d\left((h_\lambda)_*\nu\right) = \int_{J_{\lambda_0}} \log|f'_\lambda \circ h_\lambda| d\nu.$$

We observe that, as the critical points in the Julia set are preserved by the holomorphic motion, for every $z \in J_{\lambda_0}$ the holomorphic function $\lambda \mapsto f'_\lambda \circ h_\lambda(z)$ is either never zero or constantly equal to zero. Hence, the same arguments as those in the proof of [BH24, Lemma 4.2] give that Ly_{λ_0} is harmonic, hence in particular real-analytic.

As a consequence of the analyticity of both Ly_{λ_0} and δ (by Corollary 3.2), we see that also the function $G_{\lambda_0}\colon U(\lambda_0) \to \mathbb{R}$ defined as

$$G_{\lambda_0}(\lambda) := \delta(\lambda)\mathrm{Ly}_{\lambda_0}(\lambda) = \delta(\lambda) \int_{J_{\lambda_0}} \log|f'_\lambda \circ h_\lambda| d\nu$$

is real-analytic. The same proof as that of [BH24, Proposition 4.5] gives that G_{λ_0} has a minimum at λ_0. Therefore we have $DG_{\lambda_0}(\lambda_0) = 0$ and the Hessian $G''_{\lambda_0}(\lambda_0)\colon T_{\lambda_0}\Omega \times T_{\lambda_0}\Omega \to \mathbb{R}$ is well-defined at λ_0 and it gives a positive semi-definite symmetric bilinear form $\langle\cdot,\cdot\rangle_G$ on the tangent space $T_{\lambda_0}\Omega$. Namely, we can set

$$\langle \vec{u}, \vec{v}\rangle_G := (G''_{\lambda_0}(\lambda_0))(\vec{u},\vec{v}) \quad \text{for every} \quad \vec{u},\vec{v} \in T_{\lambda_0}\Omega.$$

For every $\vec{v} \in T_{\lambda_0}\Omega$, we will also denote $\|\vec{v}\|_G := \sqrt{\langle \vec{v}, \vec{v} \rangle_G}$. A direct computation shows that if $\gamma(t), t \in (-1,1)$ is a smooth path in $U(0)$ with $\gamma(0) = \lambda_0$ and $\gamma'(0) = \vec{v} \in T_{\lambda_0}\Omega$, we have

$$||\vec{w}||_G^2 = \frac{d^2}{dt^2}\bigg|_{t=0} G_{\lambda_0}(\gamma(t)).$$

This characterization is used to prove that the metric, although defined in $\mathrm{Rat}_D^{\mathrm{cm}}$, indeed descends to the quotient in $\mathrm{rat}_D^{\mathrm{cm}}$; see for instance [BH24, Section 4.2].

Now we recall the construction of the pressure form $\langle \cdot, \cdot \rangle_{\mathcal{P}}$. Recall that $\delta(\lambda)$ is the unique real number satisfying $p(\delta(\lambda), \lambda) = 0$. Recall that the pressure function $\mathcal{P}(\phi)$ is defined as $\mathcal{P}(\phi) := \sup_\mu h_\mu(f_{\lambda_0}) + \int_{J_0} \phi d\mu$ where the supremum is taken over all the f_{λ_0}-invariant probability measures on J_0. By [PRS04], the definition of $\mathcal{P}(-t \log |f'_\lambda|)$ coincides with $p(t, \lambda)$ as in (2).

Let $\mathcal{C}(J_{\lambda_0})$ be the set of cohomology classes of Hölder continuous functions with pressure 0 with respect to f_{λ_0}, that is,

$$\mathcal{C}(J_{\lambda_0}) := \{\phi : \phi \in C^\alpha(J_{\lambda_0}, \mathbb{R}) \text{ for some } \alpha > 0, \mathcal{P}(\phi) = 0\}/\sim$$

where $\phi_1 \sim \phi_2$ if ϕ_1 and ϕ_2 are C^0-cohomologous on J_{λ_0}, i.e., there exists a continuous function $g \colon J_{\lambda_0} \to \mathbb{R}$ such that $\phi_1 - \phi_2 = g - g \circ f_{\lambda_0}$. For every $\lambda \in \Omega$, we denote by $\mathscr{E}(\lambda)$ the class $[-\delta(\lambda) \log |f'_\lambda \circ h_\lambda|] \in \mathcal{C}(J_{\lambda_0})$ and by ν_λ the unique equilibrium state for any representative ϕ of $\mathscr{E}(\lambda)$, whose existence and uniqueness is guaranteed by Proposition 3.12. It is a standard fact (see for instance [McM08, p. 375]) that the tangent space of $\mathcal{C}(J_{\lambda_0})$ at $\mathscr{E}(\lambda)$ can be identified with

$$T_{\mathscr{E}(\lambda)}\mathcal{C}(J_{\lambda_0}) = \left\{\psi \colon \psi \in C^\alpha(J_{\lambda_0}, \mathbb{R}) \text{ for some } \alpha > 0, \int_{J_{\lambda_0}} \psi d\nu_\lambda = 0\right\} / \sim,$$

where we used the fact that, by definition, the pressure is constant on $\mathcal{C}(J_{\lambda_0})$. Following [McM08, p. 375], we define the *pressure form* $|| \cdot ||_{pm}$ *on* $\mathscr{E}(\Omega) \subset \mathcal{C}(J_{\lambda_0})$ as

$$||[\psi]||_{pm}^2 := \frac{\mathrm{Var}(\psi, \nu_\lambda)}{\int_{J_{\lambda_0}} \phi d\nu_\lambda} \quad \text{for every} \quad \psi \in T_{\mathscr{E}(\lambda)}\mathcal{C}(J_{\lambda_0}),$$

where

$$\mathrm{Var}(\psi, \nu_\lambda) := \lim_{n\to\infty} \frac{1}{n} \int_{J_{\lambda_0}} \left(\sum_{i=0}^{n-1} \psi \circ f^i(x)\right)^2 d\nu_\lambda \in [0, +\infty].$$

Finally, we define the *pressure form* $\| \cdot \|_{\mathcal{P}}$ *on* Ω as the pull-back of $|| \cdot ||_{pm}$ by the map $\mathscr{E}$. Namely, given $\vec{w} \in T_\lambda\Omega$, let $\gamma(t) := [f_t]$, $t \in (-1, 1)$ be a smooth path in Ω with $\gamma(0) = \lambda$ and $\gamma'(0) = \vec{w}$. Letting ϕ_t be a representative of the class $\mathscr{E}(\gamma(t))$, we define

$$\|\vec{w}\|_{\mathcal{P}} := ||\dot{\phi}_0||_{pm} = \frac{\mathrm{Var}(\dot{\phi}_0, \nu_{\lambda_0})}{\int_{J_{\lambda_0}} \phi_0 d\nu_{\lambda_0}}.$$

Observe that $|| \cdot ||_{\mathcal{P}}$ is positive semi-definite on $T_\lambda\Omega$ since we have $\mathrm{Var}(\dot{\phi}_0, \nu_\lambda) \geq 0$ and $\int_{J_{\lambda_0}} \phi_0 d\nu_{\lambda_0} > 0$.

A computation as in [BH24, Proposition 4.12] gives the equality

$$||\vec{w}||_{\mathcal{P}}^2 = \frac{||\vec{w}||_G^2}{\int_{J_{\lambda_0}} \phi_0 d\nu_{\lambda_0}}.$$

Namely, the Hessian form $\langle \cdot, \cdot \rangle_G$ is conformal equivalent to the pressure form $\langle \cdot, \cdot \rangle_{\mathcal{P}}$. We also see that $||\vec{w}||_G = 0$ if and only if $||\vec{w}||_{\mathcal{P}} = 0$, if and only if $\mathrm{Var}(\dot{\phi}_0, \nu) = 0$. It follows from standard facts in thermodynamical formalism that these conditions are also equivalent to the fact that $\dot{\phi}_0(x)$

is a C^0-coboundary, i.e., it is C^0-cohomologous to zero. As in [HN23], we deduce from this last equivalence the following lemma.

Lemma 4.1. *If $||\vec{w}||_G = 0$, there exists a constant $K \in \mathbb{R}$ such that, for every $n \in \mathbb{N}$, we have*

$$\frac{d}{dt}\Big|_{t=0} S_n\big(\log|f_t' \circ h_{\gamma(t)}(x)|\big) = K \cdot S_n\big(\log|f' \circ h_{\gamma(0)}(x)|\big)$$

for all n-periodic points x of f in J_{λ_0}. Here $S_n\phi := \phi + \phi \circ f + \cdots + \phi \circ f^{n-1}$ denotes the Birkhoff sum of ϕ.

Given a C^1-path $\gamma\colon (0,1) \to \Omega$, we define the *length* $\ell_G(\gamma)$ of γ as

$$\ell_G(\gamma) := \int_0^1 \|\gamma'(t)\|_G dt.$$

Proposition 4.2. *Let Ω be a bounded Λ-hyperbolic component of a Misiurewicz family Λ in* poly_D^{cm}. *We have $\ell_G(\gamma) > 0$ for any non-trivial C^1-path $\gamma\colon (0,1) \to \Omega$.*

Proof. The proof follows the same arguments as that of [BH24, Proposition 5.3]. We give a sketch here to show how Lemma 4.1 is used and to highlight that this is the place where we use the assumption that Ω is a *bounded* component of a family $\Lambda \subset \mathrm{poly}_D^{cm}$ (which implies that the Lyapunov exponent of the measure of maximal entropy is constant on Ω; see (1)), as well as the need of the more refined equidistribution property for the multipliers as in Lemma 2.1 (1).

Suppose by contradiction that we have $\ell_G(\gamma) = 0$. We denote by $\{x_i(\lambda)\}_{i\geq 1}$ the set of maps parametrizing the repelling periodic points on Ω, and let n_i be the period of the corresponding cycle. Fix $s_0 \in (0,1)$ and an index i_0. It follows from Lemma 4.1 that, for every $i \geq 1$ and every $s \in (0,1)$, we have

$$S_{n_{i_0}n_i}\big(\log|f'_{\gamma(s)}(x_i(\gamma(s)))|\big) = a_{i,i_0} e^{\widetilde{K}(s,s_0)} S_{n_{i_0}n_i}\big(\log|f'_{\gamma(s_0)}(x_{i_0}(\gamma(s_0)))|\big)$$

for some strictly positive (as both $x_i(\gamma(s))$ and $x_{i_0}(\gamma(s_0))$ are repelling) constants a_{i,i_0}.

As Ω is bounded, the Lyapunov exponent of the measure of maximal entropy is constantly equal to $\log D$ on Ω by (1). Hence, it follows from Lemma 2.1 (1) that, for every $s \in (0,1)$, we have

$$\begin{aligned}\log D &= \lim_{n\to\infty} \frac{1}{n_{i_0}n} \frac{1}{D^{n_{i_0}n}} \sum_{x_i\colon n_i = n_{i_0}n} a_{i,i_0} e^{\widetilde{K}(s,s_0)} S_{n_{i_0}n}\big(\log|f'_{\gamma(s_0)}(x_{i_0}(\gamma(s_0)))|\big) \\ &= e^{\widetilde{K}(s,s_0)} \cdot \left(\lim_{n\to\infty} \frac{1}{D^{n_{i_0}n}} \sum_{x_i\colon n_i = n} a_{i,i_0}\right) \cdot \frac{1}{n_{i_0}} S_{n_{i_0}}\big(\log|f'_{\gamma(s_0)}(x_{i_0}(\gamma(s_0)))|\big).\end{aligned}$$

We deduce that the function $\widetilde{K}(s,s_0)$ is independent of s. This shows that the absolute values of all the multipliers of the $x_i(\gamma(s))$'s are constant along γ, which contradicts the rigidity results of [JX23]. □

5. Analyticity of the metric and a distance function on Ω

The goal of this section is to prove Theorem 1.2. Our proof follows the general framework of [BH24, Section 5]. For the case of Misiurewicz maps, the main point is to prove that the 2-form $\langle\cdot,\cdot\rangle_G$ is analytic on the unit tangent bundle of Ω. For every $k \in \mathbb{N}$, $z \in \mathbb{C}^k$, and $r > 0$, we denote by $D_k(z,r)$ the k-dimensional polydisk in $\mathbb{C}^k$ centered at z with radius r. Recall that $\ell := \dim_{\mathbb{C}}\Omega$.

Let $\{\vec{v}_s\}_{s\in D_1(0,R_0)}$ be a holomorphic family of elements of $\mathbb{C}^\ell \setminus \{\vec{0}\}$, i.e., we assume that the map $s \mapsto \vec{v}_s \in \mathbb{C}^\ell \setminus \{\vec{0}\}$ is holomorphic. Observe also that we can identify $T_\lambda\Omega$ with $\mathbb{C}^\ell$ for every

$\lambda \in D_\ell(0, R_0)$. Consider the map $\gamma\colon D_\ell(0, R_0) \times D_1(0, R_0) \times D_1(0, R_0) \to \mathbb{C}^\ell$ given by

$$\gamma(\lambda, t, s) = \lambda + t\vec{v}_s.$$

Up to shrinking R_0, we can assume that the image of γ is contained in Ω. Moreover, it is clear from the definition that γ satisfies

$$\gamma(\lambda, 0, s) = \lambda \quad \text{and} \quad \frac{d}{dt}\Big|_{t=0} \gamma(\lambda, t, s) = \vec{v}_s \in T_\lambda\Omega \quad \text{for all } \lambda \in D_\ell(0, R_0) \text{ and } s \in D_1(0, R_0).$$

For every $(\theta, \lambda, t, s) \in \mathbb{R} \times D_\ell(0, R_0) \times D_1(0, R_0) \times D_1(0, R_0)$, we also define

$$\phi_{(\theta,\lambda,t,s)} := -\delta(\lambda) \log |f'_\lambda| + \theta \log |f'_{\gamma(\lambda,t,s)} \circ \Psi_{\lambda,t,s}|\colon J_\lambda \to \mathbb{R}$$

where we denote by $\Psi_{\lambda,t,s}\colon J_\lambda \to J_{\lambda,t,s}$ the conjugacy map induced by the holomorphic motion on Ω. Using notations as in Section 3, the pressure of $\phi_{(\theta,\lambda,t,s)}$ can be written as

$$\mathcal{P}(\phi_{(\theta,\lambda,t,s)}) = P(t_1, \lambda_1, t_2, \lambda_2)$$

where $\lambda_1 = \lambda$, $\lambda_2 = \gamma(\lambda, t, s)$ (which is analytic in the variables), $t_1 = \delta(\lambda_1)$ (which is analytic in λ_1 and so in λ), and $t_2 = \theta$.

Proposition 5.1. *There exists* $0 < R < R_0$ *such that the map* $(-R, R) \times D_\ell(0, R) \times (-R, R) \times D_1(0, R) \ni (\theta, \lambda, t, s) \mapsto \mathcal{P}(\phi_{(\theta,\lambda,t,s)})$ *is real-analytic. In particular, the map* $D_\ell(0, R) \times D_1(0, R) \ni (\lambda, s) \mapsto (G''_\lambda(\lambda))(\vec{v}_s, \vec{v}_s)$ *is real-analytic.*

Proof. The first assertion is a direct consequence of Theorem 3.1. The second one conclusion follows by taking derivatives of the pressure with respect to θ and then along tangent vectors, as in the proof of [BH24, Corollary 5.2]. This completes the proof. □

We can now give a proof of Theorem 1.2.

Proof of Theorem 1.2. By Proposition 5.1, the pseudo-metric d_G is determined by a family of positive semi-definite bilinear forms on $T_\lambda\Omega$ depending analytically on $\lambda \in \Omega$. By the same induction argument as in [BH24, Section 5.4], we see that any continuous path between two points x and y having zero length must lie in an analytic real one-dimensional submanifold of Ω. If such a path existed, it would contradict Proposition 4.2. The assertion follows. □

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