

JOINING RIGIDITY FOR RATIONAL MAPS

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ABSTRACT. We initiate a joining rigidity theory for rational maps on the Riemann sphere $\mathbb{P}^1 = \mathbb{P}^1(\mathbb{C})$. Let $f_1, f_2: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be rational maps of degree at least 2, and μ_1, μ_2 their respective measures of maximal entropy, whose supports are the Julia sets $J(f_1)$ and $J(f_2)$. We study ergodic joinings of the systems $(J(f_1), f_1, \mu_1)$ and $(J(f_2), f_2, \mu_2)$, namely ergodic probability measures on $J(f_1) \times J(f_2)$ which are invariant under $f_1 \times f_2$ and whose marginals are μ_1 and μ_2 .

Our main theorem shows that a positive-mass local holomorphic relation forces algebraic rigidity. More precisely, if the joining charges the graph of a local biholomorphism, then that local relation globalizes to an invariant algebraic curve and yields either a finite cycle of rational graph or transpose-graph relations, or a genuinely multi-valued invariant algebraic correspondence. If no local biholomorphic graph has positive joining measure, then the joining generates a compact non-discrete family of local holomorphic relations.

The proof introduces normalized inverse branch transfer maps and studies their cluster limits. Starting from a local biholomorphic graph of positive joining measure, recurrence and contraction of inverse branches produce recurrent local intertwining relations. These are promoted to an algebraic relation by a local-to-global rigidity argument in the non-Lattès case and by affine uniformization in the Lattès case. In the absence of any positive-mass local biholomorphic graph, the cluster family must be infinite, and its non-discrete closure gives the second alternative.

1. INTRODUCTION

1.1. Joinings in dynamical systems. The study of joinings in dynamical systems began with Furstenberg's foundational work [Fur67]. A joining of two measure-preserving systems is an invariant measure on the product space whose coordinate marginals are the original invariant measures. While the product measure is always a joining, non-product joinings often reveal an underlying geometric or algebraic relation between the two systems. Furstenberg introduced this language to formalize the idea that two systems are dynamically unrelated when their only joining is the product joining, a property he called *disjointness*. Since then, joinings have become one of the central tools in measurable dynamics. They provide a way to detect common factors, rigidity and multiple recurrence phenomena; see, for example, the survey of de la Rue [Rue06] and Glasner's monograph [Gla03]. Furstenberg's joining theory forms part of a broader measure-rigidity perspective, also exemplified by the $\times 2, \times 3$ conjecture [Fur67], in which multiple dynamical invariances are expected to force algebraic structure. A *self-joining* is a joining of a system with itself and provides a natural measure-theoretic means of detecting its internal symmetries. For example, a measure-preserving map commuting with the dynamics determines a self-joining supported on its graph, while more general self-joinings may be supported on invariant correspondences.

A central theme in joining theory is *rigidity*. In many natural systems, every ergodic joining is either the product joining or is forced by an explicit algebraic or geometric relation. Ratner's joining theorem for horocycle flows is a fundamental example. In her work on horocycle flows and rigidity of products [Rat83], Ratner proved that non-trivial joinings of finite-volume horocycle flows arise from algebraic correspondences, providing a foundational model for joining rigidity in homogeneous dynamics. A higher-rank counterpart was developed by Einsiedler and Lindenstrauss. They classified joinings for broad classes of higher-rank diagonalizable actions on locally homogeneous spaces

[EL07] and later proved that joinings of higher-rank torus actions on S -arithmetic homogeneous spaces are algebraic [EL19].

Joining rigidity has also been extended to infinite-volume rank-one spaces. Mohammadi and Oh [MO16] classified locally finite joinings of horospherical subgroup actions on $\Gamma \backslash G$, where $G = \mathrm{PSL}_2(\mathbb{R})$ or $\mathrm{PSL}_2(\mathbb{C})$ and Γ is geometrically finite and Zariski dense, extending Ratner's theorem from lattices to infinite-volume hyperbolic geometry. Pan [Pan18] obtained a joining classification for horocycle flows on certain abelian covers, while related work of Oh–Pan [OP19] established local mixing and classified horospherically invariant measures on abelian covers.

The goal of this paper is to initiate a joining rigidity theory in holomorphic dynamics.

1.2. Statement of results. For $i = 1, 2$, let $f_i: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be a rational map of degree at least 2, $J(f_i)$ its Julia set and μ_i its measure of maximal entropy [Lyu83; FLM83]. Thus $J(f_i) = \mathrm{supp}(\mu_i)$. Set $F := f_1 \times f_2$, and denote by π_i , $i = 1, 2$, the two coordinate projections. A *joining* of f_1 and f_2 is an F -invariant probability measure ν on $J(f_1) \times J(f_2)$ such that $(\pi_1)_*\nu = \mu_1$ and $(\pi_2)_*\nu = \mu_2$. When $f_1 = f_2 = f$, we call such a measure a *self-joining* of f .

Our main result is a structure theorem for every non-product ergodic joining. We call the pair (f_1, f_2) *joining-exceptional* if there exist an integer $n \geq 1$ and an irreducible algebraic curve $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ such that $F^n(C) = C$ and both projections $\pi_1|_C$ and $\pi_2|_C$ are dominant and have generic degree strictly greater than one.

Theorem 1.1. *Let $f_1, f_2: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be rational maps of degree at least 2, and let μ_1, μ_2 be their measures of maximal entropy. Let $\nu \neq \mu_1 \times \mu_2$ be an ergodic joining of f_1 and f_2 . Then exactly one of the following two cases occurs.*

- (A) *The joining ν charges the graph of a local biholomorphism, i.e., there exist non-empty open sets $U, V \subset \mathbb{P}^1$ and a biholomorphism $\varphi: U \rightarrow V$ such that $\nu(\Gamma_\varphi) > 0$, where Γ_φ is the graph of φ . In this case exactly one of the following algebraic alternatives holds.*
 - (A1) *There exist an integer $r \geq 1$ and probability measures $\nu_0, \dots, \nu_{r-1}$ such that $\nu = r^{-1} \sum_{i=0}^{r-1} \nu_i$ and $F_*\nu_i = \nu_{i+1 \bmod r}$, where, for every i , either $\nu_i = (\mathrm{id}, A_i)_*\mu_1$ for some rational map A_i satisfying $f_2^r \circ A_i = A_i \circ f_1^r$, or $\nu_i = (B_i, \mathrm{id})_*\mu_2$ for some rational map B_i satisfying $f_1^r \circ B_i = B_i \circ f_2^r$.*
 - (A2) *The pair (f_1, f_2) is joining-exceptional, and the invariant curve C in the definition can be chosen so that $\nu(C) > 0$.*
- (B) *The joining ν charges no graph of a local biholomorphism. In this case ν gives rise, in the sense made precise in Theorem 4.1, to a compact non-discrete family $\mathcal{H} \subset \mathrm{Hol}(U, V)$ in the compact-open topology, for suitable non-empty open sets $U, V \subset \mathbb{P}^1$, such that every $h \in \mathcal{H}$ is non-constant and is locally biholomorphic, possibly after restriction to a smaller non-empty open subset.*

As an immediate consequence, if $\deg f_1 \neq \deg f_2$, the algebraic alternatives (A1) and (A2) are impossible. Hence every non-product ergodic joining charges no local biholomorphic graph and falls into case (B); see Corollary 5.1. We also note that the trivial product joining gives rise to the same non-discrete cluster phenomenon, although in that case it is automatic from the transfer-map construction; see Remark 4.2.

There is an important difference between our setting and the classical homogeneous examples discussed above. For a rational map of degree d , the maximal-entropy system is measurably isomorphic to the one-sided uniform d -Bernoulli shift [HH02; Mañ85]. Hence it admits many ergodic self-joinings having no evident complex-analytic origin, and an algebraic classification of all measurable joinings cannot hold. Theorem 1.1 instead shows that once a joining charges a local biholomorphic graph, the resulting local relation is forced to globalize algebraically. This retains the guiding principle of joining rigidity in homogeneous dynamics while accounting for the much larger measurable joining

space present in rational dynamics; see [Rat83; EL07; EL19; MO16; OP19; Pan18]. Concrete examples of case (B), with either diffuse or finite atomic conditional measures, are given in Section 6.

1.3. Diagonal dynamics and self-joinings. In the diagonal case $f_1 = f_2 = f$, Theorem 1.1 gives a structure theorem for non-product ergodic self-joinings of a single rational map. While case (B) arises from the same purely measure-theoretic phenomena as in the general setting, case (A) admits a more explicit description in the diagonal case. In alternative (A1), the rational maps appearing in the graph and transpose-graph components commute with an iterate of f . The curves in (A2) belong to the class of non-vertical and non-horizontal invariant curves studied by Pakovich [Pak23]. Indeed, setting $A := f^n$, the curve C is (A, A) -invariant and, since both projections are dominant, it is neither vertical nor horizontal. Pakovich’s classification gives a more explicit description of such curves when A is not a generalized Lattès map. More precisely, by [Pak23, Theorem 1.2], there exist rational maps $U_1, U_2, V_1, V_2: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ commuting with A and an integer $d \geq 0$ such that

$$U_1 \circ V_1 = U_2 \circ V_2 = A^{\circ d}, \quad V_1 \circ U_1 = V_2 \circ U_2 = A^{\circ d},$$

and the map $t \mapsto (U_1(t), U_2(t))$ parametrizes C . Thus, outside the generalized Lattès case in Pakovich’s sense, these diagonal joining-exceptional curves are described in terms of rational maps commuting with an iterate of f .

The condition (A2) is designed to capture the same phenomenon as the classical exceptional rational maps, namely the monomials, Chebyshev maps and Lattès maps, which have large symmetry or semiconjugacy structures and admit algebraic correspondences invariant under the diagonal dynamics. In particular, the classical exceptional maps fall into this framework; see Proposition 8.1.

1.4. Joining rigidity as a unifying framework in complex dynamics. The joining viewpoint provides a natural framework for several rigidity phenomena that have previously been studied separately in complex dynamics: local and global symmetries of Julia sets, semiconjugate and commuting rational maps, and the promotion of measure-preserving local relations to algebraic correspondences.

Classical work of Baker–Eremenko [BE87], Beardon [Bea92], Eremenko [Ere90], Levin [Lev90], Levin–Przytycki [LP97], and Dinh [Din00] shows that rational maps with the same Julia set, as well as local and global symmetries of Julia sets, are strongly constrained.

Semiconjugacies provide a natural source of relations between two dynamical systems; see, for example, [Pak16]. The relations in alternative (A1) are of this type. In the diagonal case, Pakovich’s classification, discussed in Section 1.3, describes the invariant curves in alternative (A2) in terms of rational maps commuting with an iterate. For general pairs of a fixed degree, [Pak25b] shows that a general pair admits a non-vertical, non-horizontal periodic curve if and only if the two maps are Möbius-conjugate.

Commuting rational maps arise as the diagonal specialization of the semiconjugacy picture. Their structure, and in particular their connection with the classical exceptional families, goes back to Ritt [Rit23] and was further developed by Eremenko and Pakovich; see [Ere90; Pak21]. Recent work of Beaumont [Bea25] on centralizers of iterates is closely related to alternative (A1). For related results on commuting endomorphisms in higher dimension, we refer to [Din01; DS02; Kau18].

More recently, Dujardin–Favre–Gauthier [DFG23] and Ji–Xie [JX23b] proved local-to-global rigidity results in which suitable local symmetries preserving Julia sets or measures of maximal entropy are promoted to algebraic correspondences. Related local-to-global rigidity phenomena also appear in [JX23a], where periodic and homoclinic data are used to obtain global rigidity statements for rational maps, including a new proof of McMullen’s multiplier-rigidity theorem [McM87].

Related measure and Julia set rigidity results occur throughout complex dynamics. For rational maps on $\mathbb{P}^1$, see [LP97; Ye15; Pak25a]. Related questions for regular polynomial endomorphisms of

$\mathbb{C}^k$ are considered in [Uen10; Zha26], while we refer to [Lam01; DF17; Vig26] and references therein for the case of automorphisms. Measure-theoretic rigidity phenomena have also been studied in the setting of random dynamics of projective surfaces [CD23].

Although the algebraic part of Theorem 1.1 belongs to this broader circle of rigidity phenomena, its starting point is different. We assume neither an equality of maximal-entropy measures nor a local analytic symmetry, semiconjugacy, or algebraic correspondence. Instead, our starting point is a *measure-theoretic* coupling of the two maximal-entropy systems, with no analytic regularity assumed. Theorem 1.1 shows that once such a coupling charges a local biholomorphic graph, this local relation globalizes to an invariant algebraic curve. Related questions and possible extensions are discussed in Section 9.

1.5. Strategy of the proof. The proof of Theorem 1.1 consists of two main parts: a procedure for extracting local holomorphic structure from an ergodic joining and a local-to-global argument applied to the resulting cluster family. When a local biholomorphic graph carries positive joining measure, this argument promotes the local graph to an invariant algebraic curve, leading to alternative (A1) or (A2) in case (A); otherwise the cluster construction itself yields the non-discrete family in case (B).

For the extraction step, we construct transfer maps by comparing inverse branches along typical pairs of inverse orbits. After normalization, the transfer maps have derivative equal to 1 at the base point, and Koebe distortion gives uniform control on fixed reference domains. Hence, after passing to fixed reference charts, the transfer maps form a normal family. For each recurrent point of the lifted joining, we consider the cluster limits of the corresponding normalized transfer maps and let $\mathcal{C}_\nu$ be the union of these cluster sets. A cluster germ is called *admissible* if it occurs over a set of positive lifted joining measure. Such a germ gives a local biholomorphic graph of positive ν -measure.

Positive-mass local relation. Suppose that ν charges the graph of a local biholomorphism φ . Recurrence and contraction of inverse branches produce an iterate $N \geq 1$, a coordinate disc D , and two distinct inverse branches $\alpha_1, \alpha_2: D \rightarrow D$ of f_1^N , with relatively compact images, such that $f_2^N \circ \varphi \circ \alpha_j = \varphi$ on D . Thus f_1^N is non-injective on the union of the two return domains; one of the contracting return branches also yields a repelling fixed point of f_1^N .

When neither map is Lattès, the generalized form of Inou's theorem [Ino11] proved by Ji–Xie [JX23b, Theorem 2.1] promotes this local intertwining relation to an invariant irreducible algebraic curve. If one map is Lattès, we show that the charged local graph first forces the other map to be Lattès as well. We then use affine uniformization and the abundance of recurrent inverse branches to show that the slope of the lifted affine relation maps a finite-index sublattice of the first translation lattice into the second. This again produces an invariant algebraic curve. In either case, the projection degrees give (A1) or (A2). Since μ_1 and μ_2 are non-atomic, so is ν ; as distinct irreducible curves have finite intersection, alternatives (A1) and (A2) are then mutually exclusive.

No positive-mass local relation. Suppose instead that ν charges no graph of a local biholomorphism. Then no cluster germ can be admissible. Since a finite cluster family necessarily contains an admissible germ, $\mathcal{C}_\nu$ must be infinite. Its closure in the ambient normal family is therefore compact and non-discrete. The uniform derivative lower bound from the normalization shows that every element of this closure is non-constant and locally biholomorphic after restriction to a smaller non-empty open set. Passing back through the fixed reference charts gives the family in case (B).

Beyond this dichotomy, we show that recurrent returns preserve these fiberwise cluster sets up to natural left and right coordinate changes; see Section 7.

1.6. Organization of the paper. Section 2 develops the normalized inverse branch construction and the associated cluster families. Sections 3 and 4 establish the two cases of Theorem 1.1 and

Section 5 completes its proof. Section 6 gives examples illustrating the necessity of case (B) for arbitrary measure-theoretic joinings. Section 7 studies the finer fiberwise cluster structure, and Section 8 shows that the classical exceptional rational maps are joining-exceptional. Finally, Section 9 discusses several directions for further study.

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2. LOCAL TRANSFER MAPS AND CLUSTER GERMS

This section carries out the first step in the proof of Theorem 1.1: extracting local holomorphic structure from an ergodic joining. Section 2.1 summarizes inverse-branch estimates along typical inverse orbits. In Section 2.2, we restrict to a positive-measure set on which the inverse branches in both coordinates are defined on a common disc and construct normalized inverse branch transfer maps. Section 2.3 passes to fixed reference charts and gives a uniform derivative lower bound for cluster limits. In Section 2.4, we define the total cluster family $\mathcal{C}_\nu$ and the admissible germs $\mathcal{A}_\nu$, and realize every admissible germ as a local graph charged by the joining.

Throughout this section, we let $f_i: \mathbb{P}^1 \rightarrow \mathbb{P}^1$, $i = 1, 2$, be rational maps of degree at least 2, μ_i their measures of maximal entropy [Lyu83; FLM83], $J(f_i) = \text{supp } \mu_i$ their Julia sets, and ν an ergodic joining of f_1 and f_2 . For $i = 1, 2$, we let $(\hat{X}_i, \hat{f}_i)$ be the natural extension of $(J(f_i), f_i)$ and $\hat{\mu}_i$ the lift of μ_i to $\hat{X}_i$. We denote by $\pi: \hat{X}_i \rightarrow J(f_i)$ the projection $\pi(\hat{x}) = x_0$. We will use without further comment that each μ_i is non-atomic and mixing; in particular, μ_i is ergodic for f_i^r for every $r \geq 1$. We will also use the standard fact that the natural extension of an ergodic invariant measure is ergodic [CFS12].

Set $F := f_1 \times f_2$ and $\hat{F} := \hat{f}_1 \times \hat{f}_2$. If ν is an F -invariant measure on $J(f_1) \times J(f_2)$, we denote by $\hat{\nu}$ its lift to $\hat{X}_1 \times \hat{X}_2$. Abusing notation, we also write $\pi: \hat{X}_1 \times \hat{X}_2 \rightarrow J(f_1) \times J(f_2)$ for the natural projection. We denote by π_i , $i = 1, 2$, the coordinate projections on $\mathbb{P}^1 \times \mathbb{P}^1$.

2.1. Inverse branches and estimates along one orbit. We begin by fixing centered conformal charts with uniform local control. Choose a finite open cover $\{W_\alpha\}_{\alpha=1}^N$ of $\mathbb{P}^1$ such that, for each α , there is a smooth family of centered conformal charts

$$\psi_{\alpha,p}: B_{\text{sph}}(p, r_0) \rightarrow \mathbb{C}, \quad \psi_{\alpha,p}(p) = 0, \quad p \in W_\alpha,$$

where $r_0 > 0$ is independent of α and p .

Choose open sets $W'_\alpha \Subset W_\alpha$ such that $\{W'_\alpha\}_{\alpha=1}^N$ still covers $\mathbb{P}^1$. Choose a measurable partition $\mathbb{P}^1 = P_1 \sqcup \cdots \sqcup P_N$ such that $P_\alpha \subset W'_\alpha$. For $p \in P_\alpha$, set $\psi_p := \psi_{\alpha,p}$. Since $P_\alpha \subset W'_\alpha \Subset W_\alpha$ and there are only finitely many chart families, after decreasing r_0 if necessary the chosen charts ψ_p and their inverses have uniform distortion bounds on slightly smaller spherical balls, independently of p . Moreover, on P_α the chosen charts belong to the fixed smooth family $p \mapsto \psi_{\alpha,p}$.

Fix $i \in \{1, 2\}$, $\hat{z} = (z_n)_{n \in \mathbb{Z}} \in \hat{X}_i$ and $n \geq 1$. On a suitable neighbourhood $U_{i,\hat{z},n}$ of z_0 , denote by $g_{i,\hat{z},n}$ the inverse branch of f_i^n along $\hat{z}$, characterized by $g_{i,\hat{z},n}(z_0) = z_{-n}$ and $f_i^n \circ g_{i,\hat{z},n} = \text{id}$. Whenever $g_{i,\hat{z},n}$ is defined on a spherical ball $B_{\text{sph}}(z_0, \rho)$, set

$$(2.1) \quad G_{i,\hat{z},n} := \psi_{z_{-n}} \circ g_{i,\hat{z},n} \circ \psi_{z_0}^{-1}.$$

The next proposition summarizes the inverse branch estimates that we will need.

Proposition 2.1. *For every $\varepsilon > 0$ and $i \in \{1, 2\}$, there exist a measurable $\hat{f}_i$ -invariant set $\hat{X}_{i,\varepsilon} \subset \hat{X}_i$ with $\hat{\mu}_i(\hat{X}_{i,\varepsilon}) = 1$ and a measurable function $\rho_{i,\varepsilon}: \hat{X}_{i,\varepsilon} \rightarrow (0, +\infty)$ such that the following hold for every $\hat{z} \in \hat{X}_{i,\varepsilon}$ and every $n \geq 1$.*

- (1) *The inverse branch $g_{i,\hat{z},n}$ is defined on $B_{\text{sph}}(z_0, \rho_{i,\varepsilon}(\hat{z}))$.*
- (2) *The map $G_{i,\hat{z},n}$ defined in (2.1) is holomorphic and univalent on $D(0, \rho_{i,\varepsilon}(\hat{z}))$.*
- (3) *For every $0 < \lambda < 1$, there exists a constant $\kappa(\lambda) \geq 1$, independent of i , such that, for every $u, v \in D(0, \lambda \rho_{i,\varepsilon}(\hat{z}))$, we have*

$$\kappa(\lambda)^{-1} \leq \left| \frac{G'_{i,\hat{z},n}(u)}{G'_{i,\hat{z},n}(v)} \right| \leq \kappa(\lambda),$$

and

$$\kappa(\lambda)^{-1} |G'_{i,\hat{z},n}(0)| \leq \frac{\text{diam } G_{i,\hat{z},n}(D(0, \lambda \rho_{i,\varepsilon}(\hat{z})))}{\lambda \rho_{i,\varepsilon}(\hat{z})} \leq \kappa(\lambda) |G'_{i,\hat{z},n}(0)|.$$

- (4) *For every $\hat{z} \in \hat{X}_{i,\varepsilon}$, we have $|G'_{i,\hat{z},n}(0)| \rightarrow 0$ as $n \rightarrow \infty$. Consequently, for every $0 < \lambda < 1$, $\text{diam } G_{i,\hat{z},n}(D(0, \lambda \rho_{i,\varepsilon}(\hat{z}))) \rightarrow 0$.*

Proof. By the standard inverse branch construction for rational maps associated to typical inverse orbits for invariant ergodic measures with positive Lyapunov exponent, for almost every $\hat{z}$ there is a positive spherical radius, depending measurably on $\hat{z}$, on which all the inverse branches $g_{i,\hat{z},n}$ are defined; see, for example, [PU10, Chapter 11] and [Ber10]. Since the centered charts were chosen from finitely many smooth local families with uniform distortion bounds, after decreasing these radii by a uniform factor we may arrange simultaneously that $g_{i,\hat{z},n}$ is defined on $B_{\text{sph}}(z_0, \rho_{i,\varepsilon}(\hat{z}))$ and that $G_{i,\hat{z},n}$ is defined and univalent on $D(0, \rho_{i,\varepsilon}(\hat{z}))$. This proves (1) and (2).

Fix $0 < \lambda < 1$. For $\hat{z} \in \hat{X}_{i,\varepsilon}$ and $n \geq 1$, consider

$$H_{i,\hat{z},n}(w) := \frac{1}{\rho_{i,\varepsilon}(\hat{z})} G_{i,\hat{z},n}(\rho_{i,\varepsilon}(\hat{z})w), \quad |w| < 1.$$

Since $G_{i,\hat{z},n}$ is univalent on $D(0, \rho_{i,\varepsilon}(\hat{z}))$, the map $H_{i,\hat{z},n}$ is univalent on the unit disc. The Koebe distortion theorem on $D(0, \lambda)$ gives the first estimate in (3), with a constant $\kappa(\lambda) \geq 1$ depending only on λ . Applying Koebe distortion and growth estimates, and enlarging $\kappa(\lambda)$ if necessary, gives the diameter estimate in (3).

Finally, intersect $\hat{X}_{i,\varepsilon}$ with the full-measure set of inverse orbits for which the Lyapunov exponent is realized, and then with all its iterates under $\hat{f}_i$. After this replacement, $\hat{X}_{i,\varepsilon}$ is $\hat{f}_i$ -invariant and, for every $\hat{z} \in \hat{X}_{i,\varepsilon}$, the derivative of the inverse branch $g_{i,\hat{z},n}$ at z_0 tends to zero as $n \rightarrow \infty$. Since the derivatives of the centered charts and their inverses are uniformly bounded, it follows that $|G'_{i,\hat{z},n}(0)| \rightarrow 0$. The diameter convergence in (4) then follows from the second estimate in (3). $\square$

2.2. Uniform domains and normalized transfer maps. The inverse-branch domains in Proposition 2.1 depend on the individual inverse orbits in the two coordinates. We now restrict to a positive-measure set on which the inverse branches in both coordinates are simultaneously defined on a common disc of uniform size.

Lemma 2.2. *Fix $\varepsilon > 0$. There exist a measurable set $E \subset \hat{X}_{1,\varepsilon} \times \hat{X}_{2,\varepsilon}$ with $\hat{\nu}(E) > 0$ and a constant $\rho_* > 0$ such that, for every $z = (\hat{x}, \hat{y}) \in E$ and every $n \geq 1$, the maps $G_{1,\hat{x},n}$ and $G_{2,\hat{y},n}$ are defined and univalent on $D(0, \rho_*)$. Moreover, if $\nu \neq \mu_1 \times \mu_2$, then E can be chosen so that*

$$(2.2) \quad \hat{\nu}(E) > (\hat{\mu}_1 \times \hat{\mu}_2)(E) > 0.$$

Proof. For $M \geq 1$, set

$$F_M := \left\{ (\hat{x}, \hat{y}) \in \hat{X}_{1,\varepsilon} \times \hat{X}_{2,\varepsilon} : \rho_{1,\varepsilon}(\hat{x}) \geq \frac{1}{M}, \quad \rho_{2,\varepsilon}(\hat{y}) \geq \frac{1}{M} \right\}.$$

The sets F_M increase to $\hat{X}_{1,\varepsilon} \times \hat{X}_{2,\varepsilon}$, which has full $\hat{\nu}$ -measure since the marginals of $\hat{\nu}$ are $\hat{\mu}_1$ and $\hat{\mu}_2$. Hence $\hat{\nu}(F_M) > 0$ for all sufficiently large M . Fix such an M and set $E := F_M$ and $\rho_* := 1/M$. The first assertion follows from Proposition 2.1.

Assume now that $\nu \neq \mu_1 \times \mu_2$. Since measurable rectangles determine a probability measure on the product, there exist measurable sets $A \subset J(f_1)$ and $B \subset J(f_2)$ such that $\nu(A \times B) > \mu_1(A)\mu_2(B)$. Disintegrating ν over the first coordinate, write $\nu = \int \nu_x d\mu_1(x)$. Then

$$\int_A \nu_x(B) d\mu_1(x) > \mu_1(A)\mu_2(B).$$

Hence there exist $\varepsilon' > 0$ and a measurable set $A' \subset A$ with $\mu_1(A') > 0$ such that $\nu_x(B) \geq \mu_2(B) + \varepsilon'$ for every $x \in A'$. Set

$$E_0 := (\pi^{-1}(A') \cap \hat{X}_{1,\varepsilon}) \times (\pi^{-1}(B) \cap \hat{X}_{2,\varepsilon}).$$

Since $\hat{\mu}_i(\hat{X}_{i,\varepsilon}) = 1$ and the marginals of $\hat{\nu}$ are $\hat{\mu}_1$ and $\hat{\mu}_2$, we have

$$\begin{aligned} \hat{\nu}(E_0) &= \nu(A' \times B) = \int_{A'} \nu_x(B) d\mu_1(x) \\ &\geq \mu_1(A')(\mu_2(B) + \varepsilon') > \mu_1(A')\mu_2(B) = (\hat{\mu}_1 \times \hat{\mu}_2)(E_0) > 0. \end{aligned}$$

Since $E_0 \cap F_M$ increases to E_0 , continuity from below gives

$$\hat{\nu}(E_0 \cap F_M) \rightarrow \hat{\nu}(E_0) \quad \text{and} \quad (\hat{\mu}_1 \times \hat{\mu}_2)(E_0 \cap F_M) \rightarrow (\hat{\mu}_1 \times \hat{\mu}_2)(E_0).$$

For M sufficiently large, this gives $\hat{\nu}(E_0 \cap F_M) > (\hat{\mu}_1 \times \hat{\mu}_2)(E_0 \cap F_M) > 0$. Setting $E := E_0 \cap F_M$ and $\rho_* := 1/M$ gives (2.2). $\square$

From now on, we fix a set $E \subset \hat{X}_{1,\varepsilon} \times \hat{X}_{2,\varepsilon}$ and a constant $\rho_* > 0$ as in Lemma 2.2. When $\nu \neq \mu_1 \times \mu_2$, we choose E so that (2.2) holds. For $z = (\hat{x}, \hat{y}) \in E$ and $n \geq 1$, define

$$a_n(z) := \frac{G'_{2,\hat{y},n}(0)}{G'_{1,\hat{x},n}(0)} \in \mathbb{C}^*$$

and the corresponding *normalized transfer map*

$$\Phi_{z,n} := G_{2,\hat{y},n}^{-1} \circ (a_n(z) \cdot \text{id}) \circ G_{1,\hat{x},n}.$$

Proposition 2.3. *There exist radii $0 < r < R < \rho_*$ such that, for every $z = (\hat{x}, \hat{y}) \in E$ and every $n \geq 1$, the map $\Phi_{z,n}: D(0, r) \rightarrow D(0, R)$ is well-defined and holomorphic, and satisfies $\Phi'_{z,n}(0) = 1$.*

Proof. By Lemma 2.2, the maps $G_{1,\hat{x},n}$ and $G_{2,\hat{y},n}$ are univalent on $D(0, \rho_*)$. Fix $0 < \lambda < 1$. By Koebe distortion and the Koebe quarter theorem, there exist constants $c_1, c_2 > 0$, depending only on λ , such that $|G_{1,\hat{x},n}(u)| \leq c_2 |G'_{1,\hat{x},n}(0)| |u|$ for $|u| < \lambda \rho_*$, while $G_{2,\hat{y},n}(D(0, \lambda \rho_*))$ contains $D(0, c_1 |G'_{2,\hat{y},n}(0)| \lambda \rho_*)$. Choose $0 < r < \lambda \rho_*$ so that $c_2 r < c_1 \lambda \rho_*$. Since $|a_n(z)| |G'_{1,\hat{x},n}(0)| = |G'_{2,\hat{y},n}(0)|$, for $|u| < r$, we have

$$|a_n(z) G_{1,\hat{x},n}(u)| \leq c_2 |G'_{2,\hat{y},n}(0)| |u| < c_1 |G'_{2,\hat{y},n}(0)| \lambda \rho_*.$$

It follows that $a_n(z) G_{1,\hat{x},n}(D(0, r)) \subset G_{2,\hat{y},n}(D(0, \lambda \rho_*))$. Thus, setting $R := \lambda \rho_*$, $\Phi_{z,n}$ is well-defined on $D(0, r)$ and takes values in $D(0, R)$. Finally, we have $\Phi'_{z,n}(0) = (G_{2,\hat{y},n}^{-1})'(0) a_n(z) G'_{1,\hat{x},n}(0) = 1$. $\square$

Corollary 2.4. *Let r and R be as in Proposition 2.3. For every $z \in E$, the family $\{\Phi_{z,n}\}_{n \geq 1}$ is normal on $D(0, r)$. In particular, every sequence in this family admits a subsequence converging locally uniformly on $D(0, r)$ to a holomorphic map $\Psi: D(0, r) \rightarrow D(0, R)$. Every such limit map is non-constant.*

Proof. The maps $\Phi_{z,n}$ take values in the bounded disc $D(0, R)$, so Montel's theorem gives normality. If $\Phi_{z,n_j} \rightarrow \Psi$ locally uniformly, then $\Phi'_{z,n_j}(0) \rightarrow \Psi'(0)$. By Proposition 2.3, $\Phi'_{z,n_j}(0) = 1$, and hence $\Psi'(0) = 1$. Thus Ψ is non-constant. Since Ψ is a locally uniform limit of maps with values in $D(0, R)$, the maximum principle gives $\Psi(D(0, r)) \subset D(0, R)$. $\square$

2.3. Fixed reference charts. The transfer maps of Proposition 2.3 are written in moving centered charts attached to (x_0, y_0) . We now conjugate them to a fixed pair of reference charts on a positive-measure subset of E .

Proposition 2.5. *There exist points $a \in J(f_1)$ and $b \in J(f_2)$, centered conformal reference charts χ_a and χ_b , spherical neighbourhoods $U \ni a$ and $V \ni b$, a measurable set $E^\sharp \subset E$ with $\hat{\nu}(E^\sharp) > 0$, radii $0 < r^\sharp < r$ and $R^\sharp > 0$, compact families $\mathcal{H} \subset \text{Hol}(D(0, r^\sharp), \mathbb{C})$ and $\mathcal{G} \subset \text{Hol}(D(0, R), \mathbb{C})$, and a constant $c_* > 0$ such that the following hold.*

- (1) *We have $\overline{\chi_a(U)} \Subset D(0, r^\sharp)$ and $\overline{D(0, R^\sharp)} \Subset \chi_b(V)$. For every $z = (\hat{x}, \hat{y}) \in E^\sharp$, we have $x_0 \in U$ and $y_0 \in V$. The map $h_{\hat{x}} := \psi_{x_0} \circ \chi_a^{-1}$ is well-defined on $D(0, r^\sharp)$ and belongs to $\mathcal{H}$, while the map $k_{\hat{y}} := \chi_b \circ \psi_{y_0}^{-1}$ is well-defined on a fixed neighbourhood of $\overline{D(0, R)}$ and its restriction to $D(0, R)$ belongs to $\mathcal{G}$.*
- (2) *For every $z \in E^\sharp$ and every $n \geq 1$, the conjugated transfer map*

$$\tilde{\Phi}_{z,n} := k_{\hat{y}} \circ \Phi_{z,n} \circ h_{\hat{x}}$$

is well-defined and holomorphic on $D(0, r^\sharp)$, with values in $D(0, R^\sharp)$.

- (3) *For every $z \in E^\sharp$, the family $\{\tilde{\Phi}_{z,n}\}_{n \geq 1}$ is normal on $D(0, r^\sharp)$ and every cluster limit $\tilde{\Psi}: D(0, r^\sharp) \rightarrow D(0, R^\sharp)$ is non-constant.*
- (4) *For every $z = (\hat{x}, \hat{y}) \in E^\sharp$, setting $q_z := \chi_a(x_0)$, we have $|k'_{\hat{y}}(0) h'_{\hat{x}}(q_z)| \geq c_*$. Consequently, if $\tilde{\Psi}$ is any cluster limit of $\{\tilde{\Phi}_{z,n}\}_{n \geq 1}$, then $|\tilde{\Psi}'(q_z)| \geq c_*$.*

Moreover, if $\hat{\nu}(E) > (\hat{\mu}_1 \times \hat{\mu}_2)(E)$, then $E^\sharp$ can be chosen so that

$$(2.3) \quad \hat{\nu}(E^\sharp) > (\hat{\mu}_1 \times \hat{\mu}_2)(E^\sharp).$$

Proof. Recall that the centered charts in Section 2.1 were chosen from finitely many smooth local families. More precisely, there are open sets $W_\alpha \subset \mathbb{P}^1$, smaller open sets W'_α with $\overline{W'_\alpha} \Subset W_\alpha$, and a measurable partition $\mathbb{P}^1 = P_1 \sqcup \dots \sqcup P_N$ such that $P_\alpha \subset W'_\alpha$ and $\psi_p = \psi_{\alpha,p}$ whenever $p \in P_\alpha$.

For $\alpha, \beta \in \{1, \dots, N\}$, set $Q_{\alpha,\beta} := (P_\alpha \times P_\beta) \cap (J(f_1) \times J(f_2))$. These sets form a finite measurable partition of $J(f_1) \times J(f_2)$. Hence

$$E = \bigsqcup_{\alpha,\beta} (E \cap \pi^{-1}(Q_{\alpha,\beta})).$$

Since $\hat{\nu}(E) > 0$, there is a pair (α, β) such that, setting $E_{\alpha,\beta} := E \cap \pi^{-1}(Q_{\alpha,\beta})$, we have $\hat{\nu}(E_{\alpha,\beta}) > 0$. If in addition $\hat{\nu}(E) > (\hat{\mu}_1 \times \hat{\mu}_2)(E)$, we choose (α, β) so that $\hat{\nu}(E_{\alpha,\beta}) > (\hat{\mu}_1 \times \hat{\mu}_2)(E_{\alpha,\beta})$; such a pair exists by the finite partition above. On the first coordinate of $E_{\alpha,\beta}$, all chosen centered charts belong to the single smooth family $p \mapsto \psi_{\alpha,p}$, while on the second coordinate they belong to the single smooth family $q \mapsto \psi_{\beta,q}$. Set

$$K_{\alpha,\beta} := (\overline{P_\alpha} \cap J(f_1)) \times (\overline{P_\beta} \cap J(f_2)).$$

Since $\overline{P_\alpha} \Subset W_\alpha$ and $\overline{P_\beta} \Subset W_\beta$, the set $K_{\alpha,\beta}$ is compactly contained in $W_\alpha \times W_\beta$.

By compactness of $K_{\alpha,\beta}$ and the smooth dependence of the two fixed chart families, we may choose finitely many points $(a_j, b_j) \in K_{\alpha,\beta}$ and spherical neighbourhoods $U_j \ni a_j$, $V'_j \ni b_j$ with $U_j \Subset W_\alpha$ and $V'_j \Subset V_j \Subset W_\beta$, such that the sets $U_j \times V'_j$ cover $K_{\alpha,\beta}$ and, setting $\chi_{a_j} := \psi_{\alpha,a_j}$ and $\chi_{b_j} := \psi_{\beta,b_j}$, the following properties hold.

There exist radii $0 < r_j^\sharp < r$ and $R_j^\sharp > 0$ such that, for every $p \in U_j$ and $q \in V'_j$, the transition map $\psi_{\alpha,p} \circ \chi_{a_j}^{-1}$ is defined on $D(0, r_j^\sharp)$, while $\chi_{b_j} \circ \psi_{\beta,q}^{-1}$ is defined on a fixed neighbourhood of $\overline{D(0, R)}$. Moreover, we have

$$\chi_{a_j}(U_j) \Subset D(0, r_j^\sharp), \quad D(0, R_j^\sharp) \Subset \chi_{b_j}(V_j),$$

and

$$(\psi_{\alpha,p} \circ \chi_{a_j}^{-1})(D(0, r_j^\sharp)) \subset D(0, r), \quad (\chi_{b_j} \circ \psi_{\beta,q}^{-1})(\overline{D(0, R)}) \subset D(0, R_j^\sharp)$$

for every $p \in U_j$ and $q \in V'_j$. The corresponding families of transition maps, restricted to the indicated fixed domains, are relatively compact in the compact-open topology.

Choose a measurable partition $Q_1, \dots, Q_s$ of $Q_{\alpha,\beta}$ so that $Q_j \subset U_j \times V'_j$. Since

$$E_{\alpha,\beta} = \bigsqcup_{j=1}^s (E_{\alpha,\beta} \cap \pi^{-1}(Q_j))$$

and $\hat{\nu}(E_{\alpha,\beta}) > 0$, there exists j_0 such that $\hat{\nu}(E_{\alpha,\beta} \cap \pi^{-1}(Q_{j_0})) > 0$. If $\hat{\nu}(E_{\alpha,\beta}) > (\hat{\mu}_1 \times \hat{\mu}_2)(E_{\alpha,\beta})$, we choose j_0 so that

$$\hat{\nu}(E_{\alpha,\beta} \cap \pi^{-1}(Q_{j_0})) > (\hat{\mu}_1 \times \hat{\mu}_2)(E_{\alpha,\beta} \cap \pi^{-1}(Q_{j_0})).$$

Set $a := a_{j_0}$, $b := b_{j_0}$, $U := U_{j_0}$, $V := V_{j_0}$, $r^\sharp := r_{j_0}^\sharp$, $R^\sharp := R_{j_0}^\sharp$, and $E^\sharp := E_{\alpha,\beta} \cap \pi^{-1}(Q_{j_0})$. Then $\hat{\nu}(E^\sharp) > 0$. If $\hat{\nu}(E) > (\hat{\mu}_1 \times \hat{\mu}_2)(E)$, the choices above also give (2.3). By construction, for every $z = (\hat{x}, \hat{y}) \in E^\sharp$, the map $h_{\hat{x}}$ is well-defined on $D(0, r^\sharp)$ and, varying $z \in E^\sharp$, the maps $h_{\hat{x}}$ form a relatively compact family there. Likewise, the maps $k_{\hat{y}}$ are well-defined on a fixed neighbourhood of $\overline{D(0, R)}$, and their restrictions to $D(0, R)$ form a relatively compact family. Let $\mathcal{H}$ and $\mathcal{G}$ be the respective compact closures. This proves (1).

By construction, we have $h_{\hat{x}}(D(0, r^\sharp)) \subset D(0, r)$ for every $z \in E^\sharp$. By the choice of $R^\sharp$, we have $k_{\hat{y}}(D(0, R)) \subset D(0, R^\sharp)$. Since Proposition 2.3 gives $\Phi_{z,n}(D(0, r)) \subset D(0, R)$, the map $\tilde{\Phi}_{z,n}$ is well-defined on $D(0, r^\sharp)$ and takes values in $D(0, R^\sharp)$. This proves (2).

The family $\{\tilde{\Phi}_{z,n}\}_{n \geq 1}$ takes values in the fixed bounded disc $D(0, R^\sharp)$, so Montel's theorem gives normality. Set $q_z := \chi_a(x_0)$. Since $h_{\hat{x}}(q_z) = 0$ and $\Phi'_{z,n}(0) = 1$, we have $\tilde{\Phi}'_{z,n}(q_z) = k'_{\hat{y}}(0) h'_{\hat{x}}(q_z)$. The right hand side is independent of n and is non-zero. Hence, if $\tilde{\Phi}_{z,n_j} \rightarrow \tilde{\Psi}$ locally uniformly, then

$$(2.4) \quad \tilde{\Psi}'(q_z) = k'_{\hat{y}}(0) h'_{\hat{x}}(q_z) \neq 0.$$

Thus every cluster limit is non-constant. Since $\tilde{\Psi}$ is a locally uniform limit of maps with values in $D(0, R^\sharp)$, the maximum principle also gives $\tilde{\Psi}(D(0, r^\sharp)) \subset D(0, R^\sharp)$. This proves (3).

To prove (4), it remains to obtain a lower bound independent of z . Since $\overline{U} \Subset W_\alpha$ and $\overline{V} \Subset W_\beta$, the function

$$(x, y) \mapsto |(\chi_b \circ \psi_{\beta,y}^{-1})'(0)(\psi_{\alpha,x} \circ \chi_a^{-1})'(\chi_a(x))|$$

is well-defined, continuous, and strictly positive on the compact set $\overline{U} \times \overline{V}$. It therefore has a positive minimum, say $c_* > 0$. For $z = (\hat{x}, \hat{y}) \in E^\sharp$, the above displayed function evaluated at (x_0, y_0) is exactly $|k'_{\hat{y}}(0) h'_{\hat{x}}(q_z)|$. Hence, we have $|k'_{\hat{y}}(0) h'_{\hat{x}}(q_z)| \geq c_*$. Combining this with (2.4) gives $|\tilde{\Psi}'(q_z)| \geq c_*$ for every cluster limit $\tilde{\Psi}$ associated to z . This proves (4) and completes the proof. $\square$

2.4. Cluster germs for the joining. We fix from now on the reference data $E^\sharp, a, b, \chi_a, \chi_b, U, V, r^\sharp$ and $R^\sharp$ obtained in Proposition 2.5. When $\nu \neq \mu_1 \times \mu_2$, we make this choice so that $\hat{\nu}(E^\sharp) > (\hat{\mu}_1 \times \hat{\mu}_2)(E^\sharp)$. The cluster families constructed below depend on these auxiliary reference data, but for simplicity we will suppress this dependence from the notation.

By Proposition 2.5, the maps $\tilde{\Phi}_{z,n}$ are defined on the fixed disc $D(0, r^\sharp)$ and take values in $D(0, R^\sharp)$. Set

$$\mathcal{K} := \overline{\{\tilde{\Phi}_{z,n} : z \in E^\sharp, n \geq 1\}} \subset \text{Hol}(D(0, r^\sharp), \mathbb{C}),$$

where the closure is taken in the compact-open topology. By Montel's theorem, $\mathcal{K}$ is compact, and it is metrizable in the compact-open topology. For each $z \in E^\sharp$, define the *cluster set*

$$C(z) := \left\{ \tilde{\Psi} \in \mathcal{K} : \tilde{\Phi}_{z,n_j} \rightarrow \tilde{\Psi} \text{ locally uniformly for some } n_j \rightarrow \infty \right\}.$$

The set $C(z)$ is non-empty and compact. Every $\tilde{\Psi} \in C(z)$ is non-constant and maps $D(0, r^\sharp)$ into $D(0, R^\sharp)$.

We now restrict to recurrent points. Set $S := \hat{F}^{-1}$. Let $E_{\text{rec}}^\sharp \subset E^\sharp$ be the recurrent part for S on $E^\sharp$, let $\tau : E_{\text{rec}}^\sharp \rightarrow \mathbb{N}$ be the first return time to $E^\sharp$, and set $T(z) := S^{\tau(z)}z = \hat{F}^{-\tau(z)}z$. By Poincaré recurrence, we have $\hat{\nu}(E_{\text{rec}}^\sharp) = \hat{\nu}(E^\sharp) > 0$.

Definition 2.6. The *cluster family associated to the joining* is

$$\mathcal{C}_\nu := \bigcup_{z \in E_{\text{rec}}^\sharp} C(z) \subset \mathcal{K}.$$

The set $\mathcal{C}_\nu$ need not be closed. Its closure $\overline{\mathcal{C}_\nu} \subset \mathcal{K}$ will play an important role when $\mathcal{C}_\nu$ is infinite.

The following definition singles out cluster germs occurring on a set of positive lifted joining measure. Section 7 will compare the set of admissible germs with the fiberwise cluster structure.

Definition 2.7. A holomorphic map $\tilde{\Psi} \in \text{Hol}(D(0, r^\sharp), D(0, R^\sharp))$ is called ν -*admissible* if

$$\hat{\nu}(\{z \in E_{\text{rec}}^\sharp : \tilde{\Psi} \in C(z)\}) > 0.$$

The set appearing in Definition 2.7 is measurable. Indeed, for every fixed $\tilde{\Psi} \in \mathcal{K}$, it is equal to

$$E_{\text{rec}}^\sharp \cap \bigcap_{m, N \geq 1} \bigcup_{n \geq N} \left\{ z \in E^\sharp : d_{\mathcal{K}}(\tilde{\Phi}_{z,n}, \tilde{\Psi}) < \frac{1}{m} \right\}.$$

For every fixed n , the map $z \mapsto \tilde{\Phi}_{z,n}$ is measurable in the compact-open topology: its evaluations on a fixed countable dense subset of $D(0, r^\sharp)$ depend measurably on the finite backward coordinates $(x_0, \dots, x_{-n}, y_0, \dots, y_{-n})$. We denote by $\mathcal{A}_\nu$ the set of all ν -admissible germs. Thus $\mathcal{A}_\nu \subset \mathcal{C}_\nu$.

The following proposition realizes an admissible cluster germ as a local holomorphic graph carrying positive joining measure.

Proposition 2.8. Take $\tilde{\Psi} \in \mathcal{A}_\nu$ and set $E_{\tilde{\Psi}}^\sharp := \{z \in E_{\text{rec}}^\sharp : \tilde{\Psi} \in C(z)\}$. Then $\varphi_{\tilde{\Psi}} := \chi_b^{-1} \circ \tilde{\Psi} \circ \chi_a$ is holomorphic on U . Moreover, for every $z = (\hat{x}, \hat{y}) \in E_{\tilde{\Psi}}^\sharp$, we have $\varphi_{\tilde{\Psi}}(x_0) = y_0$ and $\varphi'_{\tilde{\Psi}}(x_0) \neq 0$.

Consequently, we have $\pi(E_{\tilde{\Psi}}^\sharp) \subset \Gamma_{\varphi_{\tilde{\Psi}}}$ and $\nu(\Gamma_{\varphi_{\tilde{\Psi}}}) > 0$. In particular, at every $(x_0, y_0) \in \pi(E_{\tilde{\Psi}}^\sharp)$, the map $\varphi_{\tilde{\Psi}}$ is biholomorphic after restriction to sufficiently small neighbourhoods of x_0 and y_0 .

Proof. Fix $z = (\hat{x}, \hat{y}) \in E_{\tilde{\Psi}}^\sharp$. There exists $n_j \rightarrow \infty$ such that $\tilde{\Phi}_{z,n_j} \rightarrow \tilde{\Psi}$ locally uniformly. Since $h_{\hat{x}}(\chi_a(x_0)) = 0$ and $\Phi_{z,n_j}(0) = 0$, we have $\tilde{\Phi}_{z,n_j}(\chi_a(x_0)) = k_{\hat{y}}(0) = \chi_b(y_0)$. Passing to the limit gives $\varphi_{\tilde{\Psi}}(x_0) = y_0$. Moreover, Proposition 2.5 (4) gives

$$\left| \tilde{\Psi}'(\chi_a(x_0)) \right| = \left| k'_{\hat{y}}(0) h'_{\hat{x}}(\chi_a(x_0)) \right| \geq c_* > 0.$$

Hence $\varphi'_{\tilde{\Psi}}(x_0) \neq 0$.

Thus $\pi(E_{\tilde{\Psi}}^{\sharp}) \subset \Gamma_{\varphi_{\tilde{\Psi}}}$. Since $E_{\tilde{\Psi}}^{\sharp} \subset \pi^{-1}(\Gamma_{\varphi_{\tilde{\Psi}}})$ and $\pi_*\hat{\nu} = \nu$, we have

$$\nu(\Gamma_{\varphi_{\tilde{\Psi}}}) = \hat{\nu}(\pi^{-1}(\Gamma_{\varphi_{\tilde{\Psi}}})) \geq \hat{\nu}(E_{\tilde{\Psi}}^{\sharp}) > 0.$$

The final assertion follows from the inverse function theorem. $\square$

Lemma 2.9. *If $\mathcal{C}_{\nu}$ is finite, then $\mathcal{A}_{\nu}$ is finite and non-empty.*

Proof. Write $\mathcal{C}_{\nu} = \{\tilde{\Psi}_1, \dots, \tilde{\Psi}_N\}$. Since $C(z)$ is non-empty for every $z \in E_{\text{rec}}^{\sharp}$, we have

$$E_{\text{rec}}^{\sharp} = \bigcup_{j=1}^N \left\{ z \in E_{\text{rec}}^{\sharp} : \tilde{\Psi}_j \in C(z) \right\}.$$

Since $\hat{\nu}(E_{\text{rec}}^{\sharp}) > 0$, for some j we have $\hat{\nu}(\{z \in E_{\text{rec}}^{\sharp} : \tilde{\Psi}_j \in C(z)\}) > 0$. Thus $\tilde{\Psi}_j \in \mathcal{A}_{\nu}$. Since $\mathcal{A}_{\nu} \subset \mathcal{C}_{\nu}$, the set $\mathcal{A}_{\nu}$ is finite and non-empty. $\square$

Remark 2.10. If $\nu = \mu_1 \times \mu_2$, then the cluster family $\mathcal{C}_{\nu}$ is necessarily infinite, for every fixed choice of reference data as above. Indeed, if $\mathcal{C}_{\nu}$ were finite, Lemma 2.9 would give a ν -admissible cluster germ $\tilde{\Psi}$, and Proposition 2.8 would give $\nu(\Gamma_{\varphi_{\tilde{\Psi}}}) > 0$. On the other hand, since μ_2 is non-atomic, Fubini's theorem gives

$$(\mu_1 \times \mu_2)(\Gamma_{\varphi_{\tilde{\Psi}}}) = \int_U \mu_2(\{\varphi_{\tilde{\Psi}}(x)\}) d\mu_1(x) = 0.$$

This is a contradiction.

3. RIGIDITY OF POSITIVE-MASS LOCAL HOLOMORPHIC RELATIONS

The main result of this section is the following local-to-global rigidity theorem.

Theorem 3.1. *Let ν be an ergodic joining of f_1 and f_2 . Suppose that there exist non-empty open sets $U, V \subset \mathbb{P}^1$ and a biholomorphism $\varphi: U \rightarrow V$ such that $\nu(\Gamma_{\varphi}) > 0$. Then there exist $N \geq 1$ and an irreducible algebraic curve $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ such that $F^N(C) = C$ and $\nu(C) > 0$. Moreover, exactly one of alternatives (A1) and (A2) holds.*

The proof is organized as follows. In Section 3.1, we use recurrence and contraction of inverse branches to obtain many contracting local return branches from the charged graph. When neither f_1 nor f_2 is Lattès (i.e., a finite quotient of an affine endomorphism of an elliptic curve), the generalized Inou theorem [JX23b, Theorem 2.1] globalizes the resulting local relation. In Section 3.2, we treat the remaining case: if one map is Lattès, then so is the other, and affine uniformization together with a lattice-counting argument algebraizes the charged local graph. Finally, Section 3.3 classifies the resulting invariant curve and completes the proof.

For the rest of this section, we fix non-empty open sets $U, V \subset \mathbb{P}^1$ and a biholomorphism $\varphi: U \rightarrow V$ such that $\nu(\Gamma_{\varphi}) > 0$. After restricting around a point in the support of $\nu|_{\Gamma_{\varphi}}$, we may assume, keeping the same notation, that U is a connected coordinate disc, $V = \varphi(U)$, and $\varphi: U \rightarrow V$ is biholomorphic with $\nu(\Gamma_{\varphi}) > 0$.

3.1. Recurrent local relations and the non-Lattès case. The main result of this subsection is the following proposition.

Proposition 3.2. *Assume that neither f_1 nor f_2 is a Lattès map. Then there exist $N \geq 1$ and an irreducible algebraic curve $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ such that $(f_1^N \times f_2^N)(C) = C$ and $\nu(C) > 0$.*

The following lemmas use the standing charged-graph hypothesis, but do not require either map to be non-Lattès. The non-Lattès assumption is used only in the proof of Proposition 3.2 at the end of the subsection.

Lemma 3.3. *There exist coordinate discs $D^- \Subset D \Subset U$ and a compact set*

$$K \subset \{z = (\hat{x}, \hat{y}) : x_0 \in D^-, y_0 = \varphi(x_0)\}$$

of positive $\hat{\nu}$ -measure such that for every sufficiently large n and every $z = (\hat{x}, \hat{y}) \in K$, the inverse branch of f_1^n determined by the backward history $\hat{x}$ is defined on D . Moreover, the diameters of the corresponding inverse-branch images of D tend to zero uniformly on K as $n \rightarrow \infty$.

Proof. Set $\hat{\Gamma}_\varphi := \pi^{-1}(\Gamma_\varphi)$. Intersect $\hat{\Gamma}_\varphi$ with the full-measure inverse-branch set from Proposition 2.1. Since $\rho_{1,\varepsilon}(\hat{x}) > 0$ almost everywhere, there exists $\rho > 0$ such that $S := \{(\hat{x}, \hat{y}) \in \hat{\Gamma}_\varphi : \rho_{1,\varepsilon}(\hat{x}) \geq \rho\}$ has positive $\hat{\nu}$ -measure. Choose x in the support of the pushforward of $\hat{\nu}|_S$ under $(\hat{x}, \hat{y}) \mapsto x_0$. Choose coordinate discs $D^- \Subset D \Subset U$ containing x and a constant $0 < \lambda < 1$ so that $D \subset B_{\text{sph}}(x_0, \lambda\rho)$ for every $x_0 \in D^-$. Since x belongs to the support of the pushforward of $\hat{\nu}|_S$, the set $S^- := \{z \in S : x_0 \in D^-\}$ has positive $\hat{\nu}$ -measure.

For every $z = (\hat{x}, \hat{y}) \in S^-$ and every $n \geq 1$, the inverse branch of f_1^n determined by $\hat{x}$ is defined on D . By Proposition 2.1 (3)–(4), together with the uniform distortion bounds for the chosen charts, the diameters of the corresponding images of D tend to zero for every $z \in S^-$. Egorov's theorem gives a positive-measure subset on which this convergence is uniform, and regularity of $\hat{\nu}$ allows us to choose a compact subset K of positive measure. By construction, K satisfies the required properties. $\square$

Lemma 3.4. *There exist a constant $\delta > 0$ and arbitrarily large integers N for which there are a measurable set $A_N \subset D^-$ with $\mu_1(A_N) > 0$ and a collection $\mathcal{B}_N$ of at least δd_1^N distinct inverse branches $\alpha : D \rightarrow D$ of f_1^N , where $d_1 := \deg f_1$, such that $\alpha(D) \Subset D$ and $f_2^N \circ \varphi \circ \alpha = \varphi$ on D for every $\alpha \in \mathcal{B}_N$.*

Proof. Take $D^- \Subset D$ and K from Lemma 3.3. Since $\hat{F}$ is ergodic, the mean ergodic theorem gives $c > 0$ and arbitrarily large N such that $\hat{\nu}(K \cap \hat{F}^{-N}K) \geq c$. Fix one such N , sufficiently large that the uniform contraction in Lemma 3.3 applies, and set $B_N := K \cap \hat{F}^{-N}K$ and $E_N := \{x_0 : z = (\hat{x}, \hat{y}) \in B_N\}$. Since the distribution of x_0 under $\hat{\nu}$ is μ_1 , we have $\mu_1(E_N) \geq c$. For every $u \in E_N$, choosing $z = (\hat{x}, \hat{y}) \in B_N$ with $x_0 = u$ gives $u, f_1^N(u) \in D^-$ and $f_2^N(\varphi(u)) = \varphi(f_1^N(u))$.

For μ_1 -almost every x , set $m_N(x) := \#(E_N \cap (f_1^N)^{-1}(x))$, with preimages counted with multiplicity. Since $f_1^* \mu_1 = d_1 \mu_1$, we have $\int m_N d\mu_1 = d_1^N \mu_1(E_N) \geq cd_1^N$. As $0 \leq m_N \leq d_1^N$, the set $X_N := \{x : m_N(x) \geq (c/2)d_1^N\}$ has positive μ_1 -measure. Moreover, $X_N \subset D^-$ up to a μ_1 -null set, as $f_1^N(E_N) \subset D^-$.

Remove the critical values of f_1^N from X_N . Choose a simply connected coordinate disc $Q \Subset D^-$, disjoint from these critical values, such that $\mu_1(X_N \cap Q) > 0$. The inverse branches of f_1^N on Q form a finite collection. Partition $X_N \cap Q$ according to the subcollection of branches α for which $\alpha(x) \in E_N$. On one element A_N of this finite measurable partition, we have $\mu_1(A_N) > 0$ and a fixed collection $\mathcal{B}_N$ of at least $(c/2)d_1^N$ distinct inverse branches such that $\alpha(A_N) \subset E_N$ for every $\alpha \in \mathcal{B}_N$.

For $x \in A_N$ and $\alpha \in \mathcal{B}_N$, choose $z = (\hat{x}, \hat{y}) \in B_N$ with $x_0 = \alpha(x)$. Then $\hat{F}^N(z) \in K$, and its first backward history determines an inverse branch which agrees with α near x . By the uniform inverse-branch property in Lemma 3.3, this branch is defined on the fixed disc D . By uniqueness of analytic continuation, it gives an extension of α from Q to D , which we continue to denote by α . By the choice of N , the uniform contraction in Lemma 3.3 makes the diameter of each $\alpha(D)$ smaller than $\rho := \text{dist}(\overline{D^-}, \partial D) > 0$. Since $\alpha(A_N) \subset E_N \subset D^-$, we obtain $\alpha(D) \Subset D$ for every $\alpha \in \mathcal{B}_N$. In particular, $f_2^N \circ \varphi \circ \alpha$ is holomorphic on D . For $x \in A_N$, we have $\alpha(x) \in E_N$, and hence $f_2^N(\varphi(\alpha(x))) = \varphi(x)$. The holomorphic maps $f_2^N \circ \varphi \circ \alpha$ and φ therefore agree on the positive- μ_1 -measure set A_N . If they were not identical, their coincidence set would be discrete and hence μ_1 -null, since μ_1 is non-atomic. Thus $f_2^N \circ \varphi \circ \alpha = \varphi$ on D . Taking $\delta := c/2$ proves the lemma. $\square$

Lemma 3.5. *There exist an integer $N \geq 1$, a coordinate disc $D \Subset U$, and an open set $W \Subset D$ such that $f_1^N: W \rightarrow D$ is non-injective, $\varphi \circ f_1^N = f_2^N \circ \varphi$ on W , and $f_1^N|_W$ has a repelling fixed point.*

Proof. Choose a sufficiently large return time N in Lemma 3.4 so that $\delta d_1^N \geq 2$, and choose two distinct branches $\alpha_1, \alpha_2 \in \mathcal{B}_N$. Set $W := \alpha_1(D) \cup \alpha_2(D)$. Then $W \Subset D$, the map $f_1^N: W \rightarrow D$ is non-injective, and $\varphi \circ f_1^N = f_2^N \circ \varphi$ on W .

Since $\alpha_1(D) \Subset D$, the strict self-map $\alpha_1: D \rightarrow D$ has a fixed point $p \in D$. By Schwarz–Pick, $|\alpha_1'(p)| < 1$. Since α_1 is an inverse branch of f_1^N , we have $f_1^N(p) = p$ and $|(f_1^N)'(p)| = |\alpha_1'(p)|^{-1} > 1$. Thus $p \in W$ is a repelling fixed point of f_1^N . $\square$

We can now prove Proposition 3.2.

Proof of Proposition 3.2. Take D, W, φ and N from Lemma 3.5. The open set D is connected, $W \subset D$, the map $f_1^N: W \rightarrow D$ is non-injective and has a repelling fixed point by Lemma 3.5, and φ is non-constant with $\varphi \circ f_1^N = f_2^N \circ \varphi$ on W .

Apply [JX23b, Theorem 2.1] with $f = f_1^N$, $g = f_2^N$, $U = D$, $U' = W$, and $h = \varphi|_D$. Since f_1^N and f_2^N are non-Lattès, the theorem gives an irreducible algebraic curve $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ invariant under $f_1^N \times f_2^N$ and containing the graph of φ on D . Moreover, the compact set K in Lemma 3.3 has positive $\hat{\nu}$ -measure and projects into the graph of φ over $D^- \Subset D$. Therefore $\nu(\Gamma_{\varphi|_D}) \geq \hat{\nu}(K) > 0$, and hence $\nu(C) > 0$. $\square$

3.2. The Lattès case. We now assume that one of f_1, f_2 is Lattès. The charged local graph first forces the other map to be Lattès as well; see Lemma 3.7. We then use affine uniformizations and the many-return statement in Lemma 3.4 to algebraize the local relation. The affine-lattice part of the argument is reminiscent of rigidity for higher-rank algebraic actions, although those results do not apply to the present rank-one, non-invertible setting; see, for example, [KK02; KS05]. The main result of this subsection is the following proposition.

Proposition 3.6. *Assume that one of f_1 and f_2 is a Lattès map. Then there exist $N \geq 1$ and an irreducible algebraic curve $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ such that $F^N(C) = C$ and $\nu(C) > 0$.*

Lemma 3.7. *Let ν be an ergodic joining of $(J(f_1), f_1, \mu_1)$ and $(J(f_2), f_2, \mu_2)$. Suppose that ν charges the graph of a local biholomorphism. If one of f_1, f_2 is a Lattès map, then so is the other.*

Proof. By symmetry, suppose that f_1 is Lattès. Let $\varphi: U \rightarrow V$ be a biholomorphic restriction of the charged local graph and set

$$\eta := \nu|_{\Gamma_\varphi}, \quad \alpha := (\pi_1)_*\eta, \quad \beta := (\pi_2)_*\eta.$$

Then $\eta \neq 0$ and $0 < \alpha \leq \mu_1$, $0 < \beta \leq \mu_2$, while $\beta = \varphi_*\alpha$.

Let m denote the spherical area measure on $\mathbb{P}^1$. Since f_1 is Lattès, its maximal-entropy measure is absolutely continuous with respect to m . Hence $\alpha \ll m$. Since φ is locally biholomorphic, it sends m -null sets to m -null sets locally, and therefore $\beta = \varphi_*\alpha \ll m$. Thus μ_2 has a non-zero absolutely continuous component with respect to m .

Write the Lebesgue decomposition $\mu_2 = \mu_2^{\text{ac}} + \mu_2^{\text{s}}$ of μ_2 with respect to m . Since $0 < \beta \leq \mu_2$ and $\beta \ll m$, we have $\mu_2^{\text{ac}} \neq 0$. Both the image and the inverse image of an m -null set under a rational map are m -null. Hence pushforward by f_2 preserves both absolute continuity and singularity with respect to m . From $(f_2)_*\mu_2 = \mu_2$ and the uniqueness of the Lebesgue decomposition, it follows that $(f_2)_*\mu_2^{\text{ac}} = \mu_2^{\text{ac}}$ and $(f_2)_*\mu_2^{\text{s}} = \mu_2^{\text{s}}$.

After normalizing μ_2^{ac} to have total mass one, we obtain an f_2 -invariant probability measure absolutely continuous with respect to the ergodic invariant probability measure μ_2 . It must therefore equal μ_2 , and hence $\mu_2 \ll m$. By Zdunik’s characterization [Zdu90], f_2 is Lattès. $\square$

For the rest of this subsection, assume that both f_1 and f_2 are Lattès. For $i = 1, 2$, choose a standard crystallographic uniformization $\Theta_i: \mathbb{C} \rightarrow \mathbb{P}^1$ with deck group Γ_i and an affine lift $A_i(z) = a_i z + b_i$, $|a_i| > 1$, satisfying $\Theta_i \circ A_i = f_i \circ \Theta_i$. We choose the presentation so that $\Gamma_i = \{z \mapsto \zeta z + \lambda : \zeta \in \mathcal{U}_i, \lambda \in \Lambda_i\}$, where Λ_i is a rank-two translation lattice and $\mathcal{U}_i$ is a finite cyclic group preserving Λ_i . In particular, $\deg f_i = |a_i|^2$.

Since the branch values of the two uniformizations are finite and the marginals of ν are non-atomic, we may restrict the charged graph to smaller connected discs, still denoted by U and V , so that $\nu(\Gamma_\varphi) > 0$, $\varphi: U \rightarrow V$ is biholomorphic, and U and V avoid the branch values of Θ_1 and Θ_2 .

Applying Lemmas 3.3 and 3.4 to this restricted graph, we retain the notation $D^- \Subset D$, K , and $\mathcal{B}_N$. We also choose N , W , and a repelling fixed point p as in Lemma 3.5.

Lemma 3.8. *Let D be the return disc from Lemma 3.5, and choose components $\tilde{D} \subset \Theta_1^{-1}(D)$ and $\tilde{V} \subset \Theta_2^{-1}(\varphi(D))$. Then the lift $H := (\Theta_2|_{\tilde{V}})^{-1} \circ \varphi \circ (\Theta_1|_{\tilde{D}})$ is the restriction of an affine map $H(z) = cz + e$ with $c \neq 0$.*

Proof. Let p be the repelling fixed point from Lemma 3.5 and set $q := \varphi(p)$. The relation $\varphi \circ f_1^N = f_2^N \circ \varphi$ near p , together with $f_1^N(p) = p$, gives $f_2^N(q) = q$. Choose lifts $\tilde{p} \in \tilde{D}$ and $\tilde{q} := H(\tilde{p})$. Since p and q are fixed by f_1^N and f_2^N , respectively, there are deck transformations $\gamma_i \in \Gamma_i$ such that $A_1^N(\tilde{p}) = \gamma_1(\tilde{p})$ and $A_2^N(\tilde{q}) = \gamma_2(\tilde{q})$. Set $B_i := \gamma_i^{-1} \circ A_i^N$. Then B_1 fixes $\tilde{p}$ and B_2 fixes $\tilde{q}$. The local relation $\varphi \circ f_1^N = f_2^N \circ \varphi$ on W lifts near $\tilde{p}$ to $H \circ B_1 = B_2 \circ H$. Indeed, both sides are lifts through Θ_2 of the same local map and agree at $\tilde{p}$.

Translate source and target coordinates so that $\tilde{p} = \tilde{q} = 0$. Since the B_i are affine and fix the origin, write $B_i(z) = \lambda_i z$. The linear part of a deck transformation has modulus one, so $|\lambda_i| = |a_i|^N > 1$. The lifted relation becomes $H(\lambda_1 z) = \lambda_2 H(z)$. Since $H'(0) \neq 0$, differentiation at zero gives $\lambda_1 = \lambda_2 =: \lambda$. Writing $H(z) = \sum_{k \geq 1} c_k z^k$, we obtain $c_k(\lambda^k - \lambda) = 0$ for every $k \geq 2$. Since $|\lambda| > 1$, all c_k with $k \geq 2$ vanish. Hence H is affine near the origin, and the identity theorem gives $H(z) = cz + e$ on the connected domain $\tilde{D}$. $\square$

Lemma 3.9. *There is a constant $C > 0$, independent of the return time N , such that every returning branch $\alpha \in \mathcal{B}_N$ has a canonical lift $\tilde{\alpha} = A_1^{-N} \circ \gamma_\alpha$, where $\gamma_\alpha(z) = \zeta_\alpha z + \lambda_\alpha$ with $\zeta_\alpha \in \mathcal{U}_1$, $\lambda_\alpha \in \Lambda_1$, and $|\lambda_\alpha| \leq C|a_1|^N$. Distinct returning branches give distinct γ_α .*

Proof. Since D avoids the branch values of Θ_1 , the restriction $\Theta_1|_{\tilde{D}}: \tilde{D} \rightarrow D$ is biholomorphic. For a returning branch $\alpha: D \rightarrow D$, set $\tilde{\alpha} := (\Theta_1|_{\tilde{D}})^{-1} \circ \alpha \circ \Theta_1|_{\tilde{D}}$. The identity $f_1^N \circ \alpha = \text{id}_D$ implies $\Theta_1 \circ A_1^N \circ \tilde{\alpha} = \Theta_1$. Since $\tilde{D}$ is connected, there is a unique $\gamma_\alpha \in \Gamma_1$ such that $A_1^N \circ \tilde{\alpha} = \gamma_\alpha$.

Fix $z_0 \in \tilde{D}$. Since $\tilde{\alpha}(\tilde{D}) \subset \tilde{D}$, we have $\gamma_\alpha(z_0) \in A_1^N(\tilde{D})$. The set $\tilde{D}$ is bounded, and if $A_1^N(z) = a_1^N z + b_{1,N}$, then $|b_{1,N}| = O(|a_1|^N)$. Thus $|\gamma_\alpha(z_0)| \leq C_0|a_1|^N$ for a constant independent of N . Writing $\gamma_\alpha(z) = \zeta_\alpha z + \lambda_\alpha$ and using the finiteness of $\mathcal{U}_1$ gives $|\lambda_\alpha| \leq C|a_1|^N$. Finally, equality of two γ_α 's implies equality of the corresponding canonical lifts and hence of the downstairs branches. $\square$

Lemma 3.10. *There is a finite-index subgroup $\Lambda'_1 \subset \Lambda_1$ such that $c\Lambda'_1 \subset \Lambda_2$, where c is the slope of the affine lift $H(z) = cz + e$.*

Proof. We first prove an auxiliary finite-index statement for a modified slope; namely, there exists $c_0 \in \mathbb{C}^*$ such that the subgroup $\Lambda'_1 := \{\lambda \in \Lambda_1 : c_0 \lambda \in \Lambda_2\}$ has rank two. In particular, Λ'_1 has finite index in Λ_1 .

For arbitrarily large return times N , choose $\mathcal{B}_N$ as in Lemma 3.4; thus $|\mathcal{B}_N| \geq \delta d_1^N = \delta|a_1|^{2N}$. For $\alpha \in \mathcal{B}_N$, write $\gamma_\alpha(z) = \zeta_\alpha z + \lambda_\alpha$ as in Lemma 3.9. Since $\mathcal{U}_1$ is finite, for each such N there exist a subcollection $\mathcal{B}'_N \subset \mathcal{B}_N$ and $\zeta_N \in \mathcal{U}_1$ such that $|\mathcal{B}'_N| \geq \delta'|a_1|^{2N}$ and $\zeta_\alpha = \zeta_N$ for every $\alpha \in \mathcal{B}'_N$, where $\delta' > 0$ is independent of N .

For $\alpha \in \mathcal{B}'_N$, the identity $f_2^N \circ \varphi \circ \alpha = \varphi$ lifts to $A_2^N \circ H \circ A_1^{-N} \circ \gamma_\alpha = \delta_\alpha \circ H$ for some $\delta_\alpha \in \Gamma_2$. Write $\delta_\alpha(z) = \xi_\alpha z + \mu_\alpha$, with $\xi_\alpha \in \mathcal{U}_2$ and $\mu_\alpha \in \Lambda_2$. Comparing linear parts and using $H(z) = cz + e$ gives $(a_2^N/a_1^N)\zeta_N = \xi_\alpha$. Thus ξ_α is independent of α ; denote it by ξ_N .

The pairs (ζ_N, ξ_N) lie in the finite set $\mathcal{U}_1 \times \mathcal{U}_2$. Passing to a sequence of return times $N_k \rightarrow \infty$, we may therefore assume that $\zeta_{N_k} = \zeta$ and $\xi_{N_k} = \xi$ are fixed. Set $c_0 := c\xi\zeta^{-1}$ and $\Lambda''_1 := \{\lambda \in \Lambda_1 : c_0\lambda \in \Lambda_2\}$. This is now a single subgroup, independent of k .

For every k and $\alpha, \beta \in \mathcal{B}'_{N_k}$, subtracting the constant terms in the two lifted identities gives $c_0(\lambda_\alpha - \lambda_\beta) = \mu_\alpha - \mu_\beta \in \Lambda_2$. Hence the parameters λ_α , $\alpha \in \mathcal{B}'_{N_k}$, lie in one translate of the fixed subgroup Λ''_1 . They are distinct and, by Lemma 3.9, satisfy $|\lambda_\alpha| \leq C|a_1|^{N_k}$.

If Λ''_1 has rank zero, each translate contains only one point, which is impossible for large k . Suppose that Λ''_1 has rank one. Since it is a fixed discrete rank-one subgroup of $\mathbb{C}$, there is $C_1 > 0$ such that every translate of Λ''_1 contains at most $C_1(1 + R)$ points in $D(0, R)$. Therefore $|\mathcal{B}'_{N_k}| \leq C_2|a_1|^{N_k}$ for a constant C_2 independent of k . This contradicts $|\mathcal{B}'_{N_k}| \geq \delta'|a_1|^{2N_k}$ as $k \rightarrow \infty$. Thus Λ''_1 has rank two, and hence finite index in Λ_1 .

By the standard crystallographic presentation, $\mathcal{U}_i$ preserves Λ_i . Hence $\zeta\Lambda_1 = \Lambda_1$ and $\xi\Lambda_2 = \Lambda_2$. Set $\Lambda'_1 := \zeta^{-1}\Lambda''_1$. This subgroup has finite index in Λ_1 . If $\lambda'' = \zeta^{-1}\lambda$ with $\lambda \in \Lambda''_1$, then $\xi\zeta^{-1}c\lambda \in \Lambda_2$. Applying ξ^{-1} and using commutativity of multiplication in $\mathbb{C}$ gives $c\lambda'' \in \Lambda_2$. $\square$

Proof of Proposition 3.6. By Lemma 3.7, both f_1 and f_2 are Lattès. Lemma 3.8 gives an affine lift $H(z) = cz + e$ of the charged local graph. Lemma 3.10 gives a finite-index subgroup $\Lambda'_1 \subset \Lambda_1$ such that $c\Lambda'_1 \subset \Lambda_2$.

Now we show that the charged local graph $\Gamma_{\varphi|_D}$ is contained in an irreducible algebraic curve $C \subset \mathbb{P}^1 \times \mathbb{P}^1$, and $\nu(C) > 0$. Set $E_i := \mathbb{C}/\Lambda_i$, let $p_i: \mathbb{C} \rightarrow E_i$ be the quotient map, and write $\Theta_i = \pi_i \circ p_i$ with $\pi_i: E_i \rightarrow \mathbb{P}^1$ finite. Set $E'_1 := \mathbb{C}/\Lambda'_1$. The inclusion $\Lambda'_1 \subset \Lambda_1$ induces an isogeny $q: E'_1 \rightarrow E_1$, while $H(z) = cz + e$ descends, by Lemma 3.10, to a holomorphic map $h: E'_1 \rightarrow E_2$. Define $\Xi := (\pi_1 \circ q, \pi_2 \circ h): E'_1 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ and set $C := \Xi(E'_1)$. Since E'_1 is a compact irreducible algebraic curve, C is an irreducible algebraic curve. On $\tilde{D}$ we have $\Theta_2 \circ H = \varphi \circ \Theta_1$, so C contains the graph of φ over D . As in the proof of Proposition 3.2, the positive-measure set K from Lemma 3.3 projects into this graph. Hence $\nu(C) > 0$.

It remains to show that there exists a return time $N \geq 1$ such that $F^N(C) = C$. By construction, C contains $\Gamma_{\varphi|_D}$. Choose a sufficiently large return time N and a corresponding branch $\alpha \in \mathcal{B}_N$ from Lemma 3.4. Then $\alpha(D) \Subset D$, $f_1^N \circ \alpha = \text{id}_D$, and $f_2^N \circ \varphi \circ \alpha = \varphi$ on D .

For every $x \in D$, the point $(\alpha(x), \varphi(\alpha(x)))$ belongs to C , and its image under F^N is $(x, \varphi(x))$. Hence $\Gamma_{\varphi|_D} \subset F^N(C)$. The image $F^N(C)$ is an irreducible algebraic curve. Since C and $F^N(C)$ contain the same non-empty analytic graph, they coincide. Thus $F^N(C) = C$. $\square$

3.3. Classification and conclusion of the proof of Theorem 3.1.

Proposition 3.11. *Let $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ be an irreducible algebraic curve with $\nu(C) > 0$ and $F^N(C) = C$ for some $N \geq 1$. Then exactly one of alternatives (A1) and (A2) holds.*

Proof. Both coordinate projections of C are dominant. Indeed, if $\pi_1(C)$ were a point, then C would be vertical and $\nu(C) \leq \mu_1(\{a\}) = 0$, contradicting $\nu(C) > 0$; the same argument applies to π_2 .

If both projection degrees are greater than one, C gives alternative (A2). We now assume that at least one projection has degree one.

Let $r \geq 1$ be minimal with $F^r(C) = C$, and set $C_j := F^j(C)$ for $0 \leq j < r$. The curves C_j are distinct and F cyclically permutes them. Set $\mathcal{U} := \bigcup_{j=0}^{r-1} C_j$. Since $F(\mathcal{U}) = \mathcal{U}$ and $\nu(\mathcal{U}) \geq \nu(C) > 0$, invariance and ergodicity of ν give $\nu(\mathcal{U}) = 1$.

The joining ν is non-atomic as its marginals are non-atomic. Since distinct irreducible curves meet in finitely many points, the intersections $C_j \cap C_k$, for $j \neq k$, are ν -null. Since $\nu(\mathcal{U}) = 1$ and,

modulo the finite intersections of distinct curves, the only component of $\mathcal{U}$ mapped into C_{j+1} is C_j , invariance of ν gives $F_*(\nu|_{C_j}) = \nu|_{C_{j+1(\bmod r)}}$. Consequently, $\nu(C_j) = 1/r$ for every j .

If some C_j has both projection degrees greater than one, then $F^r(C_j) = C_j$ and alternative (A2) holds. Otherwise, for every j , at least one coordinate projection of C_j has degree one. Hence C_j is either the graph of a rational map or the transpose of such a graph.

Set $\nu_j := r \nu|_{C_j}$. Then each ν_j is a probability measure, $F_*\nu_j = \nu_{j+1(\bmod r)}$, and $(F^r)_*\nu_j = \nu_j$. Suppose that $C_j = \Gamma_{A_j}$. The first marginal of ν_j is an f_1^r -invariant probability measure absolutely continuous with respect to μ_1 . Since μ_1 is mixing, and hence ergodic for f_1^r , this marginal equals μ_1 . Therefore $\nu_j = (\text{id}, A_j)_*\mu_1$. Moreover, $F^r(C_j) = C_j$ gives $f_2^r \circ A_j = A_j \circ f_1^r$.

If C_j is the transpose of the graph of a rational map B_j , the same argument using the second marginal gives $\nu_j = (B_j, \text{id})_*\mu_2$ and $f_1^r \circ B_j = B_j \circ f_2^r$. Finally, $\nu = \frac{1}{r} \sum_{j=0}^{r-1} \nu_j$, so these measures give precisely the cyclic decomposition in alternative (A1).

Finally, alternatives (A1) and (A2) are mutually exclusive. Indeed, in (A1) the joining is supported on a finite union of graph or transpose-graph curves. An irreducible curve as in (A2) has finite intersection with each such component unless they coincide, while coincidence is impossible as both projection degrees in (A2) are greater than one. Since ν is non-atomic, such a curve cannot have positive ν -measure. $\square$

Proof of Theorem 3.1. If neither f_1 nor f_2 is Lattès, Proposition 3.2 gives an invariant irreducible algebraic curve of positive joining measure. If one of f_1, f_2 is Lattès, Proposition 3.6 gives such a curve. In either case, Proposition 3.11 gives exactly one of alternatives (A1) and (A2). $\square$

4. JOININGS WITHOUT CHARGED LOCAL HOLOMORPHIC GRAPHS

We now treat joinings for which no local biholomorphic graph has positive joining measure. The main result is the following theorem.

Theorem 4.1. *Let ν be an ergodic joining of f_1 and f_2 . Suppose that $\nu(\Gamma_\varphi) = 0$ for every bi-holomorphism $\varphi: U' \rightarrow V'$ between non-empty open subsets of $\mathbb{P}^1$. Then $\mathcal{C}_\nu$ is infinite, and the family*

$$\mathcal{H}_\nu := \left\{ \chi_b^{-1} \circ \tilde{\Psi} \circ \chi_a : \tilde{\Psi} \in \overline{\mathcal{C}_\nu} \right\} \subset \text{Hol}(U, V)$$

is compact and non-discrete. Every element of $\mathcal{H}_\nu$ is non-constant and is locally biholomorphic after restriction to a non-empty open subset.

Remark 4.2. Theorem 4.1 also applies to the product joining, which is mixing and hence ergodic. Indeed, since μ_2 is non-atomic, Fubini's theorem shows that $(\mu_1 \times \mu_2)(\Gamma_\varphi) = 0$ for every local measurable map φ . Thus the product joining satisfies the hypothesis of Theorem 4.1, which produces a compact non-discrete family of local holomorphic maps.

For the product joining this phenomenon is automatic from the transfer-map construction and does not reflect any dependence between the two coordinates. For the non-product joinings considered in Theorem 1.1, the reference set $E^\sharp$ can be chosen to satisfy (2.3), so that the reference data themselves witness a genuine dependence between the two coordinates.

Lemma 4.3. *Assume that $\mathcal{C}_\nu$ is infinite. Then $\overline{\mathcal{C}_\nu} \subset \mathcal{K}$ is compact and non-discrete. Moreover, every $\tilde{\Psi} \in \overline{\mathcal{C}_\nu}$ is a non-constant holomorphic map $\tilde{\Psi}: D(0, r^\sharp) \rightarrow D(0, R^\sharp)$, and is locally biholomorphic after restriction to a non-empty open subset of $\chi_a(U)$.*

Proof. Since $\mathcal{K}$ is compact and metrizable, $\overline{\mathcal{C}_\nu}$ is compact. As $\mathcal{C}_\nu$ is infinite, its closure has an accumulation point and is therefore non-discrete. It remains to prove the assertions about the elements of $\overline{\mathcal{C}_\nu}$. Let $\tilde{\Psi}_j \in \mathcal{C}_\nu$ converge locally uniformly on $D(0, r^\sharp)$ to $\tilde{\Psi} \in \overline{\mathcal{C}_\nu}$. For each j , choose $z_j = (\hat{x}_j, \hat{y}_j) \in E_{\text{rec}}^\sharp$ such that $\tilde{\Psi}_j \in C(z_j)$, and set $q_j := \chi_a(x_{j,0})$. By Proposition 2.5 (1), the points

q_j belong to the compact set $\overline{\chi_a(U)} \in D(0, r^\sharp)$. After passing to a subsequence, we may assume that $q_j \rightarrow q \in \overline{\chi_a(U)}$.

Since $\tilde{\Psi}_j \in C(z_j)$, Proposition 2.5 (4) gives $|\tilde{\Psi}'_j(q_j)| \geq c_*$ for every j . Local uniform convergence of holomorphic maps implies local uniform convergence of their derivatives on compact subsets. Hence $\tilde{\Psi}'_j(q_j) \rightarrow \tilde{\Psi}'(q)$, and therefore $|\tilde{\Psi}'(q)| \geq c_* > 0$. Thus $\tilde{\Psi}$ is non-constant. By the inverse function theorem, $\tilde{\Psi}$ is biholomorphic on a neighbourhood W of q . Since $q \in \overline{\chi_a(U)}$, the open set $W \cap \chi_a(U)$ is non-empty, so $\tilde{\Psi}$ is locally biholomorphic on a non-empty open subset of $\chi_a(U)$.

Finally, every $\tilde{\Psi}_j$ takes values in $D(0, R^\sharp)$. Hence the locally uniform limit $\tilde{\Psi}$ takes values in $\overline{D(0, R^\sharp)}$. Since $\tilde{\Psi}$ is non-constant, the maximum principle shows that $\tilde{\Psi}(D(0, r^\sharp)) \subset D(0, R^\sharp)$. This proves the lemma. $\square$

Proof of Theorem 4.1. Suppose, toward a contradiction, that $\mathcal{C}_\nu$ is finite. By Lemma 2.9, there exists $\tilde{\Psi} \in \mathcal{A}_\nu$. Let $\varphi := \varphi_{\tilde{\Psi}}$ be the corresponding holomorphic map given by Proposition 2.8.

For every $z = (\hat{x}, \hat{y}) \in E_{\tilde{\Psi}}^\sharp$, Proposition 2.8 gives $\varphi'(x_0) \neq 0$. Hence, by the inverse function theorem, there is an open neighbourhood U_z of x_0 on which φ is biholomorphic onto its image. By second countability, there are countably many such open sets $U_j, j \geq 1$, whose union contains x_0 for every $z = (\hat{x}, \hat{y}) \in E_{\tilde{\Psi}}^\sharp$. Therefore

$$E_{\tilde{\Psi}}^\sharp \subset \bigcup_{j \geq 1} \pi^{-1}(\Gamma_{\varphi|_{U_j}}).$$

Since $\hat{\nu}(E_{\tilde{\Psi}}^\sharp) > 0$, there exists j such that $\hat{\nu}(\pi^{-1}(\Gamma_{\varphi|_{U_j}})) > 0$. As $\pi_*\hat{\nu} = \nu$, this gives $\nu(\Gamma_{\varphi|_{U_j}}) > 0$, contradicting the hypothesis that ν charges no graph of a local biholomorphism. Hence $\mathcal{C}_\nu$ is infinite. By Lemma 4.3, the closure $\overline{\mathcal{C}_\nu} \subset \mathcal{K}$ is compact and non-discrete. Moreover, every $\tilde{\Psi} \in \overline{\mathcal{C}_\nu}$ is non-constant and is locally biholomorphic after restriction to a non-empty open subset of $\chi_a(U)$.

For $\tilde{\Psi} \in \overline{\mathcal{C}_\nu}$, define $\varphi_{\tilde{\Psi}} := \chi_b^{-1} \circ \tilde{\Psi} \circ \chi_a$. By Proposition 2.5, these maps are defined on the fixed reference neighbourhood U and take values in V . Hence $\mathcal{H}_\nu = \{\varphi_{\tilde{\Psi}} : \tilde{\Psi} \in \overline{\mathcal{C}_\nu}\} \subset \text{Hol}(U, V)$. Consider the map

$$\Theta : \overline{\mathcal{C}_\nu} \rightarrow \text{Hol}(U, V), \quad \Theta(\tilde{\Psi}) = \varphi_{\tilde{\Psi}} = \chi_b^{-1} \circ \tilde{\Psi} \circ \chi_a.$$

Here $\tilde{\Psi}$ is restricted to the relatively compact domain $\chi_a(U) \in D(0, r^\sharp)$. The map Θ is continuous for the compact-open topologies. It is also injective: if $\Theta(\tilde{\Psi}_1) = \Theta(\tilde{\Psi}_2)$ on U , then $\tilde{\Psi}_1 = \tilde{\Psi}_2$ on the non-empty open set $\chi_a(U)$, hence on $D(0, r^\sharp)$ by the identity theorem.

Since $\overline{\mathcal{C}_\nu}$ is compact and $\text{Hol}(U, V)$ is Hausdorff, Θ is a homeomorphism onto its image. Therefore $\mathcal{H}_\nu = \Theta(\overline{\mathcal{C}_\nu})$ is compact and non-discrete. Moreover, every $h \in \mathcal{H}_\nu$ is non-constant. Since every $\tilde{\Psi} \in \overline{\mathcal{C}_\nu}$ is locally biholomorphic after restriction to a non-empty open subset of $\chi_a(U)$, the same is true of the corresponding map $h = \chi_b^{-1} \circ \tilde{\Psi} \circ \chi_a$. This completes the proof. $\square$

5. PROOF OF THEOREM 1.1

We can now complete the proof of Theorem 1.1.

Proof of Theorem 1.1. If ν charges the graph of a local biholomorphism, then case (A) holds, and Theorem 3.1 gives exactly one of alternatives (A1) and (A2). Otherwise case (B) holds, and Theorem 4.1 gives the asserted compact non-discrete family. $\square$

Theorem 1.1 gives an immediate consequence when the degrees are different.

Corollary 5.1. *Assume that $\deg f_1 \neq \deg f_2$. Then every non-product ergodic joining of f_1 and f_2 belongs to case (B) of Theorem 1.1.*

Proof. Alternative (A1) forces $\deg f_1 = \deg f_2$ by taking degrees in $f_2^T \circ A = A \circ f_1^T$, or in the corresponding transpose-graph relation. For (A2), let C be an invariant curve as in that alternative, let $\sigma: \tilde{C} \rightarrow C$ be its normalization, and for $i = 1, 2$, set $p_i := \pi_i \circ \sigma$. Since $(f_1^n \times f_2^n)(C) = C$, the restriction of $f_1^n \times f_2^n$ to C lifts to a self-map $g: \tilde{C} \rightarrow \tilde{C}$ satisfying $p_1 \circ g = f_1^n \circ p_1$ and $p_2 \circ g = f_2^n \circ p_2$. Taking degrees gives $\deg g = (\deg f_1)^n = (\deg f_2)^n$. Thus, if the degrees are different, the algebraic case (A) cannot occur, and the joining belongs to case (B). $\square$

The conclusion of Corollary 5.1 is not vacuous; see Remark 6.1.

6. MEASURABLE EXAMPLES

In this section, we give two constructions of ergodic joinings in case (B) of Theorem 1.1 with very different conditional measures. The first has diffuse conditional measures, while the second has finite atomic conditional measures.

Denote by $\Sigma_d := \{0, \dots, d-1\}^{\mathbb{N}}$ the one-sided full shift with the uniform Bernoulli measure β_d and left shift σ . By the Bernoulli theorem [HH02; Mañ85], for every rational map f of degree d , the system $(J(f), f, \mu_f)$ is measurably isomorphic to $(\Sigma_d, \sigma, \beta_d)$.

6.1. Diffuse Bernoulli couplings. Fix $0 < \rho < 1$ and give each pair of symbols (i, j) the probability

$$p_\rho(i, j) = \rho d^{-1} \mathbf{1}_{\{i=j\}} + (1 - \rho) d^{-2}.$$

Taking these pairs independently over time gives an ergodic non-product self-joining λ_ρ of the uniform d -shift with uniform marginals. Indeed, summing $p_\rho(i, j)$ over either coordinate gives the uniform distribution on the alphabet, so both coordinate marginals of λ_ρ are equal to β_d .

The conditional measures of λ_ρ over either coordinate are non-atomic. Indeed, after conditioning on $x = (x_n) \in \Sigma_d$, the coordinates of the second sequence remain independent, and every one-symbol conditional probability is at most $\rho + \frac{1-\rho}{d} < 1$. Hence every individual sequence has conditional measure zero. Consequently, if $E \subset \Sigma_d$ is measurable and $B: E \rightarrow \Sigma_d$ is any measurable map, then

$$\lambda_\rho(\Gamma_B) = \int_E (\lambda_\rho)_x(\{B(x)\}) d\beta_d(x) = 0.$$

Let $\theta: (\Sigma_d, \sigma, \beta_d) \rightarrow (J(f), f, \mu_f)$ be a Bernoulli isomorphism and set $\nu_\rho := (\theta \times \theta)_* \lambda_\rho$. Then ν_ρ is an ergodic non-product self-joining of f . We note that since $\theta_* \beta_d = \mu_f$, the two marginals of ν_ρ are both μ_f . Moreover, the conditional measures of ν_ρ over either coordinate are, almost everywhere, the pushforwards under θ of the corresponding conditional measures of λ_ρ , and are therefore non-atomic. Hence, if $E \subset J(f)$ is measurable and $B: E \rightarrow J(f)$ is any measurable map, then $\nu_\rho(\Gamma_B) = 0$. In particular, ν_ρ charges no graph of a local biholomorphism and therefore belongs to case (B) of Theorem 1.1.

Remark 6.1. A modification of the above example also shows that Corollary 5.1 is not vacuous. Indeed, suppose that $d_1 \neq d_2$ have a common divisor $c > 1$. Identify the uniform d_i -symbol shift with the product of the uniform c -symbol shift and the uniform (d_i/c) -symbol shift. Coupling the two common c -symbol coordinates diagonally and taking the two remaining coordinates independently gives a non-product ergodic joining of the uniform d_1 - and d_2 -symbol shifts. Transporting this joining through Bernoulli isomorphisms gives a non-product ergodic joining of any pair of rational maps of degrees d_1 and d_2 , which belongs to case (B) by Corollary 5.1.

6.2. A finite-to-one measurable joining. We next show that case (B) can also occur for joinings with finite atomic conditional measures.

By [Pak25b, Theorem 1.2], for every $d \geq 2$ there exists a non-empty Zariski open subset $\mathcal{U} \subset \text{Rat}_d$ such that, for $f_1, f_2 \in \mathcal{U}$, the product map $f_1 \times f_2$ admits an irreducible periodic curve which is neither vertical nor horizontal if and only if f_1 and f_2 are Möbius-conjugate. Choose non-conjugate

maps $f_1, f_2 \in \mathcal{U}$. Let $\theta_i: (\Sigma_d, \sigma, \beta_d) \rightarrow (J(f_i), f_i, \mu_i)$, $i = 1, 2$, be Bernoulli isomorphisms. Identify the alphabet with $\mathbb{Z}/d\mathbb{Z}$ and define $T: \Sigma_d \rightarrow \Sigma_d$ by

$$T(x)_n = x_n + x_{n+1} \pmod{d}.$$

Then $T \circ \sigma = \sigma \circ T$. Moreover, $T_*\beta_d = \beta_d$. Indeed, for every block $(y_0, \dots, y_{N-1})$, choosing x_0 determines recursively a unique block $(x_0, \dots, x_N)$ satisfying $x_n + x_{n+1} = y_n$. Hence there are exactly d preimage blocks, and the corresponding cylinder has probability d^{-N} .

Set $\lambda_T := (\text{id}, T)_*\beta_d$ and $\nu_T := (\theta_1 \times \theta_2)_*\lambda_T$. Since $(\pi_1)_*\lambda_T = \beta_d$ and $(\pi_2)_*\lambda_T = T_*\beta_d = \beta_d$, the two marginals of ν_T are μ_1 and μ_2 . Hence ν_T is a joining of f_1 and f_2 . It is ergodic as λ_T is the image of the ergodic system $(\Sigma_d, \sigma, \beta_d)$ under the equivariant map $x \mapsto (x, T(x))$. It is non-product since its conditional measure over the first coordinate is a Dirac mass, whereas the corresponding conditional measure for $\mu_1 \times \mu_2$ is the non-atomic measure μ_2 .

Moreover, ν_T has finite atomic conditional measures over the second coordinate. Indeed, λ_T is supported on the graph of T , and every fiber of T consists of exactly d points. Thus the conditional measures of λ_T over the second coordinate, and hence those of ν_T , are supported on at most d points almost everywhere.

We claim that ν_T belongs to case (B) of Theorem 1.1. Suppose not. Then ν_T belongs to case (A), and Theorem 3.1 gives an integer $N \geq 1$ and an irreducible algebraic curve $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ such that $(f_1^N \times f_2^N)(C) = C$ and $\nu_T(C) > 0$. Since the marginals of ν_T are the non-atomic measures μ_1 and μ_2 , the curve C cannot be vertical or horizontal. Thus C is a non-vertical, non-horizontal periodic curve for $f_1 \times f_2$. This contradicts [Pak25b, Theorem 1.2], since f_1 and f_2 were chosen to be non-conjugate. Hence ν_T belongs to case (B).

7. RECURRENCE AND CLUSTER STRUCTURE MODULO GAUGE

The proof of Theorem 1.1 uses only the total cluster family $\mathcal{C}_\nu$. The fiberwise cluster sets $C(z)$, however, contain a finer structure coming from recurrence. The purpose of this section is to show that recurrent returns preserve this cluster structure up to natural changes of coordinates.

Theorem 7.1. *For every $z \in E_{\text{rec}}^\sharp$, the fiberwise cluster sets $C(z)$ and $C(Tz)$ are gauge equivalent. Equivalently, we have $[C(Tz)]_{\text{gau}} = [C(z)]_{\text{gau}}$.*

In Section 7.1, we compare the normalized transfer maps based at z and at its recurrent return Tz and identify the resulting left and right coordinate changes. In Section 7.2, we use these changes to define the gauge equivalence and prove Theorem 7.1.

7.1. Return maps and coordinate changes. Recall that $E_{\text{rec}}^\sharp \subset E^\sharp$ is the recurrent part for $\hat{F}^{-1}$, that $\tau: E_{\text{rec}}^\sharp \rightarrow \mathbb{N}$ is the first return time to $E^\sharp$, and that $Tz = \hat{F}^{-\tau(z)}z$ is the induced return map. The proof of Theorem 7.1 relies on the following comparison between the normalized transfer maps based at z and at Tz .

Lemma 7.2. *Fix $z = (\hat{x}, \hat{y}) \in E_{\text{rec}}^\sharp$, write $Tz = (\hat{x}^+, \hat{y}^+)$, and set $\tau := \tau(z)$. Then, after restricting to sufficiently small neighbourhoods depending on z , there exist a local biholomorphic germ B_z and local biholomorphic germs $A_{z,n}$ such that, for all sufficiently large n , we have*

$$(7.1) \quad \tilde{\Phi}_{z, n+\tau} = A_{z,n} \circ \tilde{\Phi}_{Tz,n} \circ B_z.$$

Moreover, every sequence $n_j \rightarrow \infty$ admits a subsequence along which A_{z,n_j} converges locally uniformly to a locally biholomorphic germ.

Proof. Since $Tz = \hat{F}^{-\tau}z$, we have $\hat{x}^+ = \hat{f}_1^{-\tau}\hat{x}$ and $\hat{y}^+ = \hat{f}_2^{-\tau}\hat{y}$. The inverse branches satisfy

$$G_{1, \hat{x}, n+\tau} = G_{1, \hat{x}^+, n} \circ G_{1, \hat{x}, \tau} \quad \text{and} \quad G_{2, \hat{y}, n+\tau} = G_{2, \hat{y}^+, n} \circ G_{2, \hat{y}, \tau}.$$

Recall that $a_n(z) = G'_{2,\hat{y},n}(0)/G'_{1,\hat{x},n}(0)$. Differentiating the preceding identities gives

$$a_{n+\tau}(z) = a_n(Tz)\lambda_z \quad \text{and} \quad \lambda_z := \frac{G'_{2,\hat{y},\tau}(0)}{G'_{1,\hat{x},\tau}(0)} \in \mathbb{C}^*.$$

Using the definitions of the normalized transfer maps and the cocycle relations, we obtain

$$\Phi_{z,n+\tau} = L_{z,n} \circ \Phi_{Tz,n} \circ G_{1,\hat{x},\tau}, \quad \text{where} \quad L_{z,n} := G_{2,\hat{y},\tau}^{-1} \circ G_{2,\hat{y}^+,n}^{-1} \circ (\lambda_z \cdot \text{id}) \circ G_{2,\hat{y}^+,n}.$$

All these identities are understood on sufficiently small neighbourhoods on which the compositions are defined. Passing to the fixed reference charts gives (7.1), with

$$A_{z,n} := k_{\hat{y}} \circ L_{z,n} \circ k_{\hat{y}^+}^{-1} \quad \text{and} \quad B_z := h_{\hat{x}^+}^{-1} \circ G_{1,\hat{x},\tau} \circ h_{\hat{x}}.$$

The germ B_z is a local biholomorphism depending only on the finite return segment. For fixed z , the only n -dependent part of $A_{z,n}$ is

$$H_n := G_{2,\hat{y}^+,n}^{-1} \circ (\lambda_z \cdot \text{id}) \circ G_{2,\hat{y}^+,n}.$$

By Proposition 2.1, the maps $G_{2,\hat{y}^+,n}$ are univalent on a common disc and satisfy uniform Koebe estimates there. Choose $t > 0$ sufficiently small so that $G_{2,\hat{y},\tau}^{-1}$ is defined on $D(0, t)$. Since λ_z is fixed, we may then choose $0 < s < t$, depending on z but not on n , such that

$$\lambda_z G_{2,\hat{y}^+,n}(D(0, s)) \subset G_{2,\hat{y}^+,n}(D(0, t))$$

for every n . Indeed, the Koebe distortion and growth estimates give uniform inner and outer bounds for these images in terms of $|G'_{2,\hat{y}^+,n}(0)|$. It follows that H_n is defined on $D(0, s)$ for every n and satisfies $H_n(D(0, s)) \subset D(0, t)$. Hence $\{H_n\}$ is a normal family on $D(0, s)$. Moreover, $H'_n(0) = \lambda_z$ for every n . Consequently, every locally uniform cluster limit of the H_n has non-zero derivative at the origin and is therefore locally biholomorphic after restriction.

Since $A_{z,n}$ is obtained from H_n by pre- and post-composition with fixed local biholomorphisms, after restricting to a sufficiently small fixed neighbourhood of the base point, the normality of $\{H_n\}$ gives normality of $\{A_{z,n}\}$. Every cluster limit is locally biholomorphic as the same holds for the corresponding cluster limit of $\{H_n\}$. $\square$

7.2. Gauge equivariance of fiberwise cluster sets. We now formalize the coordinate ambiguity appearing in Lemma 7.2. Recall from Proposition 2.5 (4) that, for $z = (\hat{x}, \hat{y}) \in E_{\text{rec}}^\sharp$, we set $q_z := \chi_a(x_0)$. When an element of $C(z)$ is regarded as a germ, we mean its germ at q_z . We denote by $\mathcal{G}_R$ the local pseudo-group generated by the right return germs B_z , $z \in E_{\text{rec}}^\sharp$, together with their restrictions, local inverses, and compositions whenever defined. Similarly, we denote by $\mathcal{G}_L$ the local pseudo-group generated by the germs $A_{z,n}$, for $z \in E_{\text{rec}}^\sharp$ and all sufficiently large n , and by all compact-open limits of sequences A_{z,n_j} with z fixed and $n_j \rightarrow \infty$, together with their restrictions, local inverses, and compositions whenever defined.

Definition 7.3. Two local holomorphic germs Ψ_1 and Ψ_2 in the fixed reference coordinates are *gauge equivalent* if, after restriction to sufficiently small neighbourhoods, we have $\Psi_2 = A \circ \Psi_1 \circ B$ for some $A \in \mathcal{G}_L$ and $B \in \mathcal{G}_R$. Two families K_1, K_2 of such germs are *gauge equivalent* if every element of K_1 is gauge equivalent to an element of K_2 , and conversely. We write $[K]_{\text{gau}}$ for the resulting equivalence class.

We can now prove Theorem 7.1.

Proof of Theorem 7.1. Fix $\tilde{\Psi} \in C(Tz)$. Choose $n_j \rightarrow \infty$ such that $\tilde{\Phi}_{Tz,n_j} \rightarrow \tilde{\Psi}$ locally uniformly. By Lemma 7.2, after passing to a subsequence we may assume that $A_{z,n_j} \rightarrow A$ locally uniformly for a locally biholomorphic germ A . Moreover, we have

$$\tilde{\Phi}_{z,n_j+\tau(z)} = A_{z,n_j} \circ \tilde{\Phi}_{Tz,n_j} \circ B_z$$

as germs on a fixed neighbourhood.

By normality of the family $\{\tilde{\Phi}_{z,n}\}_{n \geq 1}$, after passing to a further subsequence we may assume that $\tilde{\Phi}_{z,n_j+\tau(z)} \rightarrow \tilde{\Theta}$ locally uniformly on $D(0, r^\sharp)$ for some $\tilde{\Theta} \in C(z)$. Passing to the limit in the preceding identity gives, as germs, $\tilde{\Theta} = A \circ \tilde{\Psi} \circ B_z$. Thus every element of $C(Tz)$ is gauge equivalent to an element of $C(z)$.

Conversely, take $\tilde{\Theta} \in C(z)$ and choose $m_j \rightarrow \infty$ such that $\tilde{\Phi}_{z,m_j} \rightarrow \tilde{\Theta}$ locally uniformly. After discarding finitely many terms, write $m_j = n_j + \tau(z)$ with $n_j \rightarrow \infty$. By Lemma 7.2, after passing to a subsequence we may assume that $A_{z,n_j} \rightarrow A$ locally uniformly. Normality of the transfer maps also allows us to assume that $\tilde{\Phi}_{Tz,n_j} \rightarrow \tilde{\Psi}$ for some $\tilde{\Psi} \in C(Tz)$. Passing to the limit in

$$\tilde{\Phi}_{z,m_j} = A_{z,n_j} \circ \tilde{\Phi}_{Tz,n_j} \circ B_z$$

gives again, as germs, $\tilde{\Theta} = A \circ \tilde{\Psi} \circ B_z$. This completes the proof. $\square$

Remark 7.4. Theorem 7.1 is especially useful when $\mathcal{A}_\nu = \emptyset$. In that case no individual cluster germ occurs on a set of positive lifted joining measure, but recurrent returns still transport the fiberwise cluster sets into one another modulo gauge.

8. CLASSICAL EXCEPTIONAL MAPS ARE JOINING-EXCEPTIONAL

In this section, we specialize to the diagonal case $f_1 = f_2 = f$ and show that the classical exceptional rational maps are joining-exceptional. Recall that *classical exceptional maps* are rational maps that are Möbius-conjugate to a power map $z \mapsto z^{\pm d}$, to a signed Chebyshev map $\pm T_d$, where $d \geq 2$ and T_d is normalized by $T_d(z + z^{-1}) = z^d + z^{-d}$, or to a Lattès map (i.e., a finite quotient of an affine endomorphism of an elliptic curve).

Proposition 8.1. *Every classical exceptional rational map of degree at least 2 is joining-exceptional.*

Proof. It suffices to prove the statement for the models z^d , T_d , and the Lattès maps. Indeed, joining-exceptionality is invariant under Möbius conjugacy: if $g = M \circ f \circ M^{-1}$ and $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ satisfies the requirements in the definition of joining-exceptionality for f , then $(M \times M)(C)$ satisfies the same requirements for g . Moreover, the definition of joining-exceptionality only requires an invariant curve for some iterate. Thus the signed models $z^{\pm d}$ and $\pm T_d$ reduce to z^d and T_d , as we have $(z^{-d})^{\circ 2} = z^{d^2}$, and for the Chebyshev maps, $-T_d$ is Möbius-conjugate to T_d when d is even, while $(-T_d)^{\circ 2} = T_{d^2}$ when d is odd. Hence it remains only to construct the required invariant curves for z^d , T_d , and the Lattès maps.

First suppose that $f(z) = z^d$ with $d \geq 2$. Let $m, n > 1$ be coprime integers, and let $C_{m,n} \subset \mathbb{P}^1 \times \mathbb{P}^1$ be the image of the map $t \mapsto (t^n, t^m)$. In affine coordinates, $C_{m,n}$ is given by the equation $y^n = x^m$. Since $\gcd(m, n) = 1$, this parametrization is birational onto its image. Hence $C_{m,n}$ is irreducible. Both coordinate projections are dominant, with generic degrees $n > 1$ and $m > 1$, respectively. Moreover, $(f \times f)(C_{m,n}) = C_{m,n}$. Indeed, if $y^n = x^m$, then $(y^d)^n = (x^d)^m$, so $(f \times f)(C_{m,n}) \subset C_{m,n}$. Since $f \times f$ is finite, the image $(f \times f)(C_{m,n})$ is an irreducible algebraic curve. Hence the above inclusion is an equality. Therefore f is joining-exceptional.

Next suppose that $f = T_d$ is a Chebyshev map. Denote $\eta(z) := z + z^{-1}$ so that $\eta \circ (z \mapsto z^d) = T_d \circ \eta$. Let $C_{m,n}$ be the curve constructed above for the monomial $z \mapsto z^d$, and define

$$D_{m,n} := (\eta \times \eta)(C_{m,n}) \subset \mathbb{P}^1 \times \mathbb{P}^1.$$

Since $\eta \times \eta$ is finite and $C_{m,n}$ is irreducible, $D_{m,n}$ is irreducible. The curve $D_{m,n}$ is parametrized by $t \mapsto (\eta(t^n), \eta(t^m)) = (T_n(\eta(t)), T_m(\eta(t)))$. Hence both projections are dominant. We only need to check that the two projections have generic degree strictly greater than one. For the first projection, choose a non-trivial n -th root of unity ζ . Then t and ζt have the same first coordinate,

since $\eta((\zeta t)^n) = \eta(t^n)$. On the other hand, since $\gcd(m, n) = 1$, we have $\zeta^m \neq 1$, and for generic t the two second coordinates $\eta(t^m)$ and $\eta(\zeta^m t^m)$ are distinct. Hence the first projection is not generically one-to-one, and so it has generic degree strictly greater than one. The same argument, exchanging m and n , applies to the second projection. Moreover, since the product map $(z \mapsto z^d) \times (z \mapsto z^d)$ preserves $C_{m,n}$ and $\eta \circ (z \mapsto z^d) = T_d \circ \eta$, we have $(T_d \times T_d)(D_{m,n}) = D_{m,n}$. Thus T_d is joining-exceptional.

Finally suppose f is a Lattès map. Then there exist an elliptic curve E , a finite holomorphic map $\Theta: E \rightarrow \mathbb{P}^1$ and an affine endomorphism $L: E \rightarrow E$ such that $\Theta \circ L = f \circ \Theta$. Since $\deg f \geq 2$, the affine map L has a fixed point; indeed, writing $L(t) = \alpha(t) + c$, the endomorphism $\text{id} - \alpha$ is non-zero and hence surjective. After translating the origin of E to such a fixed point, and replacing Θ by the corresponding translated covering map, we may assume that L fixes the origin of E , so that L is a group endomorphism. Set

$$T_\Theta := \{a \in E : \Theta(t + a) = \Theta(t) \text{ for every } t \in E\}.$$

This is a finite subgroup of E , since it acts freely by translations on every fiber of the finite map Θ . For $k \geq 1$, let $[k]: E \rightarrow E$ denote multiplication by k , and let $E[k] := \ker[k]$ be the k -torsion subgroup of E . Since T_Θ is finite, we can choose coprime integers $m, n > 1$ such that $E[m] \not\subset T_\Theta$ and $E[n] \not\subset T_\Theta$. Set $M_1 := [m]$ and $M_2 := [n]$. Since multiplication maps commute with L , both M_1 and M_2 commute with L . Set

$$\tilde{C} := \{(M_1(t), M_2(t)) : t \in E\} \subset E \times E \quad \text{and} \quad C := (\Theta \times \Theta)(\tilde{C}) \subset \mathbb{P}^1 \times \mathbb{P}^1.$$

The curve C is irreducible as it is the image of the irreducible curve E under the finite map $\Theta \times \Theta$. Both projections are dominant. Moreover, neither projection is generically one-to-one. Indeed, suppose that $\pi_1|_C$ has generic degree one. Then, for generic $t \in E$ and every $s \in E[m]$, the points t and $t + s$ give the same first coordinate on C , hence also the same second coordinate. Thus $\Theta(n(t + s)) = \Theta(nt)$ for generic t , and therefore for every $t \in E$. Since $[n]: E \rightarrow E$ is surjective, it follows that $ns \in T_\Theta$ for every $s \in E[m]$. Since $\gcd(m, n) = 1$, multiplication by n permutes $E[m]$, and hence $E[m] \subset T_\Theta$, a contradiction. Therefore $\deg(\pi_1|_C) > 1$. The same argument, exchanging m and n , gives $\deg(\pi_2|_C) > 1$. Finally, since M_1 and M_2 commute with L and L is surjective, we have $(L \times L)(\tilde{C}) = \tilde{C}$. Applying $\Theta \times \Theta$ and using $\Theta \circ L = f \circ \Theta$, we obtain $(f \times f)(C) = C$. Thus f is joining-exceptional. This completes the proof. $\square$

9. FURTHER DIRECTIONS AND QUESTIONS

The results of this paper suggest several directions in which joining rigidity in rational dynamics may be developed further. We first discuss the boundary between measurable and holomorphic rigidity and the joining-exceptional alternative, and then turn to higher-order joinings and higher-dimensional holomorphic dynamics.

9.1. Measurable versus holomorphic rigidity. Theorem 1.1 naturally raises the question of recognizing when a purely measurable joining charges the graph of a local biholomorphism. The examples in Section 6 show why this question is genuinely analytic rather than merely a question about the existence of measurable dependence.

Another problem is to understand more precisely the joining-exceptional alternative. In the diagonal case $f_1 = f_2 = f$, one natural problem is to characterize joining-exceptional maps more explicitly in dynamical terms. Pakovich's classification of diagonal invariant curves, applied to a suitable iterate of f , provides a natural starting point for this question; see [Pak23] and Section 1.3.

For a general pair (f_1, f_2) , the corresponding problem is to characterize those pairs for which an iterate of $f_1 \times f_2$ preserves an irreducible curve whose two coordinate projections have degree

greater than one. Pakovich's work gives a structural description of invariant curves for broad non-exceptional classes of pairs [Pak23]. It would be interesting to reformulate this description as a more direct dynamical criterion for this property and to understand the remaining cases.

Another question is to understand the non-discrete local analytic families arising in case (B). In particular, one may ask which additional dynamical conditions force one member of such a family to occur on positive joining measure, and hence force the algebraic conclusion of Theorem 3.1.

9.2. Higher-order joinings. Another direction is to consider higher-order joinings; see, for example, [GHR92; JR08]. For instance, a three-fold self-joining ν on $J(f)^3$ has three pairwise projections, each of which is an ordinary self-joining, but these pairwise projections need not determine ν .

At the purely measurable level, higher-order phenomena can already occur without being visible in the pairwise projections: the Bernoulli shift on two symbols with equal weights admits an ergodic non-product three-fold joining all of whose two-coordinate projections are products. Thus a useful higher-order theory should distinguish between such purely measurable joinings and joinings which produce analytic or algebraic relations.

A natural geometric question is whether ergodic three-fold self-joinings carrying positive-mass local analytic relations can be classified in terms of invariant algebraic subvarieties of $(\mathbb{P}^1)^3$. A natural graph-type model, after passing to an iterate f^r , is a subvariety of the form $\{(x, A(x), B(x)) : x \in \mathbb{P}^1\}$, where A and B commute with an iterate of f . More generally, for joinings of several possibly different rational maps, one may expect compatible systems of local or algebraic correspondences among the coordinates.

9.3. Higher-dimensional holomorphic dynamics. Finally, it is natural to seek an analogue of Theorem 1.1 for holomorphic endomorphisms of $\mathbb{P}^k$, $k \geq 2$, and their equilibrium measures. As in dimension one, the corresponding measure-preserving systems are Bernoulli; see, e.g., [DS10, Section 1.6] and the references therein. One may ask whether positive joining mass on a local biholomorphic relation forces an invariant algebraic correspondence, and whether, in the absence of such a relation, normalized transfer maps still produce a compact non-discrete family of holomorphic cluster maps.

The first difficulty appears already in the construction of transfer maps. In dimension one, if $G_{1,\hat{x},n}$ and $G_{2,\hat{y},n}$ are inverse branches in the two coordinates, the normalization

$$G_{2,\hat{y},n}^{-1} \circ \left(\frac{G'_{2,\hat{y},n}(0)}{G'_{1,\hat{x},n}(0)} \text{id} \right) \circ G_{1,\hat{x},n}$$

has derivative equal to 1 at the base point. More importantly, Koebe distortion converts this exact derivative normalization into uniform control on fixed domains.

In higher dimension, one can formally normalize the derivatives in a similar way and may consider a linear normalization involving $DG_{2,\hat{y},n}(0)DG_{1,\hat{x},n}(0)^{-1}$. The difficulty is that such a normalization does not by itself give uniform geometric control. The derivative cocycles may have several Lyapunov directions with very different contraction rates, and the normalizing linear maps may become highly anisotropic. Thus the one-dimensional argument based on univalence and Koebe distortion does not apply directly.

Higher-dimensional inverse-branch estimates, such as those of Berteloot–Dupont–Molino [BDM08] and Berteloot–Dupont [BD19], provide substitutes for Koebe distortion adapted to the Lyapunov splitting, but involve tempered coordinate changes and small exponential errors. To adapt the transfer-map argument to joinings, one would need to choose compatible normalizations for the inverse branches in the two coordinates and prove compactness on fixed reference domains.

The positive-mass local-relation case would also require a higher-dimensional local-to-global rigidity theorem. In dimension one, the generalized Inou theorem [JX23b, Theorem 2.1] used in Section 3 promotes the local intertwining relation between inverse branches to an invariant algebraic curve. In higher dimension, one would need conditions under which an analogous local relation extends

to an invariant algebraic correspondence in $\mathbb{P}^k \times \mathbb{P}^k$. Thus a higher-dimensional joining rigidity theorem would likely require both compactness for suitably normalized transfer maps and a higher-dimensional local-to-global rigidity result.

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