A MAÑÉ-MANNING FORMULA FOR EXPANDING MEASURES FOR ENDOMORPHISMS OF $\mathbb{P}^k$

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Abstract. Let $k \geq 1$ be an integer and f a holomorphic endomorphism of $\mathbb{P}^k(\mathbb{C})$ of algebraic degree $d \geq 2$. We introduce a *volume dimension* for ergodic f-invariant probability measures with strictly positive Lyapunov exponents. In particular, this class of measures includes all ergodic measures whose measure-theoretic entropy is strictly larger than $(k-1)\log d$, a natural generalization of the class of measures of positive measure-theoretic entropy in dimension 1. The volume dimension is equivalent to the Hausdorff dimension when $k = 1$, but depends on the dynamics of f to incorporate the possible failure of Koebe's theorem and the non-conformality of holomorphic endomorphisms for $k \geq 2$.

If ν is an ergodic f-invariant probability measure with strictly positive Lyapunov exponents, we prove a generalization of the Mañé-Manning formula relating the volume dimension, the measure-theoretic entropy, and the sum of the Lyapunov exponents of ν. As a consequence, we give a characterization of the first zero of a natural pressure function for such expanding measures in terms of their volume dimensions. For hyperbolic maps, such zero also coincides with the volume dimension of the Julia set, and with the exponent of a natural (volume-)conformal measure. This generalizes results by Denker-Urbański and McMullen in dimension 1 to any dimension $k \geq 1$.

Our methods mainly rely on a theorem by Berteloot-Dupont-Molino, which gives a precise control on the distortion of inverse branches of endomorphisms along generic inverse orbits with respect to measures with strictly positive Lyapunov exponents.

1. Introduction

Let $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ be a rational map of degree $d \geq 2$ and ν an ergodic f-invariant probability measure whose Lyapunov exponent is strictly positive. Such a measure is necessarily supported on the Julia set $J(f)$ of f. There is a well-known relation between the Hausdorff dimension $\mathrm{HD}(\nu)$, the measure-theoretic entropy $h_\nu(f)$, and the Lyapunov exponent $\chi_\nu(f)$ of ν; namely, we have

$$\mathrm{HD}(\nu) = \frac{h_\nu(f)}{\chi_\nu(f)}. \tag{1.1}$$

This formula is usually referred to as the *Mañé-Manning formula*; see [Man84; Mañ88]. Hofbauer and Raith [HR92] proved a version of (1.1) for piecewise monotone maps on the unit interval with bounded variation; see also [Led81]. The fact that (1.1) holds in one-dimensional complex dynamics crucially relies on distortion estimates for univalent holomorphic maps coming from Koebe's theorem; see Section 1.2.

For smooth dynamical systems in higher dimensions, related formulas are known to hold in a number of settings. If $f : M \to M$ is a diffeomorphism of a compact manifold M and ν is an ergodic probability measure on M which is absolutely continuous with respect to the Lebesgue measure, Pesin [Pes77] proved that

$$h_\nu(f) = \chi_\nu^+(f)$$

Date: August 5, 2023.

where $\chi_\nu^+(f)$ is the sum of the non-negative Lyapunov exponents of f counted with multiplicity; see also [Mañ81]. When M is a surface, Young [You82] proved that

$$\mathrm{HD}(\nu) = \frac{h_\nu(f)}{\chi_+} + \frac{h_\nu(f)}{|\chi_-|}$$

when ν is ergodic and $\chi_- < 0 < \chi_+$ are its Lyapunov exponents. This formula has been generalized to the case of diffeomorphisms in any dimension; see [LY85] and [BPS99]. Such systems display attracting and repelling directions, and one decomposes the problem into two problems, one for f (along unstable manifolds) and one for f^{-1} (along stable manifolds). The Mañé-Manning formula (1.1) can be seen as a version of Young's result in (complex) dimension 1 where the system is not invertible. In this paper, we address the validity of (1.1) in several complex variables, and more specifically for expanding measures for (non-invertible) holomorphic endomorphisms of projective spaces in any dimension.

Let $k \geq 1$ be an integer and denote $\mathbb{P}^k := \mathbb{P}^k(\mathbb{C})$. If $f : \mathbb{P}^k \to \mathbb{P}^k$ is a holomorphic endomorphism of algebraic degree $d \geq 2$, it is not hard to find examples where (1.1), with χ_ν replaced by the sum of the Lyapunov exponents of ν (the natural generalization of the expansion rate along generic orbits), does not hold. For instance, one can consider product self-maps of $\mathbb{C}^2$ of the form $(z, w) \mapsto (z^2 + a_1, w^2 + a_2)$, where $a_i \in \mathbb{C}$ are such that the measures of maximal entropy of each component have different Hausdorff dimensions.

In [BD03], Binder-DeMarco proposed a conjectural formula for the Hausdorff dimension of the measure of maximal entropy μ of an endomorphism of $\mathbb{P}^k$ as follows:

$$\mathrm{HD}(\mu) = \frac{\log d}{\chi_1} + \cdots + \frac{\log d}{\chi_k}.$$

This conjecture has been partially settled [BD03; DD04; Dup11], and also versions of it have been proposed (and partially proved) for more general invariant measures [DD04; Dup11; Dup12; dV15; DR20]. In this paper, we introduce a natural dimension $\mathrm{VD}(\nu)$ for ergodic f-invariant measures ν with strictly positive Lyapunov exponents and show that this dimension satisfies a natural generalization of (1.1), where χ_ν is replaced by (two times) the sum of the Lyapunov exponents.

1.1. **Statement of results.** Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic endomorphism of algebraic degree $d \geq 2$. The *Julia set* $J(f)$ of f is the support of the unique measure of maximal entropy of f [Lyu83; BD01; DS10]. Let $\mathcal{M}^+(f)$ (resp. $\mathcal{M}_J^+(f)$) be the set of ergodic invariant probability measures on $\mathbb{P}^k$ (resp. on $J(f)$) with strictly positive Lyapunov exponents. The set $\mathcal{M}_J^+(f)$ contains the set $\mathcal{M}_e^+(f)$ of all ergodic probability measures whose measure-theoretic entropy is strictly larger than $(k-1)\log d$ [deT08; Dup12], which are the natural generalization of the ergodic measures with strictly positive entropy in dimension 1. Large classes of examples of measures in $\mathcal{M}_e^+(f)$ were constructed and studied in [Dup12; UZ13; SUZ14; BD23; BD22].

We introduce a *volume dimension* for measures $\nu \in \mathcal{M}^+(f)$; see Section 1.2 for an overview and Section 4 for precise definitions. The volume dimension is dynamical in nature and generalizes the notion of Hausdorff dimension in dimension 1 to higher dimensions to incorporate the failure of Koebe's theorem and the non-conformality of holomorphic endomorphisms.

For $\nu \in \mathcal{M}^+(f)$, we denote by $\mathrm{VD}(\nu)$ the volume dimension, $h_\nu(f)$ the measure-theoretic entropy, and $L_\nu(f)$ the *sum* of the Lyapunov exponents of ν. The main result of this paper relates these three quantities and generalizes the Mañé-Manning formula to any $k \geq 1$.

Theorem 1.1. *Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic endomorphism of algebraic degree $d \geq 2$. For every $\nu \in \mathcal{M}^+(f)$ we have*

$$\mathrm{VD}(\nu) = \frac{h_\nu(f)}{2L_\nu(f)}.$$

When $k = 1$, Theorem 1.1 reduces to the Mañé-Manning formula (1.1), as in this case we have $2\,\mathrm{VD}(\nu) = \mathrm{HD}(\nu)$; see Proposition 4.20. The factor $2 = 2k/k$ is due to the fact that we weight open sets of covers by their volume instead of their diameter and we have k Lyapunov exponents, counting multiplicities.

As an application of Theorem 1.1, we study a number of natural dimensions and quantities associated to an endomorphism f. In dimension 1, these quantities are already defined and well studied; see for example [DU91a; DU91b; PU10; McM00]. We first define a *dynamical dimension* $\mathrm{DD}^+_J(f)$ of f as

$$\mathrm{DD}^+_J(f) := \sup \left\{ \mathrm{VD}(\nu) \colon \nu \in \mathcal{M}^+_J(f) \right\}.$$

For $k = 1$, recall that the *pressure function* is defined as

$$P(t) := \sup_{\nu} \left\{ h_\nu(f) - t\chi_\nu(f) \right\},$$

where $t \in \mathbb{R}$ and the supremum is taken over the set of invariant probability measures on $J(f)$. In fact, the supremum can be taken over $\nu \in \mathcal{M}^+_J(f) = \mathcal{M}^+(f)$. This can be seen by combining Ruelle's inequality [Rue78] with a theorem of Przytycki [Prz93] stating that all invariant measures supported on the Julia set of a rational map have non-negative Lyapunov exponent.

For any $k \geq 1$, we define in a similar way a pressure function $P^+_J(t)$ as

$$P^+_J(t) := \sup \left\{ h_\nu(f) - tL_\nu(f) \colon \nu \in \mathcal{M}^+_J(f) \right\}.$$

By the above, we have $P^+_J(t) = P(t)$ when $k = 1$. We remark that, for any $k \geq 2$, there may exist ergodic probability measures ν on $J(f)$ with $L_\nu(f) < 0$; see Section 2.4 for examples and further comments. However, as in the case of $k = 1$, the pressure function $P^+_J(t)$ is still non-increasing and convex for all $k \geq 1$; see Lemma 2.14. We define

$$p^+_J(f) := \inf \left\{ t > 0 \colon P^+_J(t) \leq 0 \right\}.$$

As a consequence of Theorem 1.1, we have the following result which generalizes a theorem due to Denker-Urbański [DU91a; DU91b] in the case of rational maps to any dimension.

Theorem 1.2. *Let $f \colon \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic endomorphism of algebraic degree $d \geq 2$. Then we have*

$$2\,\mathrm{DD}^+_J(f) = p^+_J(f).$$

Finally, in the spirit of the celebrated Bowen-Ruelle formula for hyperbolic maps [Bow79; Rue82], we give an interpretation of $p^+_J(f)$, when f is hyperbolic (i.e., uniformly expanding on $J(f)$; see Section 2.1) in terms of (volume-)conformal measures. Given $t \geq 0$, we say that a probability measure ν on $J(f)$ is *t-volume-conformal on* $J(f)$ if, for every Borel subset $A \subset J(f)$ on which f is invertible, we have

$$\nu(f(A)) = \int_A |\operatorname{Jac} f|^t d\nu$$

and define

$$\delta_J(f) := \inf \left\{ t \geq 0 \colon \text{ there exists a } t\text{-volume-conformal measure on } J(f) \right\}.$$

For $k = 1$, the definitions of t-volume-conformal measures and $\delta_J(f)$ reduce to those of conformal measures and conformal dimension for rational maps; see [DU91a; DU91b; McM00; PU10]. In this case, owing to Bowen [Bow79], one sees that

$$\delta_J(f) = p_J^+(f) = \mathrm{HD}(J(f))$$

for every hyperbolic rational map f on $\mathbb{P}^1(\mathbb{C})$, and that there exists a unique ergodic measure ν on $J(f)$ such that $\mathrm{HD}(\nu) = \mathrm{HD}(J(f))$. We have here the following result in any dimension, which further motivates the definition of the volume dimension as a natural generalization of the Hausdorff dimension for all $k \geq 1$. Observe that, if f is hyperbolic, every invariant probability measure ν on $J(f)$ belongs to $\mathcal{M}^+(f)$.

Theorem 1.3. *Let $f\colon \mathbb{P}^k \to \mathbb{P}^k$ be a hyperbolic holomorphic endomorphism of algebraic degree $d \geq 2$. Then we have*

$$\delta_J(f) = p_J^+(f) = 2\,\mathrm{VD}(J(f))$$

and there exists a unique ergodic measure ν on $J(f)$ such that $\mathrm{VD}(\nu) = \mathrm{VD}(J(f))$.

Remark 1.4. As all our arguments will be local, our results apply more generally to the setting of *polynomial-like maps* in any dimension, i.e., proper holomorphic maps of the form $f\colon U \to V$, with $U \Subset V \Subset \mathbb{C}^k$ and V convex [DS03; DS10]. For a large class of such maps (i.e., those whose topological degree dominates all the other dynamical degrees [BDR23]), an analogue of the inclusion $\mathcal{M}_e^+(f) \subset \mathcal{M}_J^+(f)$ in this more general context has been proved in [BR22].

As every endomorphism of $\mathbb{P}^k$ lifts to a homogeneous polynomial endomorphism of $\mathbb{C}^{k+1}$, we can assume for simplicity that the maps we consider are polynomials. Observe that the Lyapunov exponents of every lifted measure are the same as those of the original measure, with the addition of an extra exponent $\log d$. Since $\log d > 0$ when $d \geq 2$, this does not change the condition on the positivity of the Lyapunov exponents.

1.2. **Volume dimensions and strategy of the proofs.** Let us first recall the idea of the proof of the Mañé-Manning formula (1.1) in dimension 1. It essentially consists of two steps.

(1) The first step consists of defining a *local dimension* at a point x by setting

$$\delta_x := \lim_{r\to 0} \frac{\log \nu(B(x,r))}{\log r}$$

(whenever the limit exists), where $B(x,r)$ denotes the balls of radius r centred and x, and proving that the limit is well-defined and equal to the ratio $h_\nu(f)/\chi_\nu(f)$ for ν-almost every x. In particular, ν is *exact-dimensional*.

(2) The second step is to prove that the Hausdorff dimension of ν must be equal to the common value of the local dimensions found in the first step [You82].

Let us describe how the one-dimensional setting plays a crucial role in Step (1). By [BK83] and [Mañ81], for ν-almost every x we have

$$h_\nu(f) = \lim_{\kappa\to 0}\lim_{n\to\infty} \frac{-\log \nu(B_n(x,\kappa))}{n},$$

where $B_n(x,\kappa)$ is the *Bowen ball of radius κ and depth n*. This is defined as

$$B_n(x,\kappa) := \left\{y\colon |f^j(y) - f^j(x)| < \kappa, 0 \leq j \leq n\right\}.$$

The crucial observation is that, for large n, the Bowen ball $B_n(x,\kappa)$ is comparable (up to precisely quantifiable errors) to the ball $B(x, \kappa\, e^{-n\chi_\nu(f)})$ of the same center and radius $\kappa\, e^{-n\chi_\nu(f)}$. Fixing a κ_0 for simplicity, and setting $n(r) \sim |\log r|/\chi_\nu(f)$, it then follows that

$$\lim_{r\to 0} \frac{\log \nu(B(x,r))}{\log r} = \lim_{r\to 0} \frac{-\log \nu(B_{n(r)}(x,\kappa_0))}{n(r)} \frac{n(r)}{-\log r} = \frac{h_\nu(f)}{\chi_\nu},$$

which in particular shows that δ_x is well-defined. The precise relation between geometric balls and Bowen balls is a consequence of Koebe's theorem and related distortion estimates, which imply that images of balls by holomorphic maps (and in particular by their inverse branches) are still comparable to balls. As a consequence, in complex dimension 1, there is a natural interplay between the Hausdorff dimension and the dynamics of a rational map. Observe in particular that one may define the Hausdorff dimension of ν by using covers consisting of Bowen balls, indexed over their depth n, and sending n to infinity; see also [CPZ19].

All the above is in sharp contrast with the higher-dimensional situation, where, due to the lack of conformality of holomophic maps, preimages of balls can be arbitrarily distorted, and far from being balls. In the best possible scenario (e.g., for hyperbolic product maps), the preimages of balls are approximately ellipses whose axes reflect the contraction rate of the inverse branches in the different directions.

On the other hand, when $\nu \in \mathcal{M}^+(f)$, a result by Berteloot-Dupont-Molino (see [BDM08; BD19] and Theorem 2.1 below) states that the best possible scenario described above is actually true, in an infinitesimal sense, for preimages of balls along generic orbits of ν. More precisely, there exists an increasing (as $\epsilon \to 0$) measurable exhaustion $\{Z^\star_\nu(\epsilon)\}_\epsilon$ of a full-measure subset $Z^\star_\nu$ of the space of orbits for f such that the preimages of sufficiently small balls along orbits in $Z^\star_\nu(\epsilon)$ are approximately ellipses, and the contraction rate for their volume is essentially given (up to further controllable error terms) by $e^{-nL_\nu(f)+nO(\epsilon)}$. This is a consequence of very refined estimates on the convexity defect of such preimages. Such property was already exploited in [BB18] to give bounds on the Hausdorff dimension of the bifurcation locus of families of endomorphisms of $\mathbb{P}^k$ [BBD18; Bia19], and in particular to prove that this is maximal near isolated Lattés maps, i.e., maps for which all the Lyapunov exponents are equal and minimal, i.e., equal to $(\log d)/2$ [BD99; BD05].

Fix $\nu \in \mathcal{M}^+(f)$. Denote by $\pi\colon Z^\star_\nu \to \mathbb{P}^k$ the projection associating to any orbit $\hat{z} = \{z_n\}_{n\in\mathbb{Z}}$ its element z_0. For $x \in \pi(Z^\star_\nu(\epsilon))$, $\kappa > 0$, and $N \in \mathbb{N}$, we consider (when well-defined) the neighbourhood $U = U(N, x, \kappa, \epsilon)$ of x satisfying

$$f^N(U) = B(f^N(x), \kappa\, e^{-NM\epsilon})$$

where e^M is a bound for the expansion of f and we require that $f^N|_U$ is injective. It follows from the above result by Berteloot-Dupont-Molino, and by further estimates that we develop in Section 2, that there exist some $r(\epsilon)$ and $n(\epsilon)$ such that, for all $x \in \pi(Z^\star_\nu(\epsilon))$, $0 < \kappa < r(\epsilon)$, and $N \geq n(\epsilon)$ the sets $U(N, x, \kappa, \epsilon)$ are indeed well-defined and approximately ellipses, of controlled geometry. We see these sets $U(N, x, \kappa, \epsilon)$ as a suitable version of the Bowen balls $B_n(x, \kappa)$ in any dimension. Let us set

$$\delta_x(\epsilon, \kappa, N) := \frac{\log \nu(U(N, x, \kappa, \epsilon))}{\log \mathrm{Vol}(U(N, x, \kappa, \epsilon))},$$

where Vol denotes the volume with respect to the Fubini-Study metric. As a first step (which corresponds to Step (1) above) towards proving Theorem 1.1, we show that every $\nu \in \mathcal{M}^+(f)$ is *exact (volume-)dimensional*; namely, for ν-almost every x, we have

$$\limsup_{\epsilon\to 0} \limsup_{\kappa\to 0} \limsup_{N\to\infty} \delta_x(\epsilon, \kappa, N) = \liminf_{\epsilon\to 0} \liminf_{\kappa\to 0} \liminf_{N\to\infty} \delta_x(\epsilon, \kappa, N) = \frac{h_\nu(f)}{2L_\nu(f)};$$

see Theorem 3.2 and Corollary 3.4. We adapt here the approach of Mañé [Mañ88] in higher dimensions, thanks to the distortion estimates developed in Section 2.

Once the local dimension of every $\nu \in \mathcal{M}^+(f)$ is well-defined as above, we give a global interpretation of this quantity by defining a *volume dimension* for these measures. The idea is to use the sets $U(N, x, \kappa, \epsilon)$ to cover the "slice" $X \cap \pi(Z^\star_\nu(\epsilon))$ of every set $X \subseteq Z^\star_\nu$. More precisely, for every

$X \subseteq \pi(Z_\nu^\star)$ and $\epsilon > 0$, setting $X^\epsilon := X \cap \pi(Z_\nu^\star(\epsilon))$, we define the quantity $\mathrm{VD}_\nu^\epsilon(X^\epsilon)$ as

$$\mathrm{VD}_\nu^\epsilon(X^\epsilon) := \sup\{\alpha : \Lambda_\alpha^\epsilon(X^\epsilon) = \infty\} = \inf\{\alpha : \Lambda_\alpha^\epsilon(X^\epsilon) = 0\},$$

where

$$\Lambda_\alpha^\epsilon(X^\epsilon) := \lim_{\kappa\to 0}\lim_{N^\star\to\infty}\inf_{\{U_i\}}\sum_{i\geq 1}\mathrm{Vol}(U_i)^\alpha.$$

Here the infimum is taken over the covers consisting of sets U_i of the form $U_i = U(N_i, x, \kappa, \epsilon)$, for some $x \in \pi(Z_\nu(\epsilon))$ and $N_i \geq N^\star$. The *volume dimensions* of X and ν are then respectively defined as

$$\mathrm{VD}_\nu(X) := \limsup_{\epsilon\to 0}\mathrm{VD}_\nu^\epsilon(X^\epsilon) \quad \text{and} \quad \mathrm{VD}(\nu) := \inf\{\mathrm{VD}_\nu(X) \colon X \subseteq \pi(Z_\nu), \nu(X) = 1\},$$

and the $\limsup_{\epsilon\to 0}$ is actually a limit; see Section 4.2. We prove in Proposition 4.26 a version of Young's criterion [You82, Proposition 2.1], relating the local volume dimensions δ_x with the volume dimensions $\mathrm{VD}_\nu(X)$ and $\mathrm{VD}(\nu)$. This corresponds to Step (2) above and, together with the exact volume-dimensionality of ν proved in the first step, completes the proof of Theorem 1.1.

1.3. **Organization of the paper.** The paper is organized as follows. In Section 2, we derive from the distortion theorem [BDM08; BD19] the estimates that we will need, and we introduce the volume-conformal measures and the pressure function $t \mapsto P_J^+(t)$. We prove the exact dimensionality of every $\nu \in \mathcal{M}^+(f)$ in Section 3. In Section 4, we define and study the volume dimensions of sets and measures. We conclude the proof of Theorem 1.1 and prove Theorems 1.2 and 1.3 in Section 5.

Acknowledgements. The authors would like to thank the University of Lille and the University of Oklahoma for the warm welcome and for the excellent work conditions.

This project has received funding from the French government through the Programme Investissement d'Avenir (LabEx CEMPI ANR-11-LABX-0007-01, ANR QuaSiDy ANR-21-CE40-0016, ANR PADAWAN ANR-21-CE40-0012-01) managed by the Agence Nationale de la Recherche.

2. Definitions and preliminary results

After fixing some notations in Section 2.1, in Section 2.2 we recall the distortion theorem by Berteloot-Dupont-Molino [BDM08; BD19] and deduce the estimates which will be essential ingredients in the proof of Theorem 1.1. We define and study basic properties about volume-conformal measures in Section 2.3, and a pressure function in Section 2.4.

2.1. **Notations.** Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic endomorphism and ν an ergodic f-invariant probability measure. By Oseledets' theorem [Ose68], one can associate to ν its *Lyapunov exponents* $\chi_{\min} := \chi_l < \ldots < \chi_1$, where $1 \leq l \leq k$. For ν-almost every $x \in \mathbb{P}^k$, there exists a stratification in complex linear subspaces $\{0\} =: (L_{l+1})_x \subset (L_l)_x \subset \ldots \subset (L_1)_x = T_x\mathbb{P}^k$ of the complex tangent space $T_x\mathbb{P}^k$ such that $Df_x(L_j)_x = (L_j)_{f(x)}$ and $\lim_{n\to\infty} n^{-1}\log\|Df_x^n v\| = \chi_j$ for all $v \in (L_j)_x \setminus (L_{j+1})_x$ for all $1 \leq j \leq l$.

Let us first assume that all the χ_j's are distinct, i.e., that we have $l = k$ and $\chi_{\min} = \chi_k < \ldots < \chi_1$. Then, $(L_j)_x$ has dimension $k - j + 1$ for all $1 \leq j \leq k$. We denote by $\mathcal{O}$ a full measure subset of the support of ν given by Oseledets' theorem. Take $x \in \mathcal{O}$. Fix a basis $(\ell_j)_x$ of the complex tangent space $T_x\mathbb{P}^k$ with the property that $(L_j)_x$ is equal to the span of $\{(\ell_j)_x, \ldots, (\ell_k)_x\}$. Denote by $\{e_j\}_{j=1}^k$ the standard basis of $\mathbb{R}^k \subset \mathbb{C}^k$. For every $r_1, \ldots, r_k \in \mathbb{R}$ (sufficiently small), we denote by $\mathcal{E}_x(r_1, \ldots, r_k)$ the image of the unit ball $\mathbb{B}^k \subset \mathbb{C}^k$ (in a given local chart at x) under the composition $e \circ \Phi : \mathbb{C}^k \to \mathbb{P}^k$, where $e \colon T_x\mathbb{P}^k \to \mathbb{P}^k$ is the standard exponential map and $\Phi \colon \mathbb{C}^k \to T_x\mathbb{P}^k \simeq \mathbb{C}^k$ is a linear map such that $\Phi\big((e_j)_x\big) = r_j(\ell_j)_x$.

If, for all $1 \le j \le k$, the argument r_j of $\mathcal{E}_x$ is a function $\phi(j)$ depending on j, we write

$$\mathcal{E}_x(\phi(j)) := \mathcal{E}_x(\phi(1), \dots, \phi(k))$$

for brevity. In particular, we will often have $\chi_{\min} > 0$ and take $\phi(j)$ of the form $\phi(j) = c_1 e^{-n(\chi_j \pm c_2 \epsilon)}$ for some $n \in \mathbb{N}$, $0 < \epsilon \ll \chi_{\min}$ sufficiently small, and some positive constants c_1 (independent of j and n) and c_2 (independent of j, n, and ϵ). We will call the sets $\mathcal{E}_x$ *dynamical ellipses* in this case.

If $l < k$, i.e., some Lyapunov exponent χ_j has multiplicity larger than 1, the above construction generalizes by taking into account the corresponding r_j with the same multiplicity. Namely, we assign the same r_j to all the directions associated to the same Lyapunov exponent which has multiplicity larger than 1.

Let $X \subseteq \mathbb{P}^k$ be a closed invariant set for f. We denote by $\mathcal{M}^+_X(f)$ the set of all ergodic f-invariant measures supported on X with strictly positive Lyapunov exponents. We drop the index X if $X = \mathbb{P}^k$. We say that X is *uniformly expanding* if there exist $\eta > 1$ and $C > 0$ such that $||Df^n_x(v)|| > C\eta^n ||v||$ for every $x \in X$, $v \in T_x\mathbb{P}^k$, and $n \in \mathbb{N}$. We say that f is *hyperbolic* if $J(f)$ is uniformly expanding.

We will consider the Fubini-Study metric on $\mathbb{P}^k$. We will denote by dist the corresponding distance, and by $B(x,r)$ the open ball centred at x and of radius r. For an open set $V \subset \mathbb{P}^k$, we denote by $\mathrm{Vol}(V)$ the volume of V with respect to the Fubini-Study metric. Given a holomorphic map $g \colon V \to \mathbb{P}^k$, we denote by $\mathrm{Jac}\, g(x)$ the Jacobian of g at $x \in V$, i.e., the determinant of the differential Dg_x.

We also fix the positive constant $M > 0$ defined as

$$M := \log \sup_{x \in \mathbb{P}^k} \sup_{v \in \mathbb{C}^k} \frac{||Df_x(v)||}{||v||} \tag{2.1}$$

and observe that e^M dominates the Lipschitz constant of f. In particular, we have $\mathrm{dist}(f(x_1), f(x_2)) \le e^M \,\mathrm{dist}(x_1, x_2)$ for every $x_1, x_2 \in \mathbb{P}^k$, and $f(B(x,r)) \subseteq B(f(x), e^M r)$ for every $x \in \mathbb{P}^k$ and $r > 0$.

2.2. **Distortion estimates along generic inverse branches.** We fix in this subsection a holomorphic endomorphism $f : \mathbb{P}^k \to \mathbb{P}^k$ of algebraic degree $d \ge 2$ and a measure $\nu \in \mathcal{M}^+(f)$. All the objects and the constants that we introduce in this subsection depend on f and ν. We denote by $\chi_1 > \ldots > \chi_l = \chi_{\min} > 0$ the (distinct) Lyapunov exponents of ν, by $k_1, \dots, k_l$ their respective multiplicities, and by $L_\nu = L_\nu(f) := \sum_{j=1}^{l} k_j \chi_j$ their sum. Recall that we have $L_\nu = \int \log |\mathrm{Jac}\, f| d\nu$ by Birkhoff's ergodic theorem.

Consider the *orbit space* of f

$$O := \left\{ \hat{x} = \{x_n\}_{n \in \mathbb{Z}} \in (\mathbb{P}^k)^{\mathbb{Z}} \colon x_{n+1} = f(x_n) \quad \forall n \in \mathbb{Z} \right\}$$

and the right shift map $T \colon O \to O$ defined as $T(\hat{x}) = \{x_{n+1}\}_{n \in \mathbb{Z}}$ for $\hat{x} = \{x_n\}_{n \in \mathbb{Z}}$. Given $\eta > 0$, a function $\phi \colon O \to (0,1]$ is said to be *η-slow* if for any $\hat{x} \in O$ we have

$$e^{-\eta} \phi(\hat{x}) \le \phi(T(\hat{x})) \le e^{\eta} \phi(\hat{x}).$$

We now recall the construction of the *lift* $\hat{\nu}$ of ν to O; see [CFS12, Section 10.4] and [PU10, Section 2.7]. For $n \in \mathbb{Z}$, we let $\pi_n \colon O \to \mathbb{P}^k$ be the projection map defined by $\pi_n(\hat{x}) = x_n$, where $\hat{x} = \{x_n\}_{n \in \mathbb{Z}}$. We write $\pi := \pi_0$ for brevity. Observe that $\pi_n \circ T = f \circ \pi_n$ for all $n \in \mathbb{Z}$.

Consider the σ-algebra $\hat{\mathcal{B}}$ on O generated by the sets of the form

$$A_{n,B} := \pi_n^{-1}(B) = \{\hat{x} \colon x_n \in B\}$$

with $n \le 0$ and $B \subseteq \mathbb{P}^k$ a Borel set. For all such sets $A_{n,B}$, set

$$\hat{\nu}(A_{n,B}) := \nu(B).$$

Then, by the invariance of ν and the fact that $x_n \in B$ if and only if $x_{n-m} \in f^{-m}(B)$ with $m \geq 0$, we see that $\hat{\nu}$ is well-defined on the sets $A_{n,B}$ as it satisfies $\hat{\nu}(A_{n,B}) = \hat{\nu}(A_{n-m,B})$ for all $m \geq 0$. Similarly, for all $m \geq 0$ and Borel sets $B_0, \ldots, B_{-m} \subseteq \mathbb{P}^k$, we have

$$\hat{\nu}\big(\{\hat{x} \colon x_0 \in B_0, \ldots, x_{-m} \in B_{-m}\}\big) = \nu\left(f^{-m}(B_0) \cap f^{-m+1}(B_1) \cap \ldots \cap B_{-m}\right).$$

We can then extend $\hat{\nu}$ to a probability measure on $\hat{\mathcal{B}}$, that we still denote by $\hat{\nu}$. By construction, $\hat{\nu}$ is T-invariant and satisfies $\pi_*(\hat{\nu}) = \nu$. As ν is ergodic, one can prove that $\hat{\nu}$ is also ergodic.

Recall that the critical set $C(f)$ of f is the set of points $x \in \mathbb{P}^k$ at which the differential Df_x is not invertible. As all the Lyapunov exponents of ν are finite, and their sum is equal to $\int \log |Df| d\nu$, we have in particular $\nu(C(f)) = 0$. Set

$$Z := \big\{\hat{x} \in O : x_n \notin C(f) \quad \forall n \in \mathbb{Z}\big\}.$$

Then the set Z is T-invariant and satisfies $\hat{\nu}(Z) = 1$. For every $\hat{x} \in Z$, we denote by $f_{\hat{x}}^{-n}$ the inverse branch of f^n defined in a neighbourhood of x_0 and such that $f_{\hat{x}}^{-n}(x_0) = x_{-n}$.

The following result is stated in [BD19, Theorem A] (see also [BDM08, Theorem 1.4]) in the case where ν is the measure of maximal entropy of f. The same statement and proof hold for any measure in $\nu \in \mathcal{M}^+(f)$, as stated at the end of the Introduction – and used in later sections – of the same paper.

Theorem 2.1. *For every $0 < 2\eta < \gamma \ll \chi_{\min}$ and $\hat{\nu}$-almost every $\hat{x} \in Z$, there exist*

(1) an integer $n_{\hat{x}} \geq 1$ and real numbers $h_{\hat{x}} \geq 1$ and $0 < r_{\hat{x}}, \rho_{\hat{x}} \leq 1$,

(2) a sequence $\{\varphi_{\hat{x},n}\}_{n \geq 0}$ of injective holomorphic maps

$$\varphi_{\hat{x},n} : B(x_{-n}, r_{\hat{x}} e^{-n(\gamma+2\eta)}) \to \mathbb{D}^k(\rho_{\hat{x}} e^{n\eta})$$

sending x_{-n} to 0 and satisfying

$$e^{n(\gamma-2\eta)} \operatorname{dist}(u,v) \leq |\varphi_{\hat{x},n}(u) - \varphi_{\hat{x},n}(v)| \leq e^{n(\gamma+3\eta)} h_{\hat{x}} \operatorname{dist}(u,v)$$

for every $n \in \mathbb{N}$ and $u, v \in B(x_{-n}, r_{\hat{x}} e^{-n(\gamma+2\eta)})$;

(3) a sequence $\{\mathcal{L}_{\hat{x},n}\}_{n \geq 0}$ of linear maps from $\mathbb{C}^k$ to $\mathbb{C}^k$ which stabilize each

$$H_j := \{0\} \times \ldots \times \mathbb{C}^{k_j} \times \ldots \times \{0\},$$

satisfy

$$e^{-n\chi_j + n(\gamma-\eta)}|v| \leq |\mathcal{L}_{\hat{x},n}(v)| \leq e^{-n\chi_j + n(\gamma+\eta)}|v| \quad \textit{for all } n \in \mathbb{N} \textit{ and } v \in H_j,$$

and such that the diagram

$$\begin{array}{ccc} B(x_0, r_{\hat{x}}) & \xrightarrow{f_{\hat{x}}^{-n}} & B(x_{-n}, r_{\hat{x}} e^{-n(\gamma+2\eta)}) \\ \downarrow{\scriptstyle \varphi_{\hat{x},0}} & & \downarrow{\scriptstyle \varphi_{\hat{x},n}} \\ \mathbb{D}^k(\rho_{\hat{x}}) & \xrightarrow{\mathcal{L}_{\hat{x},n}} & \mathbb{D}^k(\rho_{\hat{x}} e^{n\eta}) \end{array}$$

commutes for all $n \geq n_{\hat{x}}$.

Moreover, the functions $\hat{x} \mapsto h_{\hat{x}}^{-1}, r_{\hat{x}}, \rho_{\hat{x}}$ are measurable and η-slow on Z.

In particular, for every $n \in \mathbb{N}$ and $\hat{x}$ as in the statement, the inverse branch $f_{\hat{x}}^{-n}$ is well-defined on the ball $B(x_0, r_{\hat{x}})$.

Corollary 2.2. *With the same assumptions and notations as in Theorem 2.1 and Section 2.1, for $\hat{\nu}$-almost all $\hat{x} \in Z$ and all $t, t_1, \ldots, t_k \in (0,1]$, $n \geq n_{\hat{x}}$, and $y, w \in B(x_0, r_{\hat{x}})$, we have*

(1) $e^{-n(L_\nu + 10k\eta)} \leq |\operatorname{Jac} f_{\hat{x}}^{-n}(y)| \leq e^{-n(L_\nu - 10k\eta)}$;

(2) $e^{-20kn\eta} \leq |\operatorname{Jac} f_{\hat{x}}^{-n}(y)| \cdot |\operatorname{Jac} f_{\hat{x}}^{-n}(w)|^{-1} \leq e^{20kn\eta}$;

(3) $\mathcal{E}_{x_{-n}}(t_j r_{\hat{x}} h_{\hat{x}}^{-1} e^{-n(\chi_j+10\eta)}) \subset f_{\hat{x}}^{-n}(\mathcal{E}_{x_0}(t_j r_{\hat{x}})) \subset \mathcal{E}_{x_{-n}}(t_j r_{\hat{x}} h_{\hat{x}} e^{-n(\chi_j-10\eta)})$;
(4) $(tr_{\hat{x}})^{2k} e^{-2n(L_\nu+10k\eta)} \le \mathrm{Vol}(\mathcal{E}_{x_{-n}}(tr_{\hat{x}} e^{-n\chi_j})) \le (tr_{\hat{x}})^{2k} e^{-2n(L_\nu-10k\eta)}$;
(5) $(tr_{\hat{x}} h_{\hat{x}}^{-1})^{2k} e^{-2n(L_\nu+20k\eta)} \le \mathrm{Vol}(f_{\hat{x}}^{-n}(B(x_0, tr_{\hat{x}}))) \le (tr_{\hat{x}} h_{\hat{x}})^{2k} e^{-2n(L_\nu-20k\eta)}$.

Proof. The assertions (1) and (3) follow directly from Theorem 2.1 (2) and (3). The assertion (2) follows from (1). The assertion (4) follows from the fact that the distances in $\mathbb{P}(T_{x_{-n}}\mathbb{P}^k) \simeq \mathbb{P}^{k-1}$ between the directions associated to distinct Lyapunov exponents at x_{-n} are larger (up to a multiplicative constant independent of n) than $e^{-5n\eta}$, again by Theorem 2.1 (2). This allows one to compare the volume of $\mathcal{E}_{x_{-n}}(tr_{\hat{x}} e^{-n\chi_j})$ with that of an ellipse in $\mathbb{C}^k$, whose axes are parallel to the coordinate planes. The assertion (5) is a consequence of (3), applied with $t_j = t$ for all $j = 1, \ldots, k$, and (4). □

Definition 2.3. We define $Z_\nu \subseteq Z$ to be the full $\hat{\nu}$-measure set of elements $\hat{x} \in Z$ satisfying the conditions in Theorem 2.1 and Corollary 2.2. For $\alpha > 0$, we also define

$$Z_{\nu,\alpha} := \{\hat{x} \in Z_\nu : n_{\hat{x}} < \alpha^{-1}, r_{\hat{x}} > \alpha, h_{\hat{x}} < \alpha^{-1}\}.$$

It follows from the definition that, as $\alpha \to 0$, the sets $Z_{\nu,\alpha}$ increase to Z_ν. In particular, we have $\hat{\nu}(Z_{\nu,\alpha}) \to 1$ as $\alpha \to 0$.

Corollary 2.4. *For every $0 < \epsilon \ll \chi_{\min}$ sufficiently small, there exist $Z'_\nu(\epsilon) \subseteq Z_\nu$, $n'(\epsilon) \ge 1$, and $r(\epsilon) \in (0,1)$ such that*
(1) $\hat{\nu}(Z(\epsilon)) > 1 - \epsilon$;
(2) $n_{\hat{x}} \le n(\epsilon)$ *and* $r_{\hat{x}} \ge r(\epsilon)$ *for all* $\hat{x} \in Z'_\nu(\epsilon)$;
(3) for all $t, t_1, \ldots, t_k \in (0,1]$, $n \ge n(\epsilon)$, $\hat{x} \in Z'_\nu(\epsilon)$, *and* $y, w \in B(x_0, r(\epsilon))$ *we have*
(a) $e^{-n(L_\nu+k\epsilon)} \le |\operatorname{Jac} f_{\hat{x}}^{-n}(y)| \le e^{-n(L_\nu-k\epsilon)}$;
(b) $e^{-kn\epsilon} \le |\operatorname{Jac} f_{\hat{x}}^{-n}(y)| \cdot |\operatorname{Jac} f_{\hat{x}}^{-n}(w)|^{-1} \le e^{kn\epsilon}$;
(c) $\mathcal{E}_{x_{-n}}(t_j r(\epsilon) e^{-n(\chi_j+\epsilon)}) \subset f_{\hat{x}}^{-n}(\mathcal{E}_{x_0}(t_j r(\epsilon))) \subset \mathcal{E}_{x_{-n}}(t_j r(\epsilon) e^{-n(\chi_j-\epsilon)})$;
(d) $(tr(\epsilon))^{2k} e^{-2n(L_\nu+k\epsilon)} \le \mathrm{Vol}(\mathcal{E}_{x_{-n}}(tr(\epsilon) e^{-n\chi_j})) \le (tr(\epsilon))^{2k} e^{-2n(L_\nu-k\epsilon)}$;
(e) $(tr(\epsilon))^{2k} e^{-2n(L_\nu+k\epsilon)} \le \mathrm{Vol}(f_{\hat{x}}^{-n}(B(x_0, tr(\epsilon)))) \le (tr(\epsilon))^{2k} e^{-2n(L_\nu-k\epsilon)}$,

where $n_{\hat{x}}$, $h_{\hat{x}}$, and $r_{\hat{x}}$ are as in Theorem 2.1.

Proof. By choosing $\alpha = \alpha(\epsilon)$ sufficiently small, Corollary 2.2 and the Definition 2.3 of $Z_{\nu,\alpha}$ give the existence of a set $Z''_\nu(\epsilon) := Z_{\nu,\alpha(\epsilon)}$ and numbers $r(\epsilon)$, $n'(\epsilon)$ satisfying the properties in the statement, with (3c) and (3e) replaced by

$$\mathcal{E}_{x_{-n}}(t_j r(\epsilon)\alpha(\epsilon) e^{-n(\chi_j+\epsilon/2)}) \subset f_{\hat{x}}^{-n}(\mathcal{E}_{x_0}(t_j r(\epsilon))) \subset \mathcal{E}_{x_{-n}}(t_j r(\epsilon)\alpha(\epsilon)^{-1} e^{-n(\chi_j-\epsilon/2)})$$

and

$$(tr(\epsilon)\alpha(\epsilon))^{2k} e^{-2n(L_\nu+k\epsilon/2)} \le \mathrm{Vol}(f_{\hat{x}}^{-n}(B(x_0, tr(\epsilon)))) \le (tr(\epsilon)\alpha(\epsilon)^{-1})^{2k} e^{-2n(L_\nu-k\epsilon/2)},$$

respectively. Since all the Lyapunov exponents of ν are strictly positive, the assertion follows up to increasing $n'(\epsilon)$. □

Lemma 2.5. *For every $0 < \epsilon \ll \chi_{\min}$ sufficiently small, there exist $n(\epsilon) \in \mathbb{N}$, a subset $Z_\nu(\epsilon) \subseteq Z'_\nu(\epsilon)$ with $\hat{\nu}(Z'_\nu(\epsilon) \setminus Z_\nu(\epsilon)) < \epsilon$, and, for all $\hat{x} \in Z_\nu(\epsilon)$, a sequence $\{n_l\}_{l\ge0} = \{n_l(\hat{x})\}_{l\ge0}$ such that*
(1) $n'(\epsilon) \le n_0 \le n(\epsilon)$;
(2) $n_{l+1} - n_l < \epsilon n_l$ *for all* $l \ge 0$;
(3) $T^{n_l}(\hat{x}) \in Z'_\nu(\epsilon)$ *for all* $l \ge 0$,

where $Z'_\nu(\epsilon)$ and $n'(\epsilon)$ are as in Corollary 2.4.

A version of Lemma 2.5 is essentially proved in [PU10, Section 11.4] in the case of $k = 1$. We will need here to further get a uniform upper bound for the element n_0 associated to any $\hat{x} \in Z_\nu(\epsilon)$.

Proof. We first show the existence of a set $Z'''_\nu(\epsilon) \subseteq Z'_\nu(\epsilon)$ with $\hat\nu(Z'_\nu(\epsilon) \setminus Z'''_\nu(\epsilon)) = 0$ and, for any $\hat x \in Z'''_\nu(\epsilon)$, of a sequence $\{n_l\}_{l\geq 0} = \{n_l(\hat x)\}_{l\geq 0}$ satisfying (2), (3), and $n_0 \geq n'(\epsilon)$.

For every $n \in \mathbb{N}$ and $\hat x \in Z$, set $S_n(\hat x) := \sum_{j=1}^{n-1} \mathbf{1}_{Z'_\nu(\epsilon)} \circ T^j(\hat x)$, where $\mathbf{1}_V$ denotes the characteristic function of $V \subset \mathbb{P}^k$. Since $\hat\nu$ is ergodic, by Birkhoff's ergodic theorem there exists a measurable set $Z'''_\nu(\epsilon) \subseteq Z'_\nu(\epsilon)$ such that $\hat\nu(Z'''_\nu(\epsilon)) = \hat\nu(Z'_\nu(\epsilon))$ and

$$\lim_{n\to\infty} n^{-1} S_n(\hat x) = \hat\nu(Z'_\nu(\epsilon)) \quad \text{for every } \hat x \in Z'''_\nu(\epsilon). \tag{2.2}$$

Take $\hat x \in Z'''_\nu(\epsilon)$. By (2.2) and the fact that $\hat\nu(Z'_\nu(\epsilon)) > 1-\epsilon$, there exists $n^\star = n^\star(\hat x) > 10/\epsilon$ such that $|n^{-1}S_n(\hat x) - \hat\nu(Z'_\nu(\epsilon))| \leq \epsilon/20$ for all $n \geq n^\star$. We define $\{n_l\}_{l\geq 0} = \{n_l(\hat x)\}_{l\geq 0}$ to be the sequence of integers $n \geq \max\{n^\star, n'(\epsilon)\}$ such that $T^n(\hat x) \in Z'_\nu(\epsilon)$.

It follows from the definitions of $S_n(\hat x)$ and of the sequence $\{n_l\}_{l\geq 0}$ that $S_{n_{l+1}}(\hat x) = S_{n_l}(\hat x) + 1$ for all $l \geq 0$. Moreover, we also have

$$S_{n_l}(\hat x) \leq n_l\big(\hat\nu(Z'_\nu(\epsilon)) + \epsilon/20\big) \quad \text{and} \quad S_{n_{l+1}}(\hat x) \geq n_{l+1}\big(\hat\nu(Z'_\nu(\epsilon)) - \epsilon/20\big) \quad \text{for all } l \geq 0.$$

We deduce from these inequalities that, again for all $l \geq 0$,

$$\frac{n_{l+1}}{n_l} \leq \frac{S_{n_{l+1}}(\hat x)}{n_l\big(\hat\nu(Z'_\nu(\epsilon)) - \epsilon/20\big)} = \frac{S_{n_l}(\hat x) + 1}{n_l\big(\hat\nu(Z'_\nu(\epsilon)) - \epsilon/20\big)} \leq \frac{\hat\nu(Z'_\nu(\epsilon)) + \epsilon/20 + 1/n_l}{\hat\nu(Z'_\nu(\epsilon)) - \epsilon/20}.$$

Hence, since $0 < \epsilon \ll \chi_{\min}$ and $n_l \geq n^\star > 10/\epsilon$ for all $l \geq 0$, we have

$$\frac{n_{l+1} - n_l}{n_l} \leq \frac{\hat\nu(Z'_\nu(\epsilon)) + \epsilon/20 + 1/n_l}{\hat\nu(Z'_\nu(\epsilon)) - \epsilon/20} - 1 \leq \frac{\epsilon/10 + \epsilon/10}{1 - \epsilon - \epsilon/20} < \epsilon.$$

This gives the existence of a set $Z'''_\nu(\epsilon)$ with the properties stated at the beginning of the proof.

For every $N > n'(\epsilon)$, set $Z^N_\nu(\epsilon) := \{\hat x \in Z'''_\nu(\epsilon) \colon n_0(\hat x) \leq N\}$. The sequence of sets $Z^N_\nu(\epsilon)$ is non-decreasing as $N \to \infty$, and satisfies $\cup_N Z^N_\nu(\epsilon) = Z'''_\nu(\epsilon)$. Fix $m^\star = m^\star(\epsilon)$ such that $\hat\nu(Z'''_\nu(\epsilon) \setminus Z^{m^\star}_\nu(\epsilon)) < \epsilon$. The assertion follows setting $Z_\nu(\epsilon) := Z^{m^\star}_\nu(\epsilon)$ and $n(\epsilon) := m^\star$. □

Recall that $\pi \colon O \to \mathbb{P}^k$ denotes the projection map defined by $\pi(\hat x) = x_0$, where $\hat x = \{x_n\}_{n\in\mathbb{Z}}$.

Remark 2.6. Observe that the sequence $\{n_l\}_{l\geq 0} = \{n_l(\hat x)\}_{l\geq 0}$ as in Lemma 2.5 only depends on $x_0 = \pi(\hat x)$. In particular, the sequence $\{n_l\}_{l\geq 0}$ as in Lemma 2.5 is well-defined for every $x \in \pi(Z_\nu(\epsilon))$.

Definition 2.7. Given $N \geq 0$, $x \in \mathbb{P}^k$, $\kappa > 0$, and $\epsilon > 0$, we denote by $U = U(N, x, \kappa, \epsilon)$ the (necessarily unique) set U, if it exists, satisfying $f^N(U) = B(f^N(x), \kappa\, e^{-NM\epsilon})$ (where M is as in (2.1)) and such that $f^N|_U$ is injective. We call $z(U)$ the *center* of U.

By Definition 2.7, we have $U(N, x_0, \kappa, \epsilon) := f^{-N}_{T^N(\hat z)}(B(f^N(x_0), \kappa\, e^{-NM\epsilon}))$ for every $\hat x$ with $\pi(\hat x) = x_0$ whenever the inverse branch $f^{-N}_{T^N(\hat z)}$ is well-defined on $B(f^N(x_0), \kappa\, e^{-NM\epsilon})$. Lemma 2.5 says that, for all $\epsilon > 0$ sufficiently small, this happens for every $x_0 \in \pi(Z_\nu(\epsilon))$, $0 < \kappa < r(\epsilon)$, and $l \geq 0$ if we take $N = n_l$, where the sequence $\{n_l\}_{l\geq 0}$ is given by that statement (and depends on x_0; see Remark 2.6). In particular, $U(n_l, x_0, \kappa, \epsilon)$ is well-defined under such conditions for all $l \geq 0$, and Corollary 2.4 can be applied with any $\hat y$ such that $y_0 = f^{n_l}(x_0)$ and $y_{-n_l} = x_0$ (observe that the factor $e^{-NM\epsilon}$ is not necessary to get this). We now aim at getting similar estimates valid for all $N \geq n(\epsilon)$. The factor $e^{-NM\epsilon}$ will need to be introduced for this reason.

In the next lemma and in the rest of the paper, we will only consider $0 < \epsilon \ll \chi_{\min}$ sufficiently small as above, and $Z'_\nu(\epsilon)$, $Z_\nu(\epsilon)$, $r(\epsilon)$, and $n(\epsilon)$ will be as in Corollary 2.4 and Lemma 2.5. Similarly, for every $\hat z \in Z_\nu(\epsilon)$ (and $z \in \pi(Z_\nu(\epsilon))$), the sequence $\{n_l\}_{l\geq 0}$ is given by Lemma 2.5 (and Remark 2.6). Recall that all these definitions depend on f and $\nu \in \mathcal{M}^+(f)$. The sets $U(N, x, \kappa, \epsilon)$ are as in Definition 2.7.

Lemma 2.8. *Fix $0 < \epsilon \ll \chi_{\min}$. Then, for all $z \in \pi(Z_\nu(\epsilon))$, $N \geq n(\epsilon)$, and $0 < \kappa < r(\epsilon)$, the set $U(N, x, \kappa, \epsilon)$ is well-defined and we have*

(1) $\mathcal{E}_z(\kappa\, e^{-N(\chi_j+(2M+1)\epsilon)}) \subseteq U(N, x, \kappa, \epsilon) \subseteq \mathcal{E}_z(\kappa\, e^{-N(\chi_j-\epsilon)})$;

(2) $\kappa^{2k} e^{-2N(L_\nu+k(2M+1)\epsilon)} \leq \mathrm{Vol}(U(N, x, \kappa, \epsilon)) \leq \kappa^{2k} e^{-2N(L_\nu-k\epsilon)}$;

(3) $e^{-N(2M+1)k\epsilon} \leq |\operatorname{Jac} f^N(y)| \cdot |\operatorname{Jac} f^N(w)|^{-1} \leq e^{N(2M+1)k\epsilon}$ *for every $y, w \in U(N, x, \kappa, \epsilon)$,*

where M is as in (2.1).

Proof. Fix $x \in \pi(Z_\nu(\epsilon))$ and $0 < \kappa < r(\epsilon)$. Then x corresponds to (at least) an orbit $\hat{x} \in Z_\nu(\epsilon)$ with $\pi(\hat{x}) = x$. By Lemma 2.5 and Remark 2.6, there exists a sequence $\{n_l\}_{l\geq 0}$ with $n_0 \leq n(\epsilon)$ and such that $n_{l+1} < (1+\epsilon)n_l$ and $f^{n_l}(x) \in \pi(Z'_\nu(\epsilon))$ for all $l \geq 0$. By Corollary 2.4 (3b), (3c), and (3e), for all $l \geq 0$ and $0 < t \leq 1$ the set

$$\widetilde{U}(n_l, x, t\kappa) := f^{-n_l}_{T^{n_l}(\hat{x})}\big(B(f^{n_l}(x_0), t\kappa)\big)$$

is well-defined and we have

$$\mathcal{E}_x(t\kappa\, e^{-n_l(\chi_j+\epsilon)}) \subseteq \widetilde{U}(n_l, x, t\kappa) \subseteq \mathcal{E}_x(t\kappa\, e^{-n_l(\chi_j-\epsilon)}), \tag{2.3}$$

$$(t\kappa)^{2k} e^{-2n_l(L_\nu+k\epsilon)} \leq \mathrm{Vol}(\widetilde{U}(n_l, x, t\kappa)) \leq (t\kappa)^{2k} e^{-2n_l(L_\nu-k\epsilon)}, \tag{2.4}$$

and

$$e^{-kn_l\epsilon} \leq |\operatorname{Jac} f^{n_l}(y)| \cdot |\operatorname{Jac} f^{n_l}(w)|^{-1} \leq e^{kn_l\epsilon} \quad \text{for every } y, w \in \widetilde{U}(n_l, x, \kappa). \tag{2.5}$$

Consider now any $N \geq n(\epsilon)$ and fix $l^\star = l^\star(N)$ such that $n_{l^\star} \leq N < n_{l^\star+1}$. Such $l^\star$ exists since $n_0 \leq n(\epsilon)$ and $n_l \to \infty$ as $l \to \infty$. It follows from the definition (2.1) of M that

$$B\big(f^N(z), \kappa\, e^{-(N-n_{l^\star})M}\big) \supseteq f^{N-n_{l^\star}}\big(B(f^{n_{l^\star}}(x), \kappa\, e^{-2(N-n_{l^\star})M})\big) \tag{2.6}$$

and

$$f^{n_{l^\star+1}-N}\Big(B\big(f^N(x), \kappa\, e^{-(n_{l^\star+1}-N)M}\big)\Big) \subseteq B(f^{n_{l^\star+1}}(x), \kappa). \tag{2.7}$$

It follows from (2.7), the second inequality in (2.3) applied with $l = l^\star + 1$ and $t = 1$, and the fact that $0 \leq n_{l^\star+1} - N \leq \epsilon n_{l^\star} \leq \epsilon N$, that $U(N, x, \kappa, \epsilon)$ is well-defined and satisfies

$$U(N, x, \kappa, \epsilon) \subseteq \widetilde{U}(n_{l^\star+1}, x, \kappa) \subseteq \mathcal{E}_x(\kappa\, e^{-n_{l^\star+1}(\chi_j-\epsilon)}) \subseteq \mathcal{E}_x(\kappa\, e^{-N(\chi_j-\epsilon)}). \tag{2.8}$$

Similarly, from (2.6), the first inequality in (2.3) applied with $l = l^\star$ and $t = e^{-2(N-n_{l^\star})M}$, and the fact that $0 \leq N - n_{l^\star} \leq \epsilon n_{l^\star} \leq \epsilon N$ we deduce that

$$\begin{aligned} U(N, x, \kappa, \epsilon) &\supseteq f^{-n_{l^\star}}_{T^{n_{l^\star}}(\hat{x})}\Big(B\big(f^{n_{l^\star}}(x), \kappa\, e^{-2(N-n_{l^\star})M}\big)\Big) \\ &\supseteq \mathcal{E}_x(\kappa\, e^{-n_{l^\star}(\chi_j-(2M+1)\epsilon)}) \supseteq \mathcal{E}_x(\kappa\, e^{-N(\chi_j-(2M+1)\epsilon)}), \end{aligned}$$

which completes the proof of the first item.

The second and the third assertions follow from similar arguments, combining (2.6) and (2.7) with (2.4) and (2.5), respectively. □

The following corollary records a special case of the above lemma when all the Lyapunov exponents of $\nu \in \mathcal{M}^+(f)$ are equal.

Corollary 2.9. *Assume that all the Lyapunov exponents of $\nu \in \mathcal{M}^+(f)$ are equal to $\chi > 0$. Then, for all $0 < \epsilon \ll \chi$, $x \in \pi(Z_\nu(\epsilon))$, $N \geq n(\epsilon)$, and $0 < \kappa < r(\epsilon)$, we have*

(1) $B(x, \kappa\, e^{-N(\chi+(2M+1)\epsilon)}) \subseteq U(N, x, \kappa, \epsilon) \subseteq B(x, \kappa\, e^{-N(\chi-\epsilon)})$;

(2) $\kappa^{2k} e^{-2kN(\chi+(2M+1)\epsilon)} \leq \mathrm{Vol}(U(N, x, \kappa, \epsilon)) \leq \kappa^{2k} e^{-2kN(\chi-\epsilon)}$.

Remark 2.10. If f is hyperbolic, then we have $Z_\nu \cap \pi^{-1}(J(f)) = Z \cap \pi^{-1}(J(f))$ for any $\nu \in \mathcal{M}_J^+(f)$. Moreover, observe that any ergodic probability measure on $J(f)$ belongs to $\mathcal{M}_J^+(f)$. In particular, we have $\pi(Z_\nu) \supseteq J(f)$ for any ergodic probability measure on $J(f)$. We also have $Z_{\nu,\alpha} \cap \pi^{-1}(J(f)) = Z_\nu \cap \pi^{-1}(J(f))$ for all α sufficiently small, which implies that we can take $Z(\epsilon) = Z'(\epsilon) = Z$ for all ϵ sufficiently small.

More generally, let $X \subseteq \mathbb{P}^k$ be a closed invariant uniformly expanding set. Take $\nu \in \mathcal{M}_X^+(f)$. Denoting by O_X the set of orbits $\{x_n\}_{n\in\mathbb{Z}} \in X^{\mathbb{Z}}$, it follows from the definition of $\hat{\nu}$ that $\hat{\nu}(O_X) = 1$ and that we can assume that $X \subseteq \pi(Z_\nu(\epsilon))$ for all $\epsilon > 0$. As $\hat{\nu}(O_X) = 1$, we can also assume that $\pi(Z_\nu(\epsilon)) \subseteq X$, hence $\pi(Z_\nu(\epsilon)) = X$, for all $\epsilon > 0$.

2.3. **Volume-conformal measures.** We again fix in this section a holomorphic endomorphism f of $\mathbb{P}^k$ of algebraic degree $d \geq 2$, and we let X be a closed invariant set for f. Recall that $f|_X : X \to X$ is *topologically exact* if for any open set $U \subset \mathbb{P}^k$ with $U \cap X \neq \emptyset$ there exists $n \geq 1$ such that $f^n(U) \supseteq X$.

Definition 2.11. Given any $t \geq 0$, a probability measure μ on X is *t-volume-conformal on X* if, for every Borel subset $A \subseteq X$ on which f is invertible, we have

$$\mu(f(A)) = \int_A |\operatorname{Jac} f|^t d\mu.$$

We define

$$\delta_X(f) := \inf \{t \geq 0 \colon \text{there exists a } t\text{-volume-conformal measure on } X\}.$$

Lemma 2.12. *Assume that $f|_X$ is topologically exact. Let μ be a probability measure on X which is t-volume-conformal on X for some $t \geq 0$. Then*

(1) the support of μ is equal to X;
(2) for every $r > 0$ there exists constants $0 < m_- = m_-(\mu, r) \leq 1$ and $0 < m_+ = m_+(\mu, r) \leq 1$ such that $m_- \leq \mu(B(x,r)) \leq m_+$ for every $x \in X$.

Proof. Assume that there exists a point $x \in X$ which does not belong to the support of μ. Take a small ball B centred at x which is disjoint from the critical set $C(f)$ of f and such that $\mu(B) = 0$. As $f|_X$ is topologically exact, we have $X \subseteq f^n(B)$ for some $n \geq 1$. Hence, it is enough to prove that $\mu(f^n(B)) = 0$ for all $n \in \mathbb{N}$. Since $B \cap C(f) = \emptyset$, this is a consequence of the volume-conformality of μ and the fact that $\mu(B) = 0$ (we need here to partition B into subsets where f^n is injective in order to apply Definition 2.11). The first assertion follows.

The second assertion is a consequence of the first and the fact that, for every probability measure μ on $\mathbb{P}^k$ and $r > 0$, there exist constants $m_\pm = m_\pm(\mu, r)$ such that (2) holds for every x in the support of μ. □

Recall that, for every $\nu \in \mathcal{M}^+(f)$ and every ϵ sufficiently small, $Z_\nu(\epsilon)$, $r(\epsilon)$, and $n(\epsilon)$ are given by Corollary 2.4 and Lemma 2.5, and the sets $U(N, x, \kappa, \epsilon)$ are defined in Definition 2.7.

Lemma 2.13. *Assume that $f|_X$ is topologically exact. Fix $\nu \in \mathcal{M}_X^+(f)$, $t \geq 0$, and $0 < \epsilon \ll \chi_{\min}$, where $\chi_{\min} > 0$ is the smallest Lyapunov exponent of ν. Then, for every $0 < \kappa < r(\epsilon)$, every t-volume-conformal probability measure μ on X, every $x \in \pi(Z_\nu(\epsilon))$, and every $N \geq n(\epsilon)$, the set $U = U(N, x, \kappa, \epsilon)$ is well-defined and satisfies*

$$\frac{m_-(\mu, \kappa\, e^{-MN\epsilon})}{C^t \kappa^{tk}} e^{-tNk(5M+2)\epsilon} \leq \frac{\mu(U)}{\operatorname{Vol}(U)^{t/2}} \leq \frac{C^t m_+(\mu, \kappa\, e^{-MN\epsilon})}{\kappa^{tk}} e^{tNk(5M+2)\epsilon},$$

where M is as in (2.1), the constants m_- and m_+ are as in Lemma 2.12, and C is a positive constant independent of κ, ϵ, x, N, ν, μ, and t.

Proof. Fix $x \in \pi(Z(\epsilon))$, $N \geq n(\epsilon)$, and $0 < \kappa < r(\epsilon)$. The set $U := U(N, x, \kappa, \epsilon)$ is well-defined by Lemma 2.8. We denote for simplicity by B the ball $B(f^N(x), \kappa\, e^{-MN\epsilon}) = f^N(U)$.

Let μ be any t-volume-conformal probability measure on X. Since X has a dense orbit, by Lemma 2.12 the support of μ is equal to X. Since f^N is injective on U, by Definition 2.11 we have

$$\mu(B) = \mu(f^N(U)) = \mu(f^N(U \cap X)) = \int_{U\cap X} |\operatorname{Jac} f^N|^t d\mu = \int_U |\operatorname{Jac} f^N|^t d\mu.$$

We deduce from Lemma 2.8 (3) that

$$e^{-tkN(2M+1)\epsilon} |\operatorname{Jac} f^N(x)|^t \mu(U) \leq \mu(B) \leq e^{tkN(2M+1)\epsilon} |\operatorname{Jac} f^N(x)|^t \mu(U).$$

It follows from the above expression that

$$\frac{e^{-tkN(2M+1)\epsilon}}{|\operatorname{Jac} f^N(x)|^t} \cdot m_- \leq \mu(U) \leq \frac{e^{tkN(2M+1)\epsilon}}{|\operatorname{Jac} f^N(x)|^t} \cdot m_+, \tag{2.9}$$

where $m_- := m_-(\mu, \kappa\, e^{-MN\epsilon})$ and $m_+ := m_+(\mu, \kappa\, e^{-MN\epsilon})$ are as in Lemma 2.12.

Again by Lemma 2.8 (3), we also have

$$\frac{e^{-2kN(2M+1)\epsilon} \operatorname{Vol}(B)}{|\operatorname{Jac} f^N(x)|^2} \leq \operatorname{Vol}(U) = \int_{B(f^N(x), \kappa\, e^{-MN\epsilon})} |\operatorname{Jac} f_{\hat{z}}^{-N}|^2 \leq \frac{e^{2kN(2M+1)\epsilon} \operatorname{Vol}(B)}{|\operatorname{Jac} f^N(x)|^2}, \tag{2.10}$$

where the integral is taken with respect to the Fubini-Study metric, $\hat{z}$ is any element in Z such that $z_0 = f^N(x)$ and $z_{-N} = x$, and we observe that $f_{\hat{z}}^{-N}$ is well-defined on $f^N(U)$ by Lemma 2.8.

Combining the inequalities (2.9) and (2.10), we see that

$$\frac{m_-}{\operatorname{Vol}(B)^{t/2}} e^{-tNk(4M+2)\epsilon} \operatorname{Vol}(U)^{t/2} \leq \mu(U) \leq \frac{m_+}{\operatorname{Vol}(B)^{t/2}} e^{tNk(4M+2)\epsilon} \operatorname{Vol}(U)^{t/2}.$$

The assertion follows from the last expression by observing that there exists a positive constant C such that $C^{-2} \leq \operatorname{Vol}(B(x,r))/r^{2k} \leq C^2$ for every $x \in \mathbb{P}^k$ and $0 < r < 1$. □

2.4. **A pressure for expanding measures.** Let f be a holomorphic endomorphism of $\mathbb{P}^k$ of algebraic degree $d \geq 2$. For any invariant probability measure ν and $t \in \mathbb{R}$, we define

$$P_\nu(t) := h_\nu(f) - t \int |\operatorname{Jac} f| d\nu = h_\nu(f) - tL_\nu(f).$$

Let $X \subseteq \mathbb{P}^k$ be a closed invariant set for f. We define a pressure function P_X^+ as

$$P_X^+(t) := \sup\{P_\nu(t) \colon \nu \in \mathcal{M}_X^+(f)\} \tag{2.11}$$

and set

$$p_X^+(f) := \inf\{t \colon P_X^+(t) = 0\}.$$

We will drop the index X when $X = \mathbb{P}^k$.

Lemma 2.14. *Let $X \subseteq \mathbb{P}^k$ be a closed invariant set for f. Assume that $\mathcal{M}_X^+(f)$ is not empty. Then we have $P_X^+(t) < \infty$ for all $t \in \mathbb{R}$ and the function $t \mapsto P_X^+(t)$ is convex and non-increasing.*

Proof. Take $\nu \in \mathcal{M}_X^+(f)$. As $L_\nu(f) > 0$ and the topological entropy of f is bounded by $k \log d$ [Gro03; DS10] we have $P_X^+(t) \leq k \log d$ for all $t \geq 0$. Take now $t < 0$. Since the function $|\operatorname{Jac} f|$ is bounded from above by a constant M' and $h_\nu(f) \leq k \log d$, we have $P_X^+(t) \leq k \log d + |t| M'$ for every $t < 0$. Hence, $P_X^+(t) < \infty$ for all $t \in \mathbb{R}$.

For any given measure $\nu \in \mathcal{M}_X^+(f)$, the function $t \mapsto P_\nu(t)$ is non-increasing. It follows from its definition (2.11) that the function $t \mapsto P_X^+(t)$ is non-increasing. It is convex as it is a supremum of affine, hence convex, functions. □

The following example illustrates that Lemma 2.14 is false (even with $X = J(f)$) if we take the supremum over the set of all ergodic probability measures, with no requirement on the Lyapunov exponents, in the definition (2.11) of the pressure function $P_X^+(t)$.

Example 2.15. It is possible to construct endomorphisms f of $\mathbb{P}^k$ admitting a saddle fixed point p_0 in the Julia set and with $|\operatorname{Jac} f(p_0)| < 1$ (and actually also equal to 0). An example of this phenomenon is given for instance by Jonsson in [Jon99, Example 9.1], see also [BDM07, Theorem 6.3], [Taf10], and [BT17, Remark 2.6] for further examples. Consider the polynomial self-map f of $\mathbb{C}^2$ defined as

$$(z, w) \mapsto \left(z^2, w^2 + 2(1 + \eta - z)w\right),$$

which extends to $\mathbb{P}^2$ as a holomorphic endomorphism. As f preserves the families of the vertical lines parallel to $\{z = 0\}$, for every $(z_0, w_0) \in \mathbb{C}^2$ the *vertical eigenvalue* of $Df_{(z_0,w_0)}$ is well-defined. It is immediate to check that, for $0 \le \eta < 1/2$, the point $p_0 = (1, 0)$ is a saddle fixed point, with vertical eigenvalue equal to 2η, and Jacobian equal to 4η. In particular, the Jacobian of f at p_0 can take any small non-negative value (including 0). The point p_0 is in $J(f)$ since $J(f)$ is closed and, for Lebesgue almost all $z_0 \in S^1$, the point $(z_0, 0)$ belongs to $J(f)$. This follows from a direct computation of the derivatives which, by Birkhoff's ergodic theorem, gives that

$$Df^n_{(z_0,0)} \sim \begin{pmatrix} 2^n & 0 \\ \star & \int_0^{2\pi} \log|1 + \eta - e^{i\theta}| d\theta \end{pmatrix} = \begin{pmatrix} 2^n & 0 \\ \star & 2^n \log|1 + \eta| \end{pmatrix},$$

and the characterization of the Julia set of f given in [Jon99, Corollary 4.4].

Consider the function

$$P_J(t) := \sup_\nu P_\nu(t)$$

where now the supremum is taken over the set of *all* invariant probability measures supported on $J(f)$. If $\nu_0 = \delta_{p_0}$ is the Dirac mass at p_0, then the function $t \mapsto P_{\nu_0}(t)$ is increasing in t and $P_{\nu_0}(0) = 0$. Hence, for such an endomorphism f, the function $P_J(t)$ is convex but it increases after some $t_0 > 0$ and has no zeroes.

Remark 2.16. One could define $\widetilde{P}_J^+(t)$ by considering the set of all ergodic probability measures with positive *sum* of Lyapunov exponents in the definition of $P_J^+(t)$. However, it is unclear to us how to generalize many of the results in this paper, and in particular Theorem 1.1, to this larger class of measures. A priori, it could be possible that the first zero of $P_J(t)$ is larger than the first zero of $P_J^+(t)$, but (possibly) equal to the first zero of $\widetilde{P}_J^+(t)$.

3. Exact volume dimension of measures in $\mathcal{M}^+(f)$

Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic endomorphism of algebraic degree $d \ge 2$. In this section we define a pointwise dynamical volume dimension for every measure $\nu \in \mathcal{M}^+(f)$ and prove that it is constant ν-almost everywhere.

Fix a measure $\nu \in \mathcal{M}^+(f)$ and let $\chi_{\min} > 0$ be the smallest Lyapunov exponent of ν. For every $0 < \epsilon \ll \chi_{\min}$, we fix $Z_\nu(\epsilon)$, $n(\epsilon)$, and $r(\epsilon)$ as given by Corollary 2.4 and Lemma 2.5. For every $x \in \pi(Z_\nu(\epsilon))$, the sequence $\{n_l\}_{l\ge0} = \{n_l(x)\}_{l\ge0}$ is also given by Lemma 2.5; see Remark 2.6. For $x \in \pi(Z_\nu(\epsilon))$, $0 < \kappa < r(\epsilon)$, and $N \ge n(\epsilon)$, we define

$$\delta_x(\epsilon, \kappa, N) := \frac{\log \nu(U(N, x, \kappa, \epsilon))}{\log \operatorname{Vol}(U(N, x, \kappa, \epsilon))}, \tag{3.1}$$

where $U(N, x, \kappa, \epsilon)$ is as in Definition 2.7. Observe that, for every ϵ, x, κ, and N as above, the definition of $\delta_x(\epsilon, \kappa, N)$ is well-posed by Lemma 2.8.

Recall that the set Z_ν (see Definition 2.3) satisfies $Z_\nu = \cup_{\epsilon>0} Z_\nu(\epsilon)$ up to a ν-negligible set, and that the family $\{Z_\nu(\epsilon)\}_{\epsilon>0}$ is non-decreasing as $\epsilon \to 0$. In particular, ν-almost every $x \in \pi(Z_\nu)$ belongs to $\pi(Z_\nu(\epsilon))$ for every $0 < \epsilon < \epsilon_0$ for some $\epsilon_0 = \epsilon_0(x)$. For every such x, we define the *upper* and the *lower local volume dimension* at x as

$$\overline{\delta}_x := \limsup_{\epsilon\to 0} \limsup_{\kappa\to 0} \limsup_{N\to\infty} \delta_x(\epsilon,\kappa,N) \quad \text{and} \quad \underline{\delta}_x := \liminf_{\epsilon\to 0} \liminf_{\kappa\to 0} \liminf_{N\to\infty} \delta_x(\epsilon,\kappa,N), \tag{3.2}$$

respectively, where $\delta_x(\epsilon,\kappa,N)$ is as in (3.1).

Definition 3.1. If $\underline{\delta}_x = \overline{\delta}_x$, we say that $\delta_x := \underline{\delta}_x = \overline{\delta}_x$ is the *local volume dimension* of ν at x. We say that $\nu \in \mathcal{M}^+(f)$ is *exact volume-dimensional* if the local volume dimension δ_x exists for ν-almost every x.

The main result of this section is the following theorem. Recall that $h_\nu(f)$, $L_\nu(f)$, and $\chi_{\min}$ denote the measure-theoretic entropy, the sum of the Lyapunov exponents, and the smallest Lyapunov exponent of ν, respectively.

Theorem 3.2. *Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic endomorphism of algebraic degree $d \geq 2$. Take $\nu \in \mathcal{M}^+(f)$ and $0 < \epsilon \ll \chi_{\min}$. Then, for ν-almost all $x \in \pi(Z_\nu(\epsilon))$ and all $0 < \kappa < r(\epsilon)$, there exists integers $m_1(\epsilon, x) \geq n(\epsilon)$ and $m_2(\epsilon,\kappa) \geq 0$ such that*

$$\frac{h_\nu(f)}{2L_\nu(f)} - c\epsilon \leq \delta_x(\epsilon,\kappa,N) \leq \frac{h_\nu(f)}{2L_\nu(f)} + c\epsilon \quad \textit{for all } N \geq m_1(\epsilon,x) + m_2(\epsilon,\kappa),$$

where $\delta_x(\epsilon,\kappa,N)$ is as in (3.1) *and $c > 0$ is a constant independent of ϵ, x, and κ.*

Remark 3.3. Although Theorem 3.2 is stated for points $x \in \pi(Z_\nu(\epsilon))$, we can associate to $\hat{\nu}$-almost every $\hat{x} \in Z_\nu(\epsilon)$ the integer $m_1(\epsilon,\hat{x}) := m_1(\epsilon, x_0)$, where, since we have $x_0 \in \pi(Z_\nu(\epsilon))$, the number $m_1(\epsilon, x_0)$ is given by Theorem 3.2.

The following consequence of Theorem 3.2 shows that every $\nu \in \mathcal{M}^+(f)$ is exact volume-dimensional.

Corollary 3.4. *Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be an endomorphism of algebraic degree $d \geq 2$ and take $\nu \in \mathcal{M}^+(f)$. For ν-almost every $x \in \mathbb{P}^k$, the local volume dimension δ_x is well-defined and equal to $(2L_\nu(f))^{-1} h_\nu(f)$.*

Proof. Recall that the family $\{Z_\nu(\epsilon)\}_{\epsilon>0}$ is non-decreasing for $\epsilon \to 0$, and that we have $Z_\nu = \cup_{\epsilon>0} Z_\nu(\epsilon)$ up to a ν-negligible set. In particular, for ν-almost every $x \in \mathbb{P}^k$ there exists $\epsilon_0 = \epsilon(x_0) > 0$ such that x belongs to $\pi(Z_\nu(\epsilon))$ for every $0 < \epsilon < \epsilon_0$. The assertion follows from the definition (3.2) of the upper and lower volume dimensions and Theorem 3.2. ☐

The rest of the section is devoted to the proof of Theorem 3.2. We will follow the general strategy presented in [PU10, Section 11.4] but we will need to use the results in Section 2.2 to replace the distortion estimates for univalent maps in dimension 1.

3.1. **Proof of Theorem 3.2: a reduction.** Fix a countable measurable partition $\mathcal{P}$ of $\mathbb{P}^k$. Up to taking the elements of the partition sufficiently small, we can assume that the *entropy* $h_\nu(f,\mathcal{P})$ of the partition $\mathcal{P}$ satisfies $h_\nu(f) - \epsilon \leq h_\nu(f,\mathcal{P}) \leq h_\nu(f)$. Recall that, by the Shannon-McMillan-Breiman Theorem [Par69; Wal00] for ν-almost every $x \in \mathbb{P}^k$ we have

$$\lim_{n\to\infty} -\frac{1}{n} \log \nu(\mathcal{P}^n(x)) =: h_\nu(f,\mathcal{P}).$$

Here $\mathcal{P}^n$ is the partition generated by $\mathcal{P}, f^{-1}\mathcal{P}, \ldots, f^{-n}\mathcal{P}$ (i.e., the partition whose elements are the sets of the form $P_0 \cap f^{-1}(P_1) \cap \ldots \cap f^{-n}(P_n)$ for $P_0, \ldots, P_n \in \mathcal{P}$), and $\mathcal{P}^n(x)$ denotes the element of the partition $\mathcal{P}^n$ containing x.

Proposition 3.5. *Fix $\nu \in \mathcal{M}^+(f)$. For every $0 < \epsilon \ll \chi_{\min}$ there exist two partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ with $h_\nu(f, \mathcal{P}_1) \geq h_\nu(f) - \epsilon$ and four constants b_E, b_F (independent of ϵ) and $c_E, c_F > 0$ (possibly depending on ϵ) such that for ν-almost every $x \in \pi(Z_\nu(\epsilon))$ there exists an integer $m(\epsilon, x) \geq n(\epsilon)$ such that for all $n \geq m(\epsilon, x)$, we have*

$$E(n) := \mathcal{E}_x(c_E e^{-n(\chi_j + b_E \epsilon)}) \subseteq \mathcal{P}_1^n(x) \quad \text{and} \quad \mathcal{P}_2^n(x) \subseteq F(n) := \mathcal{E}_x(c_F e^{-n(\chi_j - b_F \epsilon)}).$$

We prove the existence of the sequences $E(n)$ and $F(n)$ and partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ in the next two subsections. We now show how Theorem 3.2 is a consequence of Proposition 3.5.

Proof of Theorem 3.2 assuming Proposition 3.5. We fix $\epsilon > 0$ as in the statement, $x \in \pi(Z_\nu(\epsilon))$, and $0 < \kappa < r(\epsilon)$. For every $N \geq n(\epsilon)$ and $0 < \kappa < r(\epsilon)$, define the integers $n_E(N, \kappa)$ and $n_F(N, \kappa)$ as

$$n_E(N, \kappa) := \min_j \left\lfloor \frac{(\chi_j - \epsilon)N + \log c_E - \log \kappa}{\chi_j + b_E \epsilon} \right\rfloor$$

and

$$n_F(N, \kappa) := \max_j \left\lceil \frac{[\chi_j + (2M+1)\epsilon)]N + \log c_F - \log \kappa}{\chi_j - b_F \epsilon} \right\rceil,$$

where b_E, b_F, c_E, c_F are as in Proposition 3.5 and we recall that the χ_j's are the Lyapunov exponents of ν, which are strictly positive. Then, by Lemma 2.8 (1) and (2) and Proposition 3.5, for all $0 < \kappa < r(\epsilon)$, we have

(3.3)
$$\mathcal{P}_2^{n_F(N,\kappa)}(x) \subseteq F(n_F(N, \kappa)) \subseteq U(N, x, \kappa, \epsilon) \subseteq E(n_E(N, \kappa)) \subseteq \mathcal{P}_1^{n_E(N,\kappa)}(x) \text{ for all } N \geq m(\epsilon, x),$$

where $m(\epsilon, x)$ is as in Proposition 3.5, and

$$\kappa^{2k} e^{-2N(L_\nu + k(2M+1)\epsilon)} \leq \mathrm{Vol}(U(N, x, \kappa, \epsilon)) \leq \kappa^{2k} e^{-2N(L_\nu - k\epsilon)} \quad \text{for all } N \geq n(\epsilon), \tag{3.4}$$

where we recall that M is as in (2.1).

It follows from (3.3) and the Shannon-McMillan-Breiman Theorem that there exists $m'(\epsilon, x) \geq m(\epsilon, x)$ and $m''(\epsilon, \kappa) \gg 1$ such that

$$(h_\nu(f) - 2\epsilon)\Big(\lim_{N\to\infty} \frac{n_E(N,\kappa)}{N} - \epsilon\Big) \leq -\frac{\log \nu(U(N, x, \kappa, \epsilon))}{N} \leq (h_\nu(f) + \epsilon)\Big(\lim_{N\to\infty} \frac{n_F(N,\kappa)}{N} + \epsilon\Big) \tag{3.5}$$

for all $N > m'(\epsilon, x) + m''(\epsilon, \kappa)$. We used here the fact that, since $\kappa < r(\epsilon)$, the integers $n_E(N, \kappa)$ and $n_F(N, \kappa)$ are bounded below by quantities which are independent of $0 < \kappa < r(\epsilon)$. Similarly, it follows from (3.4) that there exists $m'''(\epsilon, \kappa) \gg 1$ such that

$$2(L_\nu(f) - k\epsilon) - \epsilon \leq -\frac{\log \mathrm{Vol}(U(N, x, \kappa, \epsilon))}{N} \leq 2(L_\nu(f) + (2M+1)\epsilon) + \epsilon \tag{3.6}$$

for all $N > m'''(\epsilon, \kappa) + n(\epsilon)$.

Setting

$$m_1(\epsilon, x) := m'(\epsilon, x) \geq m(\epsilon, x) \geq n(\epsilon) \quad \text{and} \quad m_2(\epsilon, \kappa) := \max\{m''(\epsilon, \kappa), m'''(\epsilon, \kappa)\},$$

the assertion follows combining (3.5), (3.6), and the definitions of $n_E(N, \kappa)$ and $n_F(N, \kappa)$. □

3.2. **Proof of Proposition 3.5: the existence of $E(n)$ and $\mathcal{P}_1$.** We will need the following lemma, see for instance [PU10, Corollary 9.1.10].

Lemma 3.6. *Let $(\mathcal{X}, \delta)$ be a compact metric space, $f : \mathcal{X} \to \mathcal{X}$ a measurable map with respect to the Borel σ-algebra on $\mathcal{X}$ and ν an f-invariant Borel probability measure. Then for every $r > 0$, there exists $\mathcal{X}_0 \subseteq \mathcal{X}$ with $\nu(\mathcal{X}_0) = 1$ and a finite partition $\mathcal{P}$ of $\mathcal{X}$ into Borel sets of positive measure ν and of diameter smaller than r such that, for every $\epsilon > 0$ and every $x \in \mathcal{X}_0$, there exists an integer $m_0 = m_0(\epsilon, x)$ such that*

$$B_{\mathcal{X}}(f^n(x), e^{-n\epsilon}) \subset \mathcal{P}(f^n(x)) \quad \text{for every } n \geq m_0,$$

where $B_{\mathcal{X}}(y, a)$ denotes the open ball in $\mathcal{X}$ of radius a and center $y \in \mathcal{X}$.

Fix $0 < \epsilon \ll \chi_{\min}$. Let $\mathcal{X}_0$, $\mathcal{P}$, and $m_0(\epsilon, x)$ be as given by Lemma 3.6 applied with $\mathcal{X} = \operatorname{Supp} \nu$. Up to taking r sufficiently small, we can assume that $h_\nu(f, \mathcal{P}) \geq h_\nu(f) - \epsilon$. Up to replacing $Z_\nu(\epsilon)$ with $\pi^{-1}(\mathcal{X}_0) \cap Z_\nu(\epsilon)$, we can assume that the conclusion of Lemma 3.6 holds for all $x \in \pi(Z_\nu(\epsilon))$.

Fix $x \in \pi(Z_\nu(\epsilon))$. In particular, there exists $\hat{x} \in Z_\nu(\epsilon)$ with $\pi(\hat{x}) = x$. Let $\{n_l\}_{l \geq 0}$ be the sequence associated to $\hat{x}$ by Lemma 2.5. We fix $l_0 \in \mathbb{N}$ such that $n_{l_0} \geq m_0(\epsilon, x)$. Recall that M is as in (2.1).

Consider an integer $n \geq n_{l_0}$ and the dynamical ellipse

$$E(n) := \mathcal{E}_x\big(Cr(\epsilon) e^{-n(\chi_j + (M+2)\epsilon)}\big),$$

where $0 < C < 1$ is a constant small enough so that

$$f^q(E(n)) \subset \mathcal{P}(f^q(x)) \quad \text{for every } q \leq n_{l_0}. \tag{3.7}$$

We now show that $f^q(E(n)) \subset \mathcal{P}(f^q(x))$ for all $n_{l_0} \leq q \leq n$. To this end, fix one such q and let $l^\star = l^\star(q) \geq l_0$ be such that $n_{l^\star} \leq q < n_{l^\star+1}$. Since $T^{n_{l^\star}}(\hat{x}) \in Z'_\nu(\epsilon)$ by Lemma 2.5 and $\pi(T^{n_{l^\star}}(\hat{x})) = f^{n_{l^\star}}(x)$, Theorem 2.1 and Corollary 2.4 (3c) imply that there exists a holomorphic inverse branch $g_{l^\star} := f^{-n_{l^\star}}_{\widehat{f^{n_{l^\star}}(x)}} : B(f^{n_{l^\star}}(x), r(\epsilon)) \to \mathbb{P}^k$ of $f^{n_{l^\star}}$ such that $g_{l^\star}(f^{n_{l^\star}}(x)) = x$ and

$$\mathcal{E}_x\big(r(\epsilon) e^{-n_{l^\star}(\chi_j + \epsilon)}\big) \subseteq g_{l^\star}\big(B(f^{n_{l^\star}}(x), r(\epsilon))\big).$$

Set

$$E'(n) := \mathcal{E}_x\big(r(\epsilon) e^{-n_{l^\star}(\chi_j + \epsilon)}\big).$$

Then $E(n) \subset E'(n)$ (by the choice of C and the inequality $n \geq n_{l^\star}$) and Corollary 2.4 (3c) gives

$$f^{n_{l^\star}}(E(n)) = f^{n_{l^\star}}\Big(\mathcal{E}_x\big(Cr(\epsilon) e^{-n(\chi_j + (M+2)\epsilon)}\big)\Big) \subseteq \mathcal{E}_{f^{n_{l^\star}}(x)}\Big(Cr(\epsilon) e^{-\chi_j(n - n_{l^\star})} e^{\epsilon n_{l^\star} - (M+2)\epsilon n}\Big).$$

Since $n \geq n_{l^\star}$, $0 \leq q - n_{l^\star} \leq \epsilon n_{l^\star}$ (by the definition of the sequence $\{n_l\}_{l \geq 0}$ in Lemma 2.5), $0 < Cr(\epsilon) < 1$ (as $0 < r(\epsilon) < 1$ by Corollary 2.4), $q \leq n$, $q < n_{l^\star+1}$, and all the χ_j's are strictly positive, by the definition (2.1) of M and the above expression we deduce that

$$\begin{aligned} f^q(E(n)) = f^{q - n_{l^\star}}(f^{n_{l^\star}}(E(n))) &\subseteq \mathcal{E}_{f^q(x)}\Big(Cr(\epsilon) e^{-\chi_j(n - n_{l^\star})} e^{\epsilon n_{l^\star} - (M+2)\epsilon n} e^{(q - n_{l^\star})M}\Big) \\ &\subseteq B\left(f^q(x), e^{\epsilon n_{l^\star}} e^{-2\epsilon n - M\epsilon n} e^{\epsilon n_{l^\star} M}\right) = B\left(f^q(x), e^{\epsilon(n_{l^\star} - n)M + \epsilon(n_{l^\star} - n) - \epsilon n}\right) \\ &\subseteq B\left(f^q(x), e^{-\epsilon n}\right) \subseteq B\left(f^q(x), e^{-\epsilon q}\right). \end{aligned}$$

As $q \geq n_{l_0} \geq m_0(\epsilon, x)$, by Lemma 3.6 we have $B\left(f^q(x), e^{-\epsilon q}\right) \subset \mathcal{P}(f^q(x))$. It follows that $f^q(E(n)) \subset \mathcal{P}(f^q(x))$ for all $n_{l_0} \leq q \leq n$. Together with (3.7), setting $\mathcal{P}_1 := \mathcal{P}$ this inclusion implies that $E(n) \subseteq \mathcal{P}_1^n(x)$, as desired.

3.3. **Proof of Proposition 3.5: the existence of $F(n)$ and $\mathcal{P}_2$.** We work with the same setting and notations as in Section 3.2. We need the following lemma, see for instance [PU10, Lemma 11.3.2]. Recall that the entropy of a countable partition $\mathcal{P} = \{P_i\}$ with respect to a probability measure μ is defined as

$$H_\mu(\mathcal{P}) := \sum_i -\mu(P_i)\log(\mu(P_i)).$$

Lemma 3.7. *Let μ be a Borel probability measure on a bounded subset A of a Euclidean space, and $\rho : A \to (0,1]$ a measurable function such that $\log\rho$ is integrable with respect to μ. There exists a countable measurable partition $\mathcal{P}$ of A such that $H_\mu(\mathcal{P}) < \infty$ and*

$$\operatorname{diam}(\mathcal{P}(x)) \le \rho(x) \quad \text{for } \mu\text{-almost every } x \in A.$$

Recall that $C(f)$ denotes the critical set of f and that $\nu(C(f)) = 0$. For $x \notin C(f)$, define the function

$$\rho(x) := c\, r(\epsilon) \min\{1, |\operatorname{Jac} f|\}, \tag{3.8}$$

where $c < 1$ is sufficiently small so that f is injective on the ball $B(x, \rho(x))$ for every $x \in \mathbb{P}^k \setminus C(f)$. Such constant exists because, since the function $\operatorname{Jac} f(x)$ is holomorphic in x, there exists a positive constant c_0 such that $|\operatorname{Jac} f(x)| \le c_0 \cdot \operatorname{dist}(x, C(f))$ for every $x \in \mathbb{P}^k$. For the same reason, the function $\log\rho$ is integrable with respect to ν, since by assumption the Lyapunov exponents of ν are not equal to $-\infty$, hence $\log|\operatorname{Jac} f|$ is integrable with respect to ν.

Consider a partition $\mathcal{P}$ given by Lemma 3.7, applied with $\mu = \nu$, $A = \operatorname{Supp}\nu$, and the function ρ as in (3.8). In particular, for ν-almost every $x \in \operatorname{Supp}\nu$ we have $\mathcal{P}(x) \subset B(x, \rho(x))$. For every $n \ge 1$, define

$$V_n(x,\rho) := \bigcap_{j=0}^{n-1} f^{-j} B(f^j(x), \rho(f^j(x))).$$

It follows from the definition of ρ that f is injective on $B(f^j(x), \rho(f^j(x)))$ for all $x \in \mathbb{P}^k$ and $0 \le j \le n-1$. As a consequence, for every $n \ge 1$ and for ν-almost every $x \in \operatorname{Supp}\nu$, the map f^n is injective on $V_n(x,\rho)$ and $\mathcal{P}^n(x) \subset V_n(x,\rho)$. It is then enough to show that, for every $n > n(\epsilon)$, the set $V_n(x,\rho)$ is contained in a set $F(n) := \mathcal{E}_x(c_F e^{-n(\chi_j - b_F\epsilon)})$, for some b_F and c_F as in the statement of Proposition 3.5.

Let $l^\star$ be the largest index of the sequence $\{n_l\}_{l\ge0}$ given by Lemma 2.5 such that $n_{l^\star} \le n-1$ (such $l^\star$ exists since $n > n(\epsilon)$). As in Section 3.2, set $g_{l^\star} := f^{-n_{l^\star}}_{\widehat{f^{n_{l^\star}}(x)}}$. Then $g_{l^\star}$ is well-defined on $B(f^{n_{l^\star}}(x), r(\epsilon))$. By the above, and in particular by the injectivity of f on V_n, we have

$$V_n(x,\rho) \subset g_{l^\star}\big(B(f^{n_{l^\star}}(x), \rho(f^{n_{l^\star}}(x)))\big).$$

By Corollary 2.4 (3c), we deduce that

$$V_n(x,\rho) \subset \mathcal{E}_x\big(Ke^{-n_{l^\star}(\chi_j-\epsilon)}\big)$$

for some constant $K > 0$ independent of x and n. Since $n-1 \le n_{l^\star+1} \le (1+\epsilon)n_{l^\star}$, we deduce that

$$V_n(x,\rho) \subset F'(n) := \mathcal{E}_x\big(Ke^{-(n-1)(1+\epsilon)^{-1}(\chi_j-\epsilon)}\big).$$

Set $F(n) := \mathcal{E}_x\big(c_F e^{-n(\chi_j - b_F\epsilon)}\big)$, where $c_F := K$ and $b_F := (1+\epsilon)^{-1}\min_j(\chi_j+1)$. The assertion follows.

This concludes the proof of Proposition 3.5 and therefore also the proof of Theorem 3.2.

4. Volume dimension of measures in $\mathcal{M}^+(f)$

Let $f\colon \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic endomorphism of algebraic degree $d \geq 2$. The goal of this section is to define volume dimensions for sets and measures and study their properties. More specifically, in Sections 4.1 and 4.2 we define and study the *volume dimension* $\mathrm{VD}(\nu)$ for measures $\nu \in \mathcal{M}^+(f)$ and $\mathrm{VD}_\nu(X)$ for subsets X of the support of ν. In Section 4.3 we prove a criterion to relate the volume dimension of a set of positive measure to the local volume dimensions defined in Section 3; see Proposition 4.26. This criterion, together with Theorem 3.2, will allow us to prove Theorem 1.1 in the next section.

4.1. **Definition of volume dimension and first properties.** Given $\epsilon > 0$, $\kappa > 0$, $N \in \mathbb{N}$, and $W \subseteq \mathbb{P}^k$, we consider the collection $\mathcal{U}_N^\kappa(W,\epsilon)$ of open subsets of $\mathbb{P}^k$ given by

$$\mathcal{U}_N^\kappa(W,\epsilon) := \{U \subset \mathbb{P}^k \colon \exists x \in W \text{ such that } U = U(N,x,\kappa,\epsilon)\}$$

where $U(N,x,\kappa,\epsilon)$ is as in Definition 2.7. Recall in particular that each $U(N,x,\kappa,\epsilon)$ (if it exists) is an open neighbourhood of x. Given $\epsilon > 0$ and $U \in \bigcup_{\kappa>0,N\geq 0} \mathcal{U}_N^\kappa(\mathbb{P}^k,\epsilon)$, we denote by $N(U), \kappa(U)$, and $z(U)$ the parameters associated to U as in that definition, i.e., such that

$$f^{N(U)}(U) = B(f^{N(U)}(z(U)), \kappa(U)\, e^{-N(U)M\epsilon}),$$

where M is as in (2.1).

Remark 4.1. Let $U_1 \neq U_2$ have the same parameters $N = N(U_i)$ and $\kappa = \kappa(U_i)$ and assume that $z(U_1)$ and $z(U_2)$ satisfy $f^N(z(U_1)) = f^N(z(U_2)) = w$. Then, we necessarily have $U_1 \cap U_2 = \emptyset$, as both U_1 and U_2 correspond to an inverse branch of f^N defined on a subset of $B(w,\kappa)$ containing w.

We fix now $\nu \in \mathcal{M}^+(f)$ and let Z_ν be as in Definition 2.3. We denote as before by $\chi_{\min} > 0$ the smallest Lyapunov exponent of ν. For every $0 < \epsilon \ll \chi_{\min}$, we fix $Z_\nu(\epsilon)$, $n(\epsilon)$, and $r(\epsilon)$ as given by Corollary 2.4 and Lemma 2.5.

By Theorem 3.2 and Remark 3.3, for $\hat\nu$-almost every $\hat x \in Z_\nu(\epsilon)$ and every $0 < \kappa < r(\epsilon)$ there exist positive integers $m_1(\epsilon,\hat x) \geq n(\epsilon)$ and $m_2(\epsilon,\kappa) \geq 1$ such that the conclusion of Theorem 3.2 holds for $N \geq m_1(\epsilon,x) + m_2(\epsilon,\kappa)$. For every $m \in \mathbb{N}$, consider the set $Z_\nu(\epsilon,m) := \{\hat x \in Z_\nu(\epsilon) \colon n(\epsilon) \leq m_1(\epsilon,\hat x) \leq m\}$. Since $\hat\nu(Z_\nu(\epsilon) \setminus Z_\nu(\epsilon,m)) \to 0$ as $m \to \infty$, for every $0 < \epsilon \ll \chi_{\min}$ there exists $m(\epsilon) \geq n(\epsilon)$ such that $\nu\left(\pi(Z_\nu(\epsilon)) \setminus \pi(Z_\nu(\epsilon,m(\epsilon)))\right) < \epsilon$. For every $0 < \epsilon \ll \chi_{\min}$, we define

$$Z_\nu^\star(\epsilon) := Z_\nu(\epsilon,m(\epsilon)) \quad \text{and} \quad Z_\nu^\star := \cup_{0<\epsilon\ll\chi_{\min}} Z_\nu^\star(\epsilon). \tag{4.1}$$

By definition, the conclusion of Theorem 3.2 holds for every $x \in \pi(Z_\nu^\star(\epsilon))$, with $m_1(\epsilon,x)$ independent of x. This fact will not be used in this subsection, but will be crucial in the proof of Proposition 4.26. Observe also that $\hat\nu(Z_\nu^\star) = 1$ and $\nu(\pi(Z_\nu^\star)) = 1$.

Remark 4.2. As in Remark 2.10, when X is uniformly expanding, for every $\nu \in \mathcal{M}_X^+(f)$ we can assume that $Z_\nu = Z_\nu^\star = Z_\nu^\star(\epsilon)$ for all $0 < \epsilon \ll \chi_{\min}$, and that $\pi(Z_\nu) = X$.

We first fix $0 < \epsilon \ll \chi_{\min}$ and define a quantity $\mathrm{VD}_\nu^\epsilon(Y)$ for every subset $Y \subseteq \pi(Z_\nu^\star(\epsilon))$. The definition will depend on both f and ν.

For $0 < \kappa < r(\epsilon)$ and $\mathbb{N} \ni N^\star \geq n(\epsilon)$, we denote by $\mathcal{U}(\epsilon,\kappa,N^\star)$ the collection of open sets

$$\mathcal{U}(\epsilon,\kappa,N^\star) := \bigcup_{N\geq N^\star} \mathcal{U}_N^\kappa(\pi(Z_\nu^\star(\epsilon)),\epsilon). \tag{4.2}$$

Lemma 4.3. *For every $0 < \epsilon \ll \chi_{\min}$, $0 < \kappa < r(\epsilon)$ and $\mathbb{N} \ni N^\star \geq n(\epsilon)$, the collection $\mathcal{U}(\epsilon,\kappa,N^\star)$ is an open cover of $\pi(Z_\nu^\star(\epsilon))$.*

Proof. It follows from Lemma 2.8 and the fact that $Z^\star_\nu(\epsilon) \subseteq Z_\nu(\epsilon)$ that $U(N, x, \kappa, \epsilon)$ is well-defined (and is an open neighbourhood of x) for all $x \in \pi(Z^\star_\nu(\epsilon))$, $0 < \kappa < r(\epsilon)$, and $N \geq n(\epsilon)$. The assertion follows. □

Definition 4.4. For every $0 < \epsilon \ll \chi_{\min}$, $Y \subseteq \pi(Z^\star_\nu(\epsilon))$, and $0 < \kappa < r(\epsilon)$, an (ϵ, κ)*-cover of* Y is a countable cover $\{U_i\}_{i\geq 1}$ of Y with the property that $U_i \in \mathcal{U}(\epsilon, \kappa, n(\epsilon))$ for all i. An ϵ*-cover* is an (ϵ, κ)-cover for some $0 < \kappa < r(\epsilon)$.

For every $\alpha \geq 0$ and $Y \subseteq \pi(Z^\star_\nu(\epsilon))$, we define $\Lambda^\epsilon_\alpha(Y) \in [0, +\infty]$ as

$$\Lambda^\epsilon_\alpha(Y) := \limsup_{\kappa\to 0} \Lambda^{\epsilon,\kappa}_\alpha(Y), \quad \text{where} \quad \Lambda^{\epsilon,\kappa}_\alpha(Y) := \lim_{N^\star\to\infty} \inf_{\{U_i\}} \sum_{i\geq 1} \mathrm{Vol}(U_i)^\alpha \tag{4.3}$$

and the infimum in the second expression is taken over all (ϵ, κ)-covers $\{U_i\}_{i\geq 1}$ of Y with $U_i \in \mathcal{U}(\epsilon, \kappa, N^\star)$ for all $i \geq 1$. Observe that the limit in the second expression above is well-defined, and equal to a supremum over $N^\star \geq n(\epsilon)$, as the $\mathcal{U}(\epsilon, \kappa, N^\star)$'s are decreasing collections of covers for $N^\star \to \infty$. We will see below that the function $\alpha \mapsto \Lambda^{\epsilon,\kappa}_\alpha(Y)$ is essentially independent of κ; see Lemma 4.11. Hence we will be able to use this approximated version of $\Lambda^\epsilon_\alpha(Y)$ in order to study its properties.

Lemma 4.5. *For every* $0 < \epsilon \ll \chi_{\min}$*,* $0 < \kappa < r(\epsilon)$*, and* $Y \subseteq \pi(Z^\star_\nu(\epsilon))$*, the following assertions hold:*

(1) the functions $\alpha \mapsto \Lambda^{\epsilon,\kappa}_\alpha(Y)$ *and* $\alpha \mapsto \Lambda^\epsilon_\alpha(Y)$ *are non-increasing;*
(2) if $\Lambda^{\epsilon,\kappa}_{\alpha_0}(Y) < \infty$ *(resp.* $\Lambda^\epsilon_{\alpha_0}(Y) < \infty$*) for some* $\alpha_0 \geq 0$*, then* $\Lambda^{\epsilon,\kappa}_\alpha(Y) = 0$ *(resp.* $\Lambda^\epsilon_\alpha(Y) = 0$*) for all* $\alpha > \alpha_0$*.*

Proof. By the definition (4.3) of $\Lambda^\epsilon_\alpha(Y)$ and $\Lambda^{\epsilon,\kappa}_\alpha(Y)$, it is enough to show the two assertions for $\Lambda^{\epsilon,\kappa}_\alpha(Y)$ for a given κ as in the statement.

The first property is clear from the definition of $\Lambda^{\epsilon,\kappa}_\alpha(Y)$ and the fact that, up to taking $N^\star$ sufficiently large, we can assume that the volume of all the U_i's is less than 1 in the definition of $\Lambda^{\epsilon,\kappa}_\alpha(Y)$; see Lemma 2.8 (2). If $\alpha_1 < \alpha_2$, then, for every $\eta > 0$ and up to taking $N^\star$ sufficiently large, for every $U \in \mathcal{U}(\epsilon, \kappa, N^\star)$ we also have

$$\mathrm{Vol}(U)^{\alpha_2} = \mathrm{Vol}(U)^{\alpha_1+(\alpha_2-\alpha_1)} \leq \eta^{(\alpha_2-\alpha_1)} \cdot \mathrm{Vol}(U)^{\alpha_1}.$$

As η can be taken arbitrarily small and $\Lambda^{\epsilon,\kappa}_{\alpha_1}(Y)$ is finite, this gives $\Lambda^{\epsilon,\kappa}_{\alpha_2}(Y) = 0$. The assertion follows. □

Because of Lemma 4.5, the following definition is well-posed.

Definition 4.6. For every $0 < \epsilon \ll \chi_{\min}$ and $Y \subseteq \pi(Z^\star_\nu(\epsilon))$, we set

$$\mathrm{VD}^\epsilon_\nu(Y) := \sup\{\alpha : \Lambda^\epsilon_\alpha(Y) = \infty\} = \inf\{\alpha : \Lambda^\epsilon_\alpha(Y) = 0\}.$$

Similarly, for every $0 < \kappa < r(\epsilon)$, we also set

$$\mathrm{VD}^{\epsilon,\kappa}_\nu(Y) := \sup\{\alpha : \Lambda^{\epsilon,\kappa}_\alpha(Y) = \infty\} = \inf\{\alpha : \Lambda^{\epsilon,\kappa}_\alpha(Y) = 0\}.$$

Remark 4.7. The definition of $\mathrm{VD}^{\epsilon,\kappa}_\nu(Y)$ will not be needed in this section, but Lemma 4.12 will be used in the proof of Proposition 5.4.

Lemma 4.8. *For every* $0 < \epsilon \ll \chi_{\min}$*,* $0 < \kappa < r(\epsilon)$*, and* $Y_1 \subseteq Y_2 \subseteq \pi(Z^\star_\nu(\epsilon))$*, we have*

$$\mathrm{VD}^{\epsilon,\kappa}_\nu(Y_1) \leq \mathrm{VD}^{\epsilon,\kappa}_\nu(Y_2) \quad \text{and} \quad \mathrm{VD}^\epsilon_\nu(Y_1) \leq \mathrm{VD}^\epsilon_\nu(Y_2).$$

Proof. For every $0 < \kappa < r(\epsilon)$, every (ϵ, κ)-cover of Y_2 is also an (ϵ, κ)-cover of Y_1. The assertion follows. □

In the next lemma, we will use the following form of Besicovitch's covering theorem [Bes45].

Theorem 4.9. *Let $n \geq 1$ be an integer. There exists a constant $b(n) > 0$ such that the following claim is true. If A is a bounded subset of $\mathbb{R}^n$, then for any function $r\colon A \to (0,\infty)$ there exists a countable subset $\{x_m\colon m \in \mathbb{N}\}$ of A such that the collection of open balls $\mathcal{B}(A,r) := \{B(x_m, r(x_m))\colon m \in \mathbb{N}\}$ covers A and can be decomposed into $b(n)$ families whose elements are disjoint.*

Lemma 4.10. *For every $0 < \epsilon \ll \chi_{\min}$, $0 < \kappa < r(\epsilon)$, and $Y \subseteq \pi(Z^\star_\nu(\epsilon))$, we have*

$$\mathrm{VD}^{\epsilon}_{\nu}(Y) \leq 1 \quad \text{and} \quad \mathrm{VD}^{\epsilon,\kappa}_{\nu}(Y) \leq 1.$$

Proof. By Lemma 4.8, it is enough to show the statement for $Y = \pi(Z^\star_\nu(\epsilon))$. By (4.3) and the Definition 4.6 of $\mathrm{VD}^{\epsilon}_{\nu}(Y)$ and $\mathrm{VD}^{\epsilon,\kappa}_{\nu}(Y)$, it is enough to show that, for any $0 < \kappa < r(\epsilon)$ and $N^\star > n(\epsilon)$, there exists an (ϵ,κ)-cover $\{U_i\}$ of $\pi(Z^\star_\nu(\epsilon))$ with $N(U_i) \geq N^\star$ for all $i \geq 1$ and such that

$$\sum_i \mathrm{Vol}(U_i) < C < \infty$$

for some constant C independent of κ. Indeed, by (4.3), this shows that $\Lambda^{\epsilon,\kappa}_1(\pi(Z^\star_\nu(\epsilon))) < \infty$, which implies that $\mathrm{VD}^{\epsilon,\kappa}_{\nu}(\pi(Z^\star_\nu(\epsilon))) \leq 1$. As C is independent of κ, this also gives $\Lambda^{\epsilon}_1(\pi(Z^\star_\nu(\epsilon))) < \infty$ and thus $\mathrm{VD}^{\epsilon}_{\nu}(\pi(Z^\star_\nu(\epsilon))) \leq 1$, as desired.

Fix $N^\star \geq n(\epsilon)$ and $0 < \kappa < r(\epsilon)$. Theorem 4.9 applied with $n = 2k$, $A := f^{N^\star}(\pi(Z^\star_\nu(\epsilon)))$, and $r \equiv \kappa\, e^{-N^\star M \epsilon}$, gives $b(2k)$ collections $\mathcal{B}_j$, $1 \leq j \leq b(2k)$, of disjoint open balls $\{B_{j,l}\}_{l \geq 1}$ centred on $f^{N^\star}(\pi(Z^\star_\nu(\epsilon)))$ and of radius r such that $A \subset \cup_{j,l} B_{j,l}$. We work here in local charts; see also Remark 1.4.

Consider an element $B_{j^\star,l^\star}$ of the collection $\{B_{j,l}\}_{j,l}$. By construction, its center belongs to $f^{N^\star}(Z^\star_\nu(\epsilon))$. Denote by $x^1_{j^\star,l^\star}, \dots, x^{m(j^\star,l^\star)}_{j^\star,l^\star}$ the preimages by $f^{N^\star}$ of the center of $B_{j^\star,l^\star}$ which belong to $\pi(Z^\star_\nu(\epsilon))$. For each $1 \leq q \leq m(j^\star,l^\star)$, choose an orbit $\hat{x} = \hat{x}(j^\star,l^\star,q) \in Z^\star_\nu(\epsilon)$ such that $\pi_0(\hat{x}(j^\star,l^\star,q)) = x^q_{j^\star,l^\star}$ (observe that $\pi_{N^\star}(\hat{x}(j^\star,l^\star,q))$ is necessarily the center of $B_{j^\star,l^\star}$). The inverse branch $f^{-N^\star}_{T^{N^\star}(\hat{x}(j^\star,l^\star,q))}$ is well-defined on the ball $B_{j^\star,l^\star}$ by the choice of the function r and Lemma 2.8. More precisely, the image of each $B_{j,l}$ under any of such branches is of the form $U(N^\star,x,\kappa,\epsilon)$ for some $x \in \pi(Z^\star_\nu(\epsilon))$. The images associated to the same $B_{j,l}$ are disjoint; see Remark 4.1. Similarly, any two such images are also disjoint whenever the corresponding balls B_{j_0,l_0} and B_{j_1,l_1} are disjoint. Observe that this in particular applies whenever $j_0 = j_1$, since each collection $\mathcal{B}_j$ consists of disjoint balls.

By construction, we have

$$\pi(Z^\star_\nu(\epsilon)) \subseteq \bigcup_{j=1}^{b(2k)} \bigcup_{l\geq 1} \bigcup_{q=1}^{m(j,l)} f^{-N^\star}_{T^{N^\star}(\hat{x}(j,l,q))}(B_{j,l}).$$

By the arguments above, we have

$$\sum_{j=1}^{b(2k)} \sum_{l\geq 1} \sum_{1\leq q\leq m(j,l)} \mathrm{Vol}\left(f^{-N^\star}_{T^{N^\star}(\hat{x}(j,l,q))}(B_{j,l})\right) \leq C' b(2k)\,\mathrm{Vol}(\mathbb{P}^k),$$

where the positive constant C' is due to the use of local charts, and is in particular independent of κ. This completes the proof. ☐

Observe that, for every $0 < \gamma < 1$, there exists a positive integer $\theta = \theta_\gamma$ with the property that it is possible to cover any ball of radius r in $\mathbb{P}^k$ with a finite number θ of open balls of radius γr (the constant θ also depends on the dimension k, but we omit this dependence since k is fixed).

For every $0 < \epsilon \ll \chi_{\min}$, define also the constant

$$\ell(\epsilon) := \frac{L_\nu + k(2M+1)\epsilon}{L_\nu - k\epsilon}. \tag{4.4}$$

Observe that $\ell(\epsilon) > 1$ for all ϵ as above, and we have $\ell(\epsilon) = 1 + O(\epsilon)$ for $\epsilon \to 0$.

Lemma 4.11. *Fix $0 < \epsilon \ll \chi_{\min}$, $0 < \kappa_1 < \kappa_2 < r(\epsilon)/3$, and $Y \subseteq \pi(Z_\nu^\star(\epsilon))$. For every $\alpha \geq 0$, we have*

$$\left(\kappa_1/\kappa_2^{\ell(\epsilon)}\right)^{2k\alpha} \Lambda^{\epsilon,\kappa_2}_{\alpha\ell(\epsilon)}(Y) \leq \Lambda^{\epsilon,\kappa_1}_{\alpha}(Y) \leq \theta_{3\kappa_2/\kappa_1} \left(\kappa_1^{\ell(\epsilon)}/\kappa_2\right)^{2k\alpha} \Lambda^{\epsilon,\kappa_2}_{\alpha\ell(\epsilon)^{-1}}(Y), \tag{4.5}$$

where $\ell(\epsilon) > 1$ is as in (4.4) *and $\theta_{3\kappa_2/\kappa_1} > 0$ is as above. In particular, for every $0 < \epsilon \ll \chi_{\min}$, $Y \subseteq \pi(Z_\nu^\star(\epsilon))$, and $0 < \kappa_1 < \kappa_2 < r(\epsilon)/3$, we have*

$$\ell(\epsilon)^{-1}\, \mathrm{VD}^{\epsilon,\kappa_2}_\nu(Y) \leq \mathrm{VD}^{\epsilon,\kappa_1}_\nu(Y) \leq \ell(\epsilon)\, \mathrm{VD}^{\epsilon,\kappa_2}_\nu(Y). \tag{4.6}$$

Proof. We first show the inequality $(\kappa_1/\kappa_2^{\ell(\epsilon)})^{2k}\Lambda^{\epsilon,\kappa_2}_{\alpha\ell(\epsilon)}(Y) \leq \Lambda^{\epsilon,\kappa_1}_\alpha(Y)$. We can assume that $\Lambda^{\epsilon,\kappa_1}_\alpha(Y) < \infty$. Fix η such that $\Lambda^{\epsilon,\kappa_1}_\alpha(Y) < \eta < \infty$. There exists an (ϵ,κ_1)-cover $\{U_i\}_{i\geq 1}$ of Y, with each U_i of the form $U_i = U(N_i, x_i, \kappa_1, \epsilon)$, such that

$$\sum_{i\geq 1} \mathrm{Vol}(U_i)^\alpha < \eta.$$

For each $i \geq 1$, set $V_i := U(N_i, x_i, \kappa_2, \epsilon)$. Since $\kappa_1 < \kappa_2 < r(\epsilon)$, we have $U_i \subset V_i$ for all i, and the collection $\{V_i\}_{i\geq 1}$ is an (ϵ,κ_2)-cover of Y. Moreover, by Lemma 2.8 (2) and the definition of $\ell(\epsilon)$, for every i we have

$$\frac{\mathrm{Vol}(V_i)^{\ell(\epsilon)}}{\mathrm{Vol}(U_i)} \leq \frac{\kappa_2^{2k\ell(\epsilon)}}{\kappa_1^{2k}} \frac{e^{-2N_i(L_\nu - k\epsilon)\ell(\epsilon)}}{e^{-2N_i(L_\nu + k(2M+1)\epsilon)}} = \left(\kappa_2^{\ell(\epsilon)}/\kappa_1\right)^{2k}.$$

It follows that

$$\sum_{i\geq 1} \mathrm{Vol}(V_i)^{\alpha\ell(\epsilon)} < \left(\kappa_2^{\ell(\epsilon)}/\kappa_1\right)^{2k\alpha} \eta.$$

By the choice of η, this shows that $\Lambda^{\epsilon,\kappa_2}_{\alpha\ell(\epsilon)}(Y) \leq (\kappa_2^{\ell(\epsilon)}/\kappa_1)^{2k\alpha}\Lambda^{\epsilon,\kappa_1}_\alpha(Y)$, as desired. By the definition of $\mathrm{VD}^{\epsilon,\kappa}_\nu(Y)$, this also shows the first inequality in (4.6).

We now show the second inequality in (4.5). As above, this also shows the second inequality in (4.6). We can assume that $\Lambda^{\epsilon,\kappa_2}_\alpha(Y) < \infty$. Fix η such that $\Lambda^{\epsilon,\kappa_2}_\alpha(Y) < \eta < \infty$. There exists an (ϵ,κ_2)-cover $\{U_i\}_{i\geq 1}$ of Y, with each U_i of the form $U_i = U(N_i, x_i, \kappa_2, \epsilon)$, such that

$$\sum_{i\geq 1} \mathrm{Vol}(U_i)^\alpha < \eta.$$

By Definition 2.7, for each $i \geq 1$, we have $A_i := f^{N_i}(U_i) = B(f^{N_i}(x_i), \kappa_2\, e^{-N_i M\epsilon})$. By the definition of $\theta_{3\kappa_2/\kappa_1}$, one can cover any ball A_i with $\theta_{3\kappa_2/\kappa_1}$ open balls of radius $(\kappa_1/3)e^{-N_i M\epsilon}$. In particular, these balls cover $f^{N_i}(Y \cap U_i) \subseteq A_i$. Up to removing from the collection the balls not intersecting $f^{N_i}(Y \cap U_i)$ and replacing all the other balls with balls of the same center and radius $\kappa_1 e^{-N_i M\epsilon}$, we see that we can cover $f^{N_i}(Y \cap U_i)$ with $\theta_{3\kappa_2/\kappa_1}$ balls of radius $\kappa_1 e^{-N_i M\epsilon}$ and centred at points of $f^{N_i}(Y \cap U_i)$. For every i, we denote by $\{B_{i,j}\}_{1\leq j\leq J_i}$ (for some $1 \leq J_i \leq \theta_{3\kappa_2/\kappa_1}$), the collection of the balls of radius $\kappa_1\, e^{-N_i M\epsilon}$ constructed above. By construction, for every i we have

$$f^{N_i}(Y \cap U_i) \subseteq \bigcup_{j=1}^{J_i} B_{i,j}.$$

For every i,j, set $V_{i,j} := g_i(B_{i,j})$, where g_i is the inverse branch of f^{N_i} defined in a neighbourhood of $f^{N_i}(z_i)$ that sends $f^{N_i}(z_i)$ to z_i. It follows that, for each i, the collection $\{V_{i,j}\}_{1\leq j\leq J_i}$ is an (ϵ,κ_1)-cover of $U_i \cap Y$. By Lemma 2.8 (2) and the definition of $\ell(\epsilon)$, for every i and $1 \leq j \leq J_i$ we have

$$\frac{\mathrm{Vol}(V_{i,j})^{\ell(\epsilon)}}{\mathrm{Vol}(U_i)} \leq \frac{\kappa_1^{2k\ell(\epsilon)}}{\kappa_2^{2k}} \frac{e^{-2N_i(L_\nu - k\epsilon)\ell(\epsilon)}}{e^{-2N_i(L_\nu + k(2M+1)\epsilon)}} = \left(\kappa_1^{\ell(\epsilon)}/\kappa_2\right)^{2k}.$$

Summing over i and j, we obtain

$$\sum_{i\geq 1}\sum_{j=1}^{J_i} \mathrm{Vol}(V_{i,j})^{\alpha\ell(\epsilon)} \leq \sum_{i\geq 1}\left(\theta_{3\kappa_2/\kappa_1}\left(\kappa_1^{\ell(\epsilon)}/\kappa_2\right)^{2k\alpha}\mathrm{Vol}(U_i)^\alpha\right)$$
$$\leq \theta_{3\kappa_2/\kappa_1}\left(\kappa_1^{\ell(\epsilon)}/\kappa_2\right)^{2k\alpha}\eta.$$

It follows that $\Lambda^{\epsilon,\kappa_1}_{\alpha\ell(\epsilon)}(Y) \leq \theta_{3\kappa_2/\kappa_1}(\kappa_1^{\ell(\epsilon)}/\kappa_2)^{2k\alpha}\Lambda^{\epsilon,\kappa_2}_{\alpha}(Y)$, as desired. This completes the proof. □

Lemma 4.12. *For every $0 < \epsilon \ll \chi_{\min}$, $0 < \kappa < r(\epsilon)/3$, and $Y \subseteq \pi(Z^\star_\nu(\epsilon))$, we have*

$$|\,\mathrm{VD}^{\epsilon,\kappa}_\nu(Y) - \mathrm{VD}^{\epsilon}_\nu(Y)| \leq (\ell(\epsilon)-1)\min\left\{\mathrm{VD}^{\epsilon}_\nu(Y), \mathrm{VD}^{\epsilon,\kappa}_\nu(Y)\right\} \leq \ell(\epsilon)-1,$$

where $\ell(\epsilon) > 1$ is as in (4.4).

Proof. Fix $0 < \epsilon \ll \chi_{\min}$ and $Y \subseteq \pi(Z^\star_\nu(\epsilon))$. It follows from Lemma 4.11 that there exists $\alpha_0 = \alpha_0(\epsilon, Y) \in [0,+\infty]$ such that

$$\alpha_0 \leq \mathrm{VD}^{\epsilon,\kappa}_\nu(Y) \leq \alpha_0\ell(\epsilon) \quad \text{for every } 0 < \kappa < r(\epsilon). \tag{4.7}$$

Take any $\eta > 0$. It follows from the definition (4.3) of $\Lambda^\epsilon_\alpha(Y)$ and $\Lambda^{\epsilon,\kappa}_\alpha(Y)$ that $\Lambda^\epsilon_{\alpha_0-\eta}(Y) = +\infty$ (assuming $\alpha_0 > 0$ and $0 < \eta < \alpha_0$) and $\Lambda^\epsilon_{\alpha_0\ell(\epsilon)+\eta}(Y) = 0$ (assuming $\alpha_0 < \infty$). Since η is arbitrary, we deduce from the definition of $\mathrm{VD}^\epsilon_\nu(Y)$ that

$$\alpha_0 \leq \mathrm{VD}^{\epsilon}_\nu(Y) \leq \alpha_0\ell(\epsilon). \tag{4.8}$$

The first inequality in the statement follows from (4.7) and (4.8). The second one follows from Lemma 4.10. □

Remark 4.13. In Definition 4.4, we do not require $z(U_i) \in Y$ for any of the U_i in an ϵ-cover of Y, but we could also define $\mathrm{VD}^\epsilon_\nu(Y)$ (and $\mathrm{VD}^{\epsilon,\kappa}_\nu(Y)$) for $Y \subseteq \pi(Z^\star_\nu(\epsilon))$ by only using sets U_i such that $z(U_i) \in Y$, rather than $z(U_i) \in \pi(Z^\star_\nu(\epsilon))$, in the definition of $\Lambda^{\epsilon,\kappa}_\alpha(Y)$. Denoting by $\overline{\Lambda}^\epsilon_\alpha(Y)$ and $\overline{\mathrm{VD}}^\epsilon_\nu(Y)$ the corresponding quantities, it is straightforward to see that $\Lambda^\epsilon_\alpha(Y) \leq \overline{\Lambda}^\epsilon_\alpha(Y)$ for all $\alpha \geq 0$. Hence, we have $\mathrm{VD}^\epsilon_\nu(Y) \leq \overline{\mathrm{VD}}^\epsilon_\nu(Y)$. On the other hand, take α such that $\Lambda^\epsilon_\alpha(Y) < \infty$ and consider an (ϵ,κ)-cover $\{U_i\}_{i\geq 1}$ of Y for which the value of $\sum_{i\geq 1}\mathrm{Vol}(U_i)^\alpha$ is close to the value of $\Lambda^\epsilon_\alpha(Y)$. By the definition of $\Lambda^\epsilon_\alpha(Y)$, we can assume that, for all i, we have $U_i \in \mathcal{U}(\epsilon,\kappa,N_0)$ for some $\kappa < r(\epsilon)/3$ and some $N_0 \geq n(\epsilon)$. Take i_0 such that $z(U_{i_0}) \notin Y$. Observe that there must exist $y \in U_{i_0} \cap Y$ and that, since $\kappa < r(\epsilon)/3$, the set $U(N(U_{i_0}), y, 3\kappa, \epsilon)$ is well-defined and contains U_{i_0}. By similar arguments as in the proof of Lemma 4.11, this shows that $\overline{\Lambda}^{\epsilon,3\kappa}_{\alpha/\ell(\epsilon)}(Y) \leq 3^{2k\alpha/\ell(\epsilon)}\Lambda^{\epsilon,\kappa}_\alpha(Y)$ for all $0 < \kappa < r(\epsilon)/3$, which gives $\overline{\mathrm{VD}}^\epsilon_\nu(Y) \leq \ell(\epsilon)\,\mathrm{VD}^\epsilon_\nu(Y)$. Similar arguments and estimates hold for $\mathrm{VD}^{\epsilon,\kappa}_\nu(Y)$.

Take now $X \subseteq \pi(Z^\star_\nu)$, and recall that $\nu(\pi(Z^\star_\nu)) = 1$. For every $0 < \epsilon \ll \chi_{\min}$, we set

$$X^\epsilon := X \cap \pi(Z^\star_\nu(\epsilon)). \tag{4.9}$$

Observe that, since $Z^\star_\nu = \cup_{0<\epsilon\ll\chi_{\min}} Z^\star_\nu(\epsilon)$, we have $\cup_{0<\epsilon\ll\chi_{\min}} X^\epsilon = X$.

Definition 4.14. For every $X \subseteq \pi(Z_\nu^\star)$, the *volume dimension* $\mathrm{VD}_\nu(X)$ of X is

$$\mathrm{VD}_\nu(X) := \limsup_{\epsilon\to 0} \mathrm{VD}_\nu^\epsilon(X^\epsilon).$$

Remark 4.15. The limsup in Definition 4.14 is actually a limit; see Section 4.2, and in particular Corollary 4.24, and Remark 5.1.

Lemma 4.16. *For every $Y_1 \subseteq Y_2 \subseteq \pi(Z_\nu^\star)$, we have $\mathrm{VD}_\nu(Y_1) \le \mathrm{VD}_\nu(Y_2)$.*

Proof. By Lemma 4.8, for any $0 < \epsilon \ll \chi_{\min}$ we have $\mathrm{VD}_\nu^\epsilon(Y_1^\epsilon) \le \mathrm{VD}_\nu^\epsilon(Y_2^\epsilon)$. The conclusion follows from Definition 4.14. □

Definition 4.17. Take $\nu \in \mathcal{M}^+(f)$. The *volume dimension* $\mathrm{VD}(\nu)$ of ν is

$$\mathrm{VD}(\nu) := \inf\big\{ \mathrm{VD}_\nu(X) : X \subseteq \pi(Z_\nu^\star), \nu(X) = 1\big\}.$$

Lemma 4.18. *For every $\nu \in \mathcal{M}^+(f)$ and $X \subseteq \pi(Z_\nu^\star)$, we have $\mathrm{VD}_\nu(X) \le 1$. In particular, we have $\mathrm{VD}(\nu) \le 1$ for every $\nu \in \mathcal{M}^+(f)$.*

Proof. The statement is an immediate consequence of Lemma 4.10 and Definitions 4.14 and 4.17. □

When $X \subseteq \mathbb{P}^k$ is a uniformly expanding closed invariant set for f, by Remark 4.2 we can assume that $\pi(Z_\nu^\star(\epsilon)) = \pi(Z_\nu^\star) = X$ for every invariant measure ν on X and every $0 < \epsilon \ll \chi_{\min}$. In particular, the following definition is well-posed and defines the term $\mathrm{VD}(J(f))$ in Theorem 1.3.

Definition 4.19. If $X \subseteq \mathbb{P}^k$ is uniformly expanding, the *volume dimension* of X is

$$\mathrm{VD}(X) := \sup_{\nu\in\mathcal{M}_X^+(f)} \mathrm{VD}_\nu(X).$$

We conclude this section with the next proposition, which in particular shows that, when $k = 1$, the volume dimension associated to any $\nu \in \mathcal{M}^+(f)$ is equivalent to the Hausdorff dimension.

Proposition 4.20. *If $\nu \in \mathcal{M}^+(f)$ is such that all the Lyapunov exponents of ν are equal to $\chi > 0$, then*

(1) $2k\,\mathrm{VD}_\nu(X) = \mathrm{HD}(X)$ for all $X \subseteq \pi(Z_\nu^\star)$;
(2) $2k\,\mathrm{VD}(\nu) = \mathrm{HD}(\nu)$,

where $\mathrm{HD}(X)$ and $\mathrm{HD}(\nu)$ denote the Hausdorff dimension of X and ν, respectively.

Proof. Recall that the Hausdorff dimension of $X \subseteq \mathbb{P}^k$ is defined as

$$\mathrm{HD}(X) := \inf\big\{\alpha\colon H_\alpha(X) = 0\big\}, \quad \text{where } H_\alpha(X) := \sup_{\delta>0}\inf_{\{B_i\}}\sum_i (\operatorname{diam} B_i)^\alpha.$$

The infimum in the second expression is taken over all countable covers of X by open balls $\{B_i\}$ whose diameter is less than δ. The Hausdorff dimension $\mathrm{HD}(\nu)$ of ν is defined as

$$\mathrm{HD}(\nu) := \inf\big\{ \mathrm{HD}(X)\colon X \subseteq \operatorname{Supp}\nu,\ \nu(X) = 1\big\}. \tag{4.10}$$

In order to prove the first assertion, it is enough to show that

$$\ell(\epsilon)^{-1}\,\mathrm{HD}(X^\epsilon) \le 2k\,\mathrm{VD}_\nu^\epsilon(X^\epsilon) \le \ell(\epsilon)\,\mathrm{HD}(X^\epsilon) \quad \text{for all } X \subseteq \pi(Z_\nu^\star) \text{ and } 0 < \epsilon \ll \chi_{\min},$$

where we recall that X^ϵ is defined as in (4.9) and the constant $\ell(\epsilon)$ is defined in (4.4). Observe that, as $L_\nu = k\chi$, we have $\ell(\epsilon) = \big(\chi + (2M+1)\epsilon\big)/(\chi - \epsilon)$.

We first prove the inequality $\mathrm{HD}(X^\epsilon) \le 2k\ell(\epsilon)\,\mathrm{VD}_\nu^\epsilon(X^\epsilon)$. Fix $\alpha_1 > \mathrm{VD}_\nu^\epsilon(X^\epsilon)$. By Lemma 4.5, we have $\Lambda_{\alpha_1}^\epsilon(X^\epsilon) = \limsup_{\kappa\to 0}\Lambda_{\alpha_1}^{\epsilon,\kappa}(X^\epsilon) = 0$. Therefore, for any $\eta > 0$ and up to taking $0 < \kappa < r(\epsilon)$ sufficiently small, there exists an (ϵ,κ)-cover $\{U_i\}_{i\ge1}$ of X^ϵ such that

$$\sum_{i\ge1} \mathrm{Vol}(U_i)^{\alpha_1} < \eta. \tag{4.11}$$

Setting $N_i := N(U_i)$, by Corollary 2.9 (1) and (2) we have

$$\frac{\operatorname{diam}(U_i)^{2k\ell(\epsilon)}}{\operatorname{Vol}(U_i)} \le \frac{2^{2k\ell(\epsilon)}\kappa^{2k\ell(\epsilon)}e^{-2kN_i(\chi-\epsilon)\ell(\epsilon)}}{\kappa^{2k}e^{-2kN_i(\chi+(2M+1)\epsilon)}} \le 2^{2k\ell(\epsilon)}\kappa^{2k(\ell(\epsilon)-1)} \le 2^{2k\ell(\epsilon)},$$

where in the last step we used the fact that $\kappa < r(\epsilon) < 1$.

For each i, define the ball $V_i := B(z(U_i), \operatorname{diam}(U_i))$ of center $z(U_i)$ and radius $\operatorname{diam}(U_i)$. Then, $U_i \subseteq V_i$ and $\{V_i\}_{i\ge1}$ is a cover of X^ϵ by balls. By the above estimates and (4.11), we have

$$\sum_{i\ge1}\operatorname{diam}(V_i)^{2k\ell(\epsilon)\alpha_1} = \sum_{i\ge1}(2\operatorname{diam}(U_i))^{2k\ell(\epsilon)\alpha_1} \le 2^{4k\ell(\epsilon)\alpha_1}\sum_{i\ge1}\operatorname{Vol}(U_i)^{\alpha_1} < 2^{4k\ell(\epsilon)\alpha_1}\eta.$$

Therefore, we have $\operatorname{HD}(X^\epsilon) \le 2k\ell(\epsilon)\alpha_1$ and the conclusion follows by taking $\alpha_1 \searrow \operatorname{VD}^\epsilon_\nu(X^\epsilon)$.

We now prove the inequality $2k\ell(\epsilon)^{-1}\operatorname{VD}^\epsilon_\nu(X^\epsilon) \le \operatorname{HD}(X^\epsilon)$. Fix α_0 such that $H_{\alpha_0}(X^\epsilon) = 0$. Then, for any $\eta > 0$, there exists a cover $\{B_i = B(x_i, r_i)\}_{i\ge1}$ of X^ϵ consisting of open balls such that

$$\sum_{i\ge1}(2r_i)^{\alpha_0} < \eta. \tag{4.12}$$

Fix any $0 < \kappa < r(\epsilon)$. By definition of $H_{\alpha_0}(X^\epsilon)$ we can assume that

$$\sup_i r_i < \kappa\, e^{-n(\epsilon)(\chi+(2M+1)\epsilon)}. \tag{4.13}$$

For each $i \ge 1$, set

$$N_i := \lfloor \frac{\log\kappa - \log r_i}{\chi + (2M+1)\epsilon} \rfloor$$

and $U_i := U(N_i, x_i, \kappa, \epsilon)$. Observe that $N_i \ge n(\epsilon)$ for all i by (4.13), hence every U_i is well-defined by Lemma 2.8. By Corollary 2.9 (1), for every i we also have

$$B(x_i, r_i) \subseteq B(x_i, \kappa\, e^{-N_i(\chi+(2M+1)\epsilon)}) \subseteq U_i \subseteq B(x_i, \kappa\, e^{-N_i(\chi-\epsilon)}).$$

In particular, the collection $\{U_i\}_{i\ge1}$ is an (ϵ,κ)-cover of X^ϵ and, for all i, we also have

$$\operatorname{Vol}(U_i)^{\ell(\epsilon)/(2k)} \le \kappa^{\ell(\epsilon)}e^{-N_i(\chi+(2M+1)\epsilon)} \le \kappa^{\ell(\epsilon)}e^{\left(1+\frac{\log r_i - \log\kappa}{\chi+(2M+1)\epsilon}\right)(\chi+(2M+1)\epsilon)} \le e^{\chi+(2M+1)\epsilon}r_i, \tag{4.14}$$

where we used the facts that $\lfloor x \rfloor \ge x - 1$ for every $x > 0$ and that $\kappa < r(\epsilon) < 1$.

It follows from (4.12) and (4.14) that

$$\sum_{i\ge1}\operatorname{Vol}(U_i)^{\alpha_0\ell(\epsilon)/(2k)} \le \sum_{i\ge1}\left(\frac{1}{2}e^{\chi+(2M+1)\epsilon}\right)^{\alpha_0}(2r_i)^{\alpha_0} < \left(\frac{1}{2}e^{\chi+(2M+1)\epsilon}\right)^{\alpha_0}\eta.$$

Therefore, for every $0 < \kappa < r(\epsilon)$, we have

$$\Lambda^{\epsilon,\kappa}_{\alpha_0\ell(\epsilon)/(2k)}(X^\epsilon) < \left(\frac{1}{2}e^{\chi+(2M+1)\epsilon}\right)^{\alpha_0}\eta < \left(\frac{1}{2}e^{\chi+(2M+1)\epsilon}\right)^{\alpha_0}\eta. \tag{4.15}$$

Taking the limsup over κ in the left hand side of (4.15), by (4.3) we obtain $\Lambda^\epsilon_{\alpha_0\ell(\epsilon)/(2k)}(X^\epsilon) < \infty$. Therefore, we have $2k\ell(\epsilon)^{-1}\operatorname{VD}^\epsilon_\nu(X^\epsilon) \le \operatorname{HD}(X^\epsilon)$. This completes the proof of the first assertion.

We now prove the second assertion. We first show the inequality $\operatorname{HD}(\nu) \le 2k\operatorname{VD}(\nu)$. For every $X \subseteq \pi(Z^\star_\nu)$, we have $2k\operatorname{VD}_\nu(X) = \operatorname{HD}(X)$ by the first assertion. By Definition 4.17, we obtain $2k\operatorname{VD}(\nu) = \inf\{\operatorname{HD}(X) : X \subseteq \pi(Z^\star_\nu), \nu(X) = 1\}$. By the definition (4.10) of $\operatorname{HD}(\nu)$, this implies that $\operatorname{HD}(\nu) \le 2k\operatorname{VD}(\nu)$.

We now prove the inequality $\operatorname{HD}(\nu) \ge 2k\operatorname{VD}(\nu)$. Take $X \subseteq \mathbb{P}^k$ with $\nu(X) = 1$. Since $\nu(\pi(Z^\star_\nu)) = 1$, we have $\nu(X \cap \pi(Z^\star_\nu)) = 1$. Then, by the first assertion and the monotonicity of the Hausdorff dimension, we have $2k\operatorname{VD}(X \cap \pi(Z^\star_\nu)) = \operatorname{HD}(X \cap \pi(Z^\star_\nu)) \le \operatorname{HD}(X)$. By the definition (4.10) of

HD(ν), we deduce that $\inf\{2k\,\mathrm{VD}(X\cap\pi(Z_\nu^\star)) : \nu(X)=1\} \le \mathrm{HD}(\nu)$. By Definition 4.17, this gives $2k\,\mathrm{VD}(\nu) \le \mathrm{HD}(\nu)$ and completes the proof. □

4.2. **An equivalent definition of** $\mathrm{VD}(\nu)$**.** We present here an equivalent definition of the volume dimension for sets $X \subset \pi(Z_\nu^\star)$. This definition in particular allows us to prove that the $\limsup_{\epsilon\to 0}$ in Definition 4.14 is always a limit; see also Remark 5.1 for sets $X \subseteq \pi(Z_\nu^\star)$ with $\nu(X) > 0$. The advantage of this definition is that we will not have the small exponential terms $e^{-NM\epsilon}$ in the definition of the sets of the covers. In particular, we work with sets which are more similar to actual Bowen balls of fixed radius. On the other hand, the collection of neighbourhoods associated to any x will in some sense depend on x. This section is not necessary in order to obtain the main results of the paper.

For every $0 < \epsilon \ll \chi_{\min}$, $0 < \kappa < r(\epsilon)$, $l \in \mathbb{N}$, and $W \subseteq \pi(Z_\nu^\star(\epsilon))$, we consider the collection $\widetilde{\mathcal{U}}_l^\kappa(W,\epsilon)$ of open subsets of $\mathbb{P}^k$ given by

$$\widetilde{\mathcal{U}}_l^\kappa(W,\epsilon) := \big\{\widetilde{U} \subset \mathbb{P}^k \colon \exists x \in W, \text{ such that } \widetilde{U} = \widetilde{U}(n_l,x,\kappa)\big\}.$$

Here, $\{n_l\}_{l\ge 0}$ is the sequence associated to $x \in \pi(Z_\nu(\epsilon))$ by Lemma 2.5, and, letting $\hat{x}$ be any element of $Z_\nu(\epsilon)$ with $x_0 = x$, we set

$$\widetilde{U}(n_l,x,\kappa) := f^{-n_l}_{T^{n_l}(\hat{x})}\big(B(f^{n_l}(x_0),\kappa)\big),$$

where the right hand side of the above expression is well-defined by Corollary 2.4. For every ϵ and κ as above and $\mathbb{N} \ni N^\star \ge n(\epsilon)$, we denote by $\widetilde{\mathcal{U}}(\epsilon,\kappa,N^\star)$ the collection of open sets

$$\widetilde{\mathcal{U}}(\epsilon,\kappa,N^\star) := \bigcup_{n_l \ge N^\star} \widetilde{\mathcal{U}}_l^\kappa(\pi(Z_\nu^\star(\epsilon)),\epsilon).$$

For every $\alpha \ge 0$ and $Y \subseteq \pi(Z_\nu^\star(\epsilon))$, we define $\widetilde{\Lambda}_\alpha^\epsilon(Y) \in [0,+\infty]$ as

$$\widetilde{\Lambda}_\alpha^\epsilon(Y) := \limsup_{\kappa\to 0} \widetilde{\Lambda}_\alpha^{\epsilon,\kappa}(Y), \quad \text{where} \quad \widetilde{\Lambda}_\alpha^{\epsilon,\kappa}(Y) := \lim_{N^\star\to\infty} \inf_{\{U_i\}} \sum_{i\ge 1} \mathrm{Vol}(\widetilde{U}_i)^\alpha \tag{4.16}$$

and the infimum is taken over all covers $\{\widetilde{U}_i\}_{i\ge 1}$ of Y with $\widetilde{U}_i \in \widetilde{\mathcal{U}}(\epsilon,\kappa,N^\star)$ for all $i \ge 1$. As in Lemma 4.5, one can show that, for every $0 < \epsilon \ll \chi_{\min}$ and $Y \subseteq \pi(Z_\nu^\star(\epsilon))$, the function $\alpha \mapsto \widetilde{\Lambda}_\alpha^\epsilon(Y)$ is non-increasing and that, if $\widetilde{\Lambda}_{\alpha_0}^\epsilon(Y) < \infty$ for some $\alpha_0 \ge 0$, then $\widetilde{\Lambda}_\alpha^\epsilon(Y) = 0$ for all $\alpha > \alpha_0$. As a consequence, the following definition is well-posed.

Definition 4.21. For every $0 < \epsilon \ll \chi_{\min}$ and $Y \subseteq Z_\nu^\star(\epsilon)$, we set

$$\widetilde{\mathrm{VD}}_\nu^\epsilon(Y) := \sup\{\alpha : \widetilde{\Lambda}_\alpha^\epsilon(Y) = \infty\} = \inf\{\alpha : \widetilde{\Lambda}_\alpha^\epsilon(Y) = 0\}.$$

Lemma 4.22. *For every* $0 < \epsilon_1 < \epsilon_2 \ll \chi_{\min}$ *and* $Y \subseteq \pi(Z_\nu^\star(\epsilon))$*, we have* $\widetilde{\mathrm{VD}}_\nu^{\epsilon_1}(Y) = \widetilde{\mathrm{VD}}_\nu^{\epsilon_2}(Y)$.

Proof. The statement is clear since the sets $\widetilde{U}(n_l,x,\kappa)$ do not depend on ϵ. □

Lemma 4.23. *For every* $0 < \epsilon \ll \chi_{\min}$ *and* $Y \subseteq \pi(Z_\nu^\star(\epsilon))$ *we have*

$$\ell(\epsilon)^{-1}\widetilde{\mathrm{VD}}_\nu^\epsilon(Y) \le \mathrm{VD}_\nu^\epsilon(Y) \le \beta(\epsilon)\widetilde{\mathrm{VD}}_\nu^\epsilon(Y),$$

where $\ell(\epsilon) > 1$ *is as in* (4.4) *and* $\beta(\epsilon) := \frac{L_\nu+k\epsilon}{L_\nu-k\epsilon}\cdot\big(\min_j \frac{\chi_j-\epsilon}{\chi_j+(2M+1)\epsilon} - \frac{1}{n(\epsilon)}\big)^{-1}$.

Observe that $\beta(\epsilon)$ as in the statement above satisfies $\beta(\epsilon) > 1$ for all $0 < \epsilon \ll \chi_{\min}$ and $\beta(\epsilon) = 1 + O(\epsilon)$ as $\epsilon \to 0$.

Proof. We first prove the first inequality. Suppose $\alpha \geq 0$ is such that $\Lambda^{\epsilon}_{\alpha}(Y) = 0$. Then for any $\eta > 0$ and $0 < \kappa < r(\epsilon)$, there exists a (κ, ϵ)-cover $\{U_i\}_{i\geq 1}$ of Y of the form $U_i = U(N_i, x_i, \kappa, \epsilon)$, with $N_i \geq N^\star \geq n(\epsilon)$ for all i, such that $\sum_{i\geq 1} \mathrm{Vol}(U_i)^\alpha < \eta$.

For each $i \geq 1$, let $l(i)$ be such that $n_{l(i)} \leq N_i < n_{l(i)+1}$. Such $l(i)$ exists since, by Lemma 2.5, we have $n_0(x) \leq n(\epsilon)$ for all $x \in \pi(Z_\nu(\epsilon))$. For every i, we then have $U_i \subset \widetilde{U}_i := \widetilde{U}(n_{l(i)+1}, x_i, \kappa)$, see also (2.8) in the proof of Lemma 2.8. In particular, we have $\{\widetilde{U}_i\} \subset \widetilde{\mathcal{U}}(\epsilon, \kappa, N^\star)$ and the sets $\{\widetilde{U}_i\}$ form a cover of Y. It follows from the definition of $\ell(\epsilon)$ that for all $i \geq 1$, we have

$$\frac{\mathrm{Vol}(\widetilde{U}(n_{l(i)+1}, x_i, \kappa))^{\ell(\epsilon)}}{\mathrm{Vol}(U(N_i, x_i, \kappa, \epsilon))} \leq \frac{\kappa^{2k\ell(\epsilon)} e^{-2n_{l(i)+1}(L_\nu - k\epsilon)\ell(\epsilon)}}{\kappa^{2k} e^{-2N_i(L_\nu + k(2M+1)\epsilon)}} = \kappa^{2k(\ell(\epsilon)-1)} e^{2(L_\nu + \kappa\epsilon(2M+1))(N_i - n_{l(i)+1})} \leq 1,$$

where in the last inequality we used the facts the $N_i < n_{l(i)+1}$ and $\kappa < r(\epsilon) < 1$. In particular, we have

$$\sum_{i\geq 1} \mathrm{Vol}\left(\widetilde{U}(n_{l(i)+1}, x_i, \kappa)\right)^{\alpha\ell(\epsilon)} \leq \sum_{i\geq 1} \mathrm{Vol}(U_i)^\alpha < \eta,$$

which gives the inequality $\widetilde{\Lambda}^{\epsilon,\kappa}_{\alpha\ell(\epsilon)}(Y) \leq \Lambda^{\epsilon,\kappa}_{\alpha}(Y)$ for any $0 < \kappa < r(\epsilon)$. Taking the limsup over κ as in the definition of $\widetilde{\Lambda}^{\epsilon}_{\alpha\ell(\epsilon)}(Y)$, we obtain $\widetilde{\Lambda}^{\epsilon}_{\alpha\ell(\epsilon)}(Y) < \infty$. By the choice of α, we deduce the desired inequality $\widetilde{\mathrm{VD}}^{\epsilon}_{\nu}(Y) \leq \ell(\epsilon)\, \mathrm{VD}^{\epsilon}_{\nu}(Y)$.

We now prove the second inequality. Suppose $\widetilde{\Lambda}^{\epsilon}_{\alpha}(Y) = 0$. Then for any $\eta > 0$ and $0 < \kappa < r(\epsilon)$, there exists a cover $\{\widetilde{U}_i\}_{i\geq 1} \subset \widetilde{\mathcal{U}}(\epsilon, \kappa, N^\star)$ of Y, which each $\widetilde{U}_i$ of the form $\widetilde{U}_i = \widetilde{U}(n_{l(i)}, x_i, \kappa)$, such that $\sum_{i\geq 1} \mathrm{Vol}(\widetilde{U}_i)^\alpha < \eta$. For each $i \geq 1$, set

$$N_i := \left\lfloor n_{l(i)} \cdot \min_j \frac{\chi_j - \epsilon}{\chi_j + (2M+1)\epsilon} \right\rfloor.$$

From the definition of N_i and Lemma 2.8, for all $i \geq 1$, we have

$$\widetilde{U}(n_{l(i)}, x_i, \kappa) \subset U(N_i, x_i, \kappa, \epsilon).$$

It follows that, for every $i \geq 1$, we have

$$\frac{\mathrm{Vol}(U(N_i, x_i, \kappa, \epsilon))^{\beta(\epsilon)}}{\mathrm{Vol}(\widetilde{U}(n_{l(i)}, x_i, \kappa))} \leq \frac{\kappa^{2k\beta(\epsilon)} e^{-2N_i(L_\nu - k\epsilon)\beta(\epsilon)}}{\kappa^{2k} e^{-2n_{l(i)}(L_\nu + k\epsilon)}} \leq 1,$$

where in the last step we used the facts that $\kappa < r(\epsilon) < 1$ and that, since $n_{l(i)} \geq n(\epsilon)$ for all $i \geq 1$ and $\lfloor r \rfloor \geq r - 1$ for all $r > 0$, we have

$$N_i \frac{L_\nu - k\epsilon}{L_\nu + k\epsilon} \beta(\epsilon) \geq \frac{n_{l(i)} \cdot \min_j \frac{\chi_j - \epsilon}{\chi_j + (2M+1)\epsilon} - 1}{\min_j \frac{\chi_j - \epsilon}{\chi_j + (2M+1)\epsilon} - \frac{1}{n(\epsilon)}} \geq n_{l(i)}.$$

Therefore, we have

$$\sum_{i\geq 1} \mathrm{Vol}(U(N_i, x_i, \kappa, \epsilon))^{\alpha\beta(\epsilon)} \leq \eta,$$

which gives the inequality $\Lambda^{\epsilon,\kappa}_{\alpha\beta(\epsilon)}(Y) \leq \widetilde{\Lambda}^{\epsilon,\kappa}_{\alpha}(Y)$ for any $0 < \kappa < r(\epsilon)$. Taking the limsup over κ as in the definition of $\Lambda^{\epsilon}_{\alpha\beta(\epsilon)}(Y)$, we obtain $\Lambda^{\epsilon}_{\alpha\beta(\epsilon)}(Y) = 0$. By the choice of α, we have $\mathrm{VD}^{\epsilon}_{\nu}(Y) \leq \beta(\epsilon)\widetilde{\mathrm{VD}}^{\epsilon}_{\nu}(Y)$. The proof is complete. □

Thanks to Lemma 4.23, one can see that the $\limsup_{\epsilon\to 0}$ in the Definition 4.6 is actually a limit. Recall that, for every $X \subseteq \pi(Z^\star_\nu)$, we denote $X^\epsilon := X \cap \pi(Z^\star_\nu(\epsilon))$.

Corollary 4.24. *For every $X \subseteq \pi(Z^\star_\nu)$, we have*

$$\mathrm{VD}_\nu(X) = \lim_{\epsilon\to 0} \mathrm{VD}^\epsilon_\nu(X^\epsilon).$$

Proof. For every $X \subseteq \pi(Z^\star_\nu)$, set $\widetilde{\mathrm{VD}}_\nu(X) := \lim_{\epsilon\to 0} \widetilde{\mathrm{VD}}^\epsilon_\nu(X^\epsilon)$. The limit is well-defined and equal to the supremum over $0 < \epsilon \ll \chi_{\min}$ since, for every $0 < \epsilon_1 < \epsilon_2 \ll \chi_{\min}$, we have $\widetilde{\mathrm{VD}}^{\epsilon_1}_\nu(X^{\epsilon_2}) = \widetilde{\mathrm{VD}}^{\epsilon_2}_\nu(X^{\epsilon_2})$ by Lemma 4.22 and $\widetilde{\mathrm{VD}}^{\epsilon_1}_\nu(X^{\epsilon_1}) \geq \widetilde{\mathrm{VD}}^{\epsilon_1}_\nu(X^{\epsilon_2})$ since $X^{\epsilon_1} \supseteq X^{\epsilon_2}$. It follows from Lemma 4.23 that $\lim_{\epsilon\to 0} \mathrm{VD}^\epsilon_\nu(X^\epsilon)$ is well-defined and equal to $\widetilde{\mathrm{VD}}_\nu(X)$. The assertion follows. ☐

Remark 4.25. A further possible (equivalent) way to define the volume dimension is the following. Take $\nu \in \mathcal{M}^+(f)$. Fix $0 < \epsilon \ll \chi_{\min}$ and take $Y \subseteq \pi(Z^\star_\nu(\epsilon))$. For every $\alpha \geq 0$, define

$$\check{\Lambda}^\epsilon_\alpha(Y) := \limsup_{\kappa\to 0} \check{\Lambda}^{\epsilon,\kappa}_\alpha(Y), \qquad \text{where} \qquad \check{\Lambda}^{\epsilon,\kappa}_\alpha(Y) := \lim_{N^\star\to\infty} \inf_{\{U_i\}} \sum_{i\geq 1} e^{-2N(U_i)L_\nu(f)\alpha} \kappa^{2k\alpha},$$

and the infimum is taken over all countable covers $\{U_i\}_{i\geq 1} \subset \mathcal{U}(\epsilon,\kappa,N^\star)$ of Y. Recall that $L_\nu(f)$ denotes the sum of the Lyapunov exponents of ν. As in Lemma 4.5, one can prove that, for every $0 < \epsilon \ll \chi_{\min}$ and $Y \subseteq \pi(Z^\star_\nu(\epsilon))$, the function $\alpha \mapsto \check{\Lambda}^\epsilon_\alpha(Y)$ is non-increasing in α, and that if $\check{\Lambda}^\epsilon_{\alpha_0}(Y)$ is finite, then $\check{\Lambda}^\epsilon_\alpha(Y) = 0$ for all $\alpha > \alpha_0$. Hence, the quantity

$$\check{\mathrm{VD}}^\epsilon_\nu(Y) := \inf\big\{\alpha\colon \check{\Lambda}^\epsilon_\alpha(Y) = 0\big\} = \sup\big\{\alpha\colon \check{\Lambda}^\epsilon_\alpha(Y) = \infty\big\}$$

is well-defined for all $0 < \epsilon \ll \chi_{\min}$ and $Y \subseteq \pi(Z^\star_\nu(\epsilon))$. For every $0 < \epsilon \ll \chi_{\min}$, define the constants

$$\ell_-(\epsilon) := \frac{L_\nu(f) - k\epsilon}{L_\nu(f)} \qquad \text{and} \qquad \ell_+(\epsilon) := \frac{L_\nu(f) + (2M+1)k\epsilon}{L_\nu(f)}.$$

Observe that, for all $0 < \epsilon \ll \chi_{\min}$, we have $\ell_-(\epsilon) < 1 < \ell_+(\epsilon)$ and $\ell_-(\epsilon), \ell_+(\epsilon) \to 1$ as $\epsilon \to 0$. One can show in this case that, for every $0 < \epsilon \ll \chi_{\min}$ and $Y \subseteq \pi(Z^\star_\nu(\epsilon))$, we have

$$\ell_-(\epsilon)\, \mathrm{VD}^\epsilon_\nu(Y) \leq \check{\mathrm{VD}}^\epsilon_\nu(Y) \leq \ell_+(\epsilon)\, \mathrm{VD}^\epsilon_\nu(Y). \tag{4.17}$$

Take now $X \subseteq \pi(Z^\star_\nu)$. As before, setting $X^\epsilon := X \cap \pi(Z^\star_\nu(\epsilon))$ for every $0 < \epsilon \ll \chi_{\min}$, we can define

$$\check{\mathrm{VD}}_\nu(X) := \limsup_{\epsilon\to 0} \mathrm{VD}^\epsilon_\nu(X^\epsilon) \qquad \text{and}$$
$$\check{\mathrm{VD}}(\nu) := \inf\big\{\check{\mathrm{VD}}_\nu(X)\colon X \subseteq \pi(Z^\star_\nu) \text{ and } \nu(X) = 1\big\}.$$

It follows from (4.17), applied with $Y = X^\epsilon$, that $\check{\mathrm{VD}}_\nu(X) = \mathrm{VD}_\nu(X)$ for all $X \subseteq \pi(Z^\star_\nu)$, and that $\check{\mathrm{VD}}(\nu) = \mathrm{VD}(\nu)$.

4.3. **From local volume dimensions to volume dimensions.** Fix a measure $\nu \in \mathcal{M}^+(f)$ and $0 < \epsilon \ll \chi_{\min}$. For $x \in \pi(Z^\star_\nu(\epsilon))$, $0 < \kappa < r(\epsilon)$, and $N \geq n(\epsilon)$ recall that $\delta_x(\epsilon,\kappa,N)$ is defined in (3.1) and well-defined by Lemma 2.8. The integer $m_1(\epsilon,x)$ in Theorem 3.2 is uniformly bounded from above for all $x \in \pi(Z^\star_\nu(\epsilon))$ by the definition (4.1) of $Z^\star_\nu(\epsilon)$. This fact is crucial in the proof of the next statement. Recall that a measure is *non-atomic* if it does not assign mass to points.

Proposition 4.26. *Let f be a holomorphic endomorphism of $\mathbb{P}^k$ of algebraic degree $d \geq 2$ and take $\nu \in \mathcal{M}^+(f)$. Assume that ν is non-atomic. Fix $\alpha_1, \alpha_2 \geq 0$ and $0 < \epsilon \ll \chi_{\min}$. Let $Y \subseteq \pi(Z^\star_\nu(\epsilon))$ be such that $\nu(Y) > 0$. Suppose that for every $0 < \kappa < r(\epsilon)$ there exists $m = m(\epsilon,\kappa) \geq n(\epsilon)$ such that*

$$\alpha_1 \leq \delta_x(\epsilon,\kappa,N) \leq \alpha_2 \quad \textit{for all } x \in Y \textit{ and } N \geq m(\epsilon,\kappa). \tag{4.18}$$

Then, we have

$$\alpha_1 \leq \mathrm{VD}^\epsilon_\nu(Y) \leq \alpha_2.$$

Proof. The proof of the proposition essentially follows the arguments in [You82, Proposition 2.1]. Recall that, for every $0 < \kappa < r(\epsilon)$ and $N^\star \geq 0$, the collection $\mathcal{U}(\epsilon, \kappa, N^\star)$ is defined in (4.2) and is a cover of Y; see Lemma 4.3. Define the quantity

$$\alpha(Y, \nu) := \inf \Big\{ \alpha : \lim_{N^\star \to \infty} \inf_{\mathcal{U}'} \sum_{U \in \mathcal{U}'} \nu(U)^\alpha = 0 \Big\},$$

where the infimum is taken over all the sub-covers $\mathcal{U}' \subset \mathcal{U}(\epsilon, \kappa, N^\star)$ of Y. Since ν is non-atomic and $\nu(Y) > 0$, we have $\alpha(Y, \nu) = 1$; see [Bil65, Section 14]. We use here the fact that, for every fixed $0 < \kappa < r(\epsilon)$, the sets $U(N, x, \kappa, \epsilon)$ shrink to $\{x\}$ as $N \to \infty$; see Lemma 2.8.

Fix $\alpha > 1$, $0 < \kappa < r(\epsilon)$, $N^\star \geq m(\epsilon, \kappa)$, and $\eta > 0$. Since $\alpha(Y, \nu) = 1$, there exists a cover $\mathcal{U}_0 \subset \mathcal{U}(\epsilon, \kappa, N^\star)$ of Y such that

$$\sum_{U \in \mathcal{U}_0} \nu(U)^\alpha < \eta.$$

By the assumption (4.18) and the choice $N^\star \geq m(\epsilon, \kappa)$, for every $U \in \mathcal{U}(\epsilon, \kappa, N^\star)$ we have $\mathrm{Vol}(U)^{\alpha_2} \leq \nu(U)$. Hence, we have

$$\sum_{U \in \mathcal{U}_0} \mathrm{Vol}(U)^{\alpha_2 \alpha} \leq \sum_{U \in \mathcal{U}_0} \nu(U)^\alpha < \eta.$$

This shows the inequality $\Lambda^{\epsilon,\kappa}_{\alpha_2 \alpha}(Y) < \eta$ for any $0 < \kappa < r(\epsilon)$. Therefore, we have $\limsup_{\kappa \to 0} \Lambda^{\epsilon,\kappa}_{\alpha_2 \alpha}(Y) = \Lambda^{\epsilon}_{\alpha_2 \alpha}(Y) < \eta$. Taking $\alpha \searrow 1$, we obtain the inequality $\mathrm{VD}^\epsilon_\nu(Y) \leq \alpha_2$.

For the other inequality, again by the assumption (4.18), for all $0 < \kappa < r(\epsilon)$ and $N^\star \geq m(\epsilon, \kappa)$ we have $\mathrm{Vol}(U)^{\alpha_1} \geq \nu(U)$ for every $U \in \mathcal{U}(\epsilon, \kappa, N^\star)$. Hence, for any cover $\mathcal{U}_0 \subset \mathcal{U}(\epsilon, \kappa, N^\star)$, we have

$$\sum_{U \in \mathcal{U}_0} \mathrm{Vol}(U)^{\alpha_1} \geq \sum_{U \in \mathcal{U}_0} \nu(U) \geq \nu(Y).$$

Therefore, we have $\Lambda^{\epsilon,\kappa}_{\alpha_1}(Y) \geq \nu(Y) > 0$ for any $0 < \kappa < r(\epsilon)$, which gives

$$\Lambda^{\epsilon}_{\alpha_1}(Y) = \limsup_{\kappa \to 0} \Lambda^{\epsilon,\kappa}_{\alpha_1}(Y) \geq \nu(Y) > 0.$$

Hence, we have $\mathrm{VD}^\epsilon_\nu(Y) \geq \alpha_1$. The proof is complete. □

Remark 4.27. Let $\nu \in \mathcal{M}^+(f)$ be non-atomic. Take $X \subseteq \pi(Z^\star_\nu)$ with $\nu(X) > 0$. Setting $X^\epsilon := X \cap \pi(Z^\star_\nu(\epsilon))$, assume that for every $0 < \epsilon \ll \chi_{\min}$ and $0 < \kappa < r(\epsilon)$ there exists $m = m(\epsilon, \kappa)$ and $\alpha^0_1, \alpha^0_2 \in \mathbb{R}$ such that

$$\alpha_1(\epsilon) \leq \delta_x(\epsilon, \kappa, N) \leq \alpha_2(\epsilon) \quad \text{for all } x \in X^\epsilon \text{ and } N \geq m(\epsilon, \kappa) \tag{4.19}$$

for some functions $\alpha_1(\epsilon) = \alpha^0_1 + O(\epsilon)$ and $\alpha_2(\epsilon) = \alpha^0_2 + O(\epsilon)$. Applying Proposition 4.26 to X^ϵ instead of Y we see that, for every $0 < \epsilon \ll \chi_{\min}$, we have $\alpha_1(\epsilon) \leq \mathrm{VD}^\epsilon_\nu(X^\epsilon) \leq \alpha_2(\epsilon)$, which gives $\alpha^0_1 \leq \mathrm{VD}_\nu(X) \leq \alpha^0_2$.

5. Proofs of Theorems 1.1, 1.2, and 1.3

In this section, $f \colon \mathbb{P}^k \to \mathbb{P}^k$ is a holomorphic endomorphism of algebraic degree $d \geq 2$.

5.1. **Proof of Theorem 1.1.** For $\nu \in \mathcal{M}^+(f)$, recall that Z_ν is defined in Definition 2.3 and for every $0 < \epsilon \ll \chi_{\min}$ (where $\chi_{\min} > 0$ is the smallest Lyapunov exponent of ν), the set $Z^\star_\nu(\epsilon)$ is defined in (4.1).

Assume first that ν is atomic. Since ν is ergodic, it gives mass only to a finite number of points,. hence it satisfies $h_\nu(f) = 0$. It also follows from the Definition 4.17 of $\mathrm{VD}(\nu)$ that $\mathrm{VD}(\nu) = 0$, since the support S_ν of ν satisfies $\nu(S_\nu) = 1$ and $\mathrm{VD}_\nu(S_\nu) = 0$, being finite. Therefore, the conclusion follows in this case.

We can then assume that ν is non-atomic. By Theorem 3.2 and Proposition 4.26, for every $0 < \epsilon \ll \chi_{\min}$ we have

$$\mathrm{VD}^\epsilon_\nu(\pi(Z^\star_\nu(\epsilon))) \le \frac{h_\nu(f)}{2L_\nu(f)} + c\epsilon,$$

where the constant c is independent of ϵ. By Definition 4.14, taking $\epsilon \searrow 0$, we obtain the inequality $\mathrm{VD}_\nu(\pi(Z_\nu)) \le (2L_\nu(f))^{-1}h_\nu(f)$. As $\nu(\pi(Z_\nu)) = 1$, by Definition 4.17, we deduce the inequality $\mathrm{VD}(\nu) \le (2L_\nu(f))^{-1}h_\nu(f)$.

In order to prove the reversed inequality, let $Y_0 \subseteq \pi(Z^\star_\nu)$ be such that $\nu(Y_0) = 1$. For any $0 < \epsilon \ll \chi_{\min}$, applying Proposition 4.26 to $Y_0 \cap \pi(Z^\star_\nu(\epsilon))$, we deduce from Theorem 3.2 that

$$\mathrm{VD}^\epsilon_\nu(Y_0 \cap \pi(Z^\star_\nu(\epsilon))) \ge \frac{h_\nu(f)}{2L_\nu(f)} - c\epsilon,$$

where again the constant c is independent of ϵ; see also Remark 4.27. By Definition 4.14, we have the inequality $\mathrm{VD}_\nu(Y_0) \ge (2L_\nu(f))^{-1}h_\nu(f)$. As Y_0 is arbitrary, it follows from Definition 4.17 that $\mathrm{VD}(\nu) \ge (2L_\nu(f))^{-1}h_\nu(f)$. The proof of Theorem 1.1 is complete.

Remark 5.1. Let $\nu \in \mathcal{M}^+(f)$ be non-atomic and take $X \subseteq \pi(Z^\star_\nu)$ with $\nu(X) > 0$. By Remark 4.27 and with similar arguments as in the proof of Theorem 3.2, it follows that the $\limsup_{\epsilon\to 0}$ in Definition 4.14 is actually a limit.

5.2. **Proof of Theorem 1.2.** Let $X \subseteq \mathbb{P}^k$ be a closed f-invariant set. Define

$$\mathrm{DD}^+_X(f) := \sup\left\{ \mathrm{VD}(\nu) \colon \nu \in \mathcal{M}^+_X(f)\right\} \tag{5.1}$$

and recall that $\delta_X(f)$, $P^+_X(t)$, and $p^+_X(f)$ are defined in Sections 2.3 and 2.4. Theorem 1.2 follows from the following proposition applied with $X = J(f)$.

Proposition 5.2. *We have $p^+_X(f) = 2\,\mathrm{DD}^+_X(f)$. In particular, the set $\left\{t \colon P^+_X(t) = 0\right\}$ is non-empty.*

Proof. We first prove the inequality $p^+_X(f) \ge 2\,\mathrm{DD}^+_X(f)$. We can assume that $\mathrm{DD}^+_X(f) > 0$. Fix $0 < t < 2\,\mathrm{DD}^+_X(f)$. By the definition (5.1) of $\mathrm{DD}^+_X(f)$, there exists $\nu \in \mathcal{M}^+_X(f)$ such that $\mathrm{VD}(\nu) > t/2$. Since $\mathrm{VD}(\nu) = (2L_\nu(f))^{-1}h_\nu(f)$ by Theorem 1.1, we have $h_\nu(f)/L_\nu(f) > t$. It follows that $h_\nu(f) - tL_\nu(f) > 0$; that is, $P^+_X(t) > 0$. Therefore we have $p^+_X(f) > t$. Since t is arbitrary, we obtain $p^+_X(f) \ge 2\,\mathrm{DD}^+_X(f)$.

Let us now prove that $2\,\mathrm{DD}^+_X(f) \ge p^+_X(f)$. Suppose that $2\,\mathrm{DD}^+_X(f) < p^+_X(f)$. Then there exists $t \in (2\,\mathrm{DD}^+_X(f), p^+_X(f))$ such that $P^+_X(t) > 0$. In particular, there exists a measure $\nu \in \mathcal{M}^+_X(f)$ with $h_\nu(f) - tL_\nu(f) > 0$. We deduce from Theorem 1.1 that

$$\mathrm{VD}(\nu) = \frac{h_\nu(f)}{2L_\nu(f)} > \frac{t}{2} > \mathrm{DD}^+_X(f).$$

This contradicts the definition of $\mathrm{DD}^+_X(f)$. Hence, we have $2\,\mathrm{DD}^+_X(f) \ge p^+_X(f)$.

By Lemma 4.18, we have $\mathrm{DD}^+_X(f) \le 1$. Since the function $t \mapsto P^+_X(t)$ is convex and non-increasing, the equality $p^+_X(f) = 2\,\mathrm{DD}^+_X(f) \le 2$ implies that the set $\{t \colon P^+_X(t) = 0\}$ is non-empty. The proof is complete. □

Remark 5.3. One can also define

$$\begin{aligned}\mathrm{DD}_e^+(f) &:= \sup\{\mathrm{VD}(\nu)\colon \nu \in \mathcal{M}_e^+(f)\},\\ P_e^+(t) &:= \sup\{h_\nu(f) - tL_\nu(f)\colon \nu \in \mathcal{M}_e^+(f)\}, \qquad \text{and}\\ p_e^+(f) &:= \inf\{t : P_e^+(f) \le 0\},\end{aligned}$$

where we recall that $\mathcal{M}_e^+(f)$ is the set of ergodic f-invariant measures whose measure-theoretic entropy is strictly larger than $(k-1)\log d$. Since $\mathcal{M}_e^+(f) \subseteq \mathcal{M}^+(f)$ for every f [deT08; Dup12], Theorem 1.1 applies in particular to every $\nu \in \mathcal{M}_e^+(f)$. The same proof as Proposition 5.2 gives $p_e^+(f) = 2\,\mathrm{DD}_e^+(f)$.

5.3. **Proof of Theorem 1.3.** Recall that if $X \subset \mathbb{P}^k$ is a uniformly expanding closed invariant set for f, the volume dimension $\mathrm{VD}(X)$ is defined as $\mathrm{VD}(X) := \sup_{\nu\in\mathcal{M}_X^+(f)} \mathrm{VD}_\nu(X)$; see Definition 4.19.

Proposition 5.4. *Let $X \subseteq \mathbb{P}^k$ be a uniformly expanding closed invariant set for f containing a dense orbit. We have $\delta_X(f) \ge 2\,\mathrm{VD}(X)$. In particular, if f is hyperbolic, we have $\delta_J(f) \ge 2\,\mathrm{VD}(J(f))$.*

Proof. We can assume that a volume-conformal measure on X exists, otherwise we have $\delta_X(f) = +\infty$ and the assertion is trivial. Let μ be a t-volume-conformal measure on X, for some $t \ge \delta_X(f)$. Since X contains a dense orbit, we have $\operatorname{Supp}\mu = X$. It suffices to prove the inequality $t \ge 2\,\mathrm{VD}_\nu(X)$ for any measure $\nu \in \mathcal{M}_X^+(f)$. By Definition 4.19, this implies that $t \ge 2\,\mathrm{VD}(X)$ and the conclusion follows by taking the infimum over t as above.

Fix $\nu \in \mathcal{M}_X^+(f)$. We can assume that $\mathrm{VD}_\nu(X) > 0$, since otherwise the assertion is trivial. In particular, recalling that all measures in $\mathcal{M}^+(f)$ are ergodic, we can assume that ν is non-atomic. Fix a constant $\gamma > 1$. Since $\mathrm{VD}_\nu(X) = \limsup_{\epsilon\to 0} \mathrm{VD}_\nu^\epsilon(X^\epsilon)$ by Definition 4.14 and $\mathrm{VD}_\nu(X) \le 1$ by Lemma 4.18, we can fix $\epsilon_0 = \epsilon_0(\gamma)$ such that $\mathrm{VD}_\nu(X) \le \gamma\,\mathrm{VD}_\nu^{\epsilon_0}(X^{\epsilon_0})$. As we can assume that $\epsilon_0(\gamma) \to 0$ as $\gamma \to 1$, it is enough to prove that $2\,\mathrm{VD}_\nu^{\epsilon_0}(X^{\epsilon_0}) \le t\ell(\epsilon_0)$, where $\ell(\epsilon)$ is as in (4.4).

By Remark 4.2, for every $0 < \epsilon \ll \chi_{\min}$ we have $X^\epsilon = X$. In particular, $X^{\epsilon_0} = X$ is compact. Fix $\alpha > 1$. As in the proof of Proposition 4.26, since $\mu(X) > 0$, for every $\eta > 0$ and $0 < \kappa < r(\epsilon)$ there exists an $N^\star$ (depending on η and κ) large enough and a cover $\{U_i\}_{i\ge1} \subset \mathcal{U}(\epsilon,\kappa,N^\star)$ of $X^{\epsilon_0} = X$ satisfying

$$\sum_i \mu(U_i)^\alpha \le \eta. \tag{5.2}$$

As $X^{\epsilon_0} = X$ is compact, we can assume that the cover $\{U_i\}$ is finite. By Lemma 2.13, we have

$$\sum_i \mathrm{Vol}(U_i)^{t\alpha/2} \le \sum_i \frac{C^{t\alpha}\kappa^{tk\alpha}e^{\alpha t N(U_i)k\epsilon(5M+2)}}{m_-(\mu,\kappa\, e^{-N(U_i)M\epsilon})^\alpha}\mu(U_i)^\alpha \le \frac{C^{t\alpha}\kappa^{tk\alpha}e^{\alpha t N^+k\epsilon(5M+2)}}{m_-(\mu,\kappa\, e^{-N^+M\epsilon})^\alpha}\sum_i \mu(U_i)^\alpha, \tag{5.3}$$

where $m_- > 0$ is as in Lemma 2.12, $C < \infty$ is as in Lemma 2.13, and N^+ is the maximum of the $N(U_i)$ (we use here that the cover $\{U_i\}$ is finite). We deduce from (5.2) and (5.3) that

$$\sum_i \mathrm{Vol}(U_i)^{t\alpha/2} < \frac{C^{t\alpha}\kappa^{tk\alpha}e^{\alpha t N^+k\epsilon(5M+2)}}{m_-(\mu,\kappa\, e^{-N^+M\epsilon})^\alpha}\eta < \infty.$$

This implies that $\mathrm{VD}_\nu^{\epsilon_0,\kappa}(X^{\epsilon_0}) \le t\alpha/2$ for all $\alpha > 1$. Taking $\alpha \searrow 1$, we have $\mathrm{VD}_\nu^{\epsilon_0,\kappa}(X^{\epsilon_0}) \le t/2$ for all $0 < \kappa < r(\epsilon)$. By Lemma 4.12, we have $\mathrm{VD}_\nu^{\epsilon_0}(X^{\epsilon_0}) \le \ell(\epsilon_0)t/2$. The proof is complete. ☐

The following result implies Theorem 1.3 by taking $X = J(f)$ if f is hyperbolic (since $J(f)$ has dense orbits).

Theorem 5.5. *Let f be an endomorphism of $\mathbb{P}^k$ of algebraic degree $d \geq 2$ and $X \subseteq \mathbb{P}^k$ a closed invariant uniformly expanding set containing a dense orbit. Then*

$$\delta_X(f) = p_X^+(f) = 2\,\mathrm{VD}(X)$$

and there exists a unique invariant probability measure μ_X supported on X and such that $\mathrm{VD}(\mu_X) = \mathrm{VD}(X)$.

Proof. It follows from the general theory of thermodynamic formalism for uniformly expanding systems (see for instance [Bow75] and [PU10, Chapters 3 and 6]) that, for every $t \in \mathbb{R}$, there exists a unique invariant probability measure μ_t on X maximizing the pressure function $P_X(t) = \sup_\nu\{h_\nu(f) - tL_\nu(f)\}$, where the supremum is taken over all invariant measures ν on X. We used here the fact that, since X is uniformly expanding, the function $-t\log|\operatorname{Jac} f|$ is Hölder continuous on X.

Let μ_X be the invariant measure μ_{t_0} associated to $t_0 = p_X^+(f)$. As $h_{\mu_X}(f) = p_X^+(f)L_{\mu_X}(f)$, it follows from Theorem 1.1 that we have $p_X^+(f) = 2\,\mathrm{VD}(\mu_X)$. Since $\mathrm{VD}(\mu_X) \leq \mathrm{VD}_{\mu_X}(X)$ by Definition 4.17 and $\mathrm{VD}(X) := \sup_{\nu\in\mathcal{M}_X^+(f)} \mathrm{VD}_\nu(X)$ by Definition 4.19, we have

$$\mathrm{VD}(\mu_X) \leq \mathrm{VD}(X).$$

We deduce from the above and Proposition 5.4 that

$$p_X^+(f) = 2\,\mathrm{VD}(\mu_X) \leq 2\,\mathrm{VD}(X) \leq \delta_X(f).$$

To complete the proof, we prove the inequality $\delta_X(f) \leq p_X^+(f)$ by constructing a $t^\star$-volume conformal measure on X for some $t^\star \leq p_X^+(f)$. Since one can follow Patterson's [Pat76] and Sullivan's [Sul83] constructions of conformal measures, we only sketch the proof and refer to those papers for more details; see also [PU10, Sections 12.1 and 12.3].

Take $x \in X$. For each $m \geq 0$, set

$$E_m := f|_X^{-m}(x).$$

Then E_m is finite and $E_{m+1} = f|_X^{-1}(E_m)$. For all $t \geq 0$, consider the sequence $\{a_m(t)\}_{m\geq 1}$ given by

$$a_m(t) := \log\Big(\sum_{x\in E_m} e^{S_m\phi_t(x)}\Big)$$

where $\phi_t(x) := -t\log|\operatorname{Jac} f(x)|$ and $S_m\phi_t(x) := \sum_{j=0}^{m-1}(\phi_t \circ f^j)(x)$. Let $c(t)$ be defined as

$$c(t) := \limsup_{m\to\infty} \frac{a_m(t)}{m}.$$

As a consequence of the expansiveness of $f|_X$ one can prove that

$$c(t) \leq P_X(t) \quad \text{for all } t \geq 0; \tag{5.4}$$

see for instance [PU10, Lemmas 12.2.3 and 12.2.4]. Moreover, the function $t \mapsto c(t)$ is continuous. Setting

$$t^\star := \inf\{t \geq 0 : c(t) \leq 0\},$$

it follows from (5.4) that $t^\star \leq p_X^+(f) < \infty$.

By [PU10, Lemma 12.1.2], there exists a sequence $\{b_m\}_{m\geq 1}$ of positive real numbers such that the quantity

$$M_s := \sum_{m=1}^{\infty} b_m e^{a_m - ms}$$

satisfies $M_s < \infty$ for $s > t^\star$ and $M_s = +\infty$ for $s \leq t^\star$. For $s > t^\star$ consider the measure ν_s defined as

$$\nu_s := \frac{1}{M_s} \sum_{m=1}^{\infty} \sum_{x \in E_m} b_m e^{a_m - ms} \delta_x.$$

One can check that any weak limit of the measures ν_s as $s \searrow t^\star$ is $t^\star$-volume-conformal; see for instance [PU10, Section 12.1]. The assertion follows. □

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