

GENERALIZED LEVI CURRENTS AND SINGULAR LOCI FOR FAMILIES OF PLURISUBHARMONIC FUNCTIONS

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ABSTRACT. We show how the formalism of Levi currents on complex manifolds, as introduced by Sibony, can be used to study the analytic structure of singular sets associated to families of plurisubharmonic functions, in the sense of Slodkowski.

1. INTRODUCTION

The theory of several complex variables originated as the study of holomorphic functions; however, soon enough, plurisubharmonic functions made their appearance and proved themselves as a useful instrument to solve problems originated in the holomorphic category.

Probably, one of the best known and oldest examples of this phenomenon is the Levi problem ([14], [36] for a survey): characterizing domains of holomorphy in terms of the pseudoconvexity of the boundary; Oka's solution [24, 25] for domains in \mathbb{C}^n highlighted the role of strictly plurisubharmonic exhaustions, whose existence was later shown to be equivalent to Steinness, by Grauert for manifolds [9] and Narasimhan for analytic spaces [21, 22].

The geometric counterpart of holomorphic functions is represented by complex analytic varieties, which are locally given as zeroes of holomorphic functions. This notion does not have a straightforward analogue for plurisubharmonic functions: they lack the rigidity of holomorphic functions, hence the geometry they describe with their level sets is not significant, in relation with complex analytic varieties; in fact, quite the opposite is true: the level sets of a strictly plurisubharmonic function do not support any kind of complex structure and are examples of B-regular sets (i.e., sets where the (restrictions of) plurisubharmonic functions are dense in the continuous ones, introduced in [31, 32]), which play an important role in the study of the regularity of the $\bar{\partial}$ -problem (see [31]).

However, there is a rigidity property shared between holomorphic and plurisubharmonic functions: holomorphic functions obey a maximum modulus property, which, for instance, characterizes algebras of holomorphic functions on plane domains (see Rudin [29]), and (pluri)subharmonic functions likewise satisfy a maximum property. Indeed, Rudin's theorem can be deduced as a special case of subharmonicity, in turn obtained via the maximum property [41].

It is quite clear that the maximum property for plurisubharmonic functions does not hold on arbitrary sets, for example it does not hold on B-regular sets; on the other hand, given an open domain of an analytic variety, plurisubharmonic functions attain

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their maximum on the boundary. It is therefore reasonable to expect that sets where the maximum property holds should bear some resemblance of complex structure; this idea originated, more or less explicitly, a number of constructions related to function algebras, such as Shilov boundary, Jensen boundary, peak points, Choquet boundaries (see [7] for a comprehensive exposition), and some of these were also employed to give more general definitions of plurisubharmonicity in the context of uniform algebras [8].

By localizing the idea of Jensen boundary, we obtain the notion of local maximum set for h -plurisubharmonic functions [37]; these sets enjoy many properties of analytic varieties, with $h + 1$ playing the role of the dimension. Moreover, local maximum sets for plurisubharmonic functions (usually just called local maximum sets) are 1-pseudoconcave, in the sense that their complement is $(n - 2)$ -pseudoconvex according to Rothstein [28], a property which is true also for complex analytic varieties of dimension at least 1. So, for instance, the largest local maximum set contained in the boundary of a pseudoconvex set will contain any germ of analytic variety and, more generally, any positive, $\partial\bar{\partial}$ -closed current of bidimension $(1, 1)$ and with compact support [23].

In particular, compact local maximum sets, just like compact analytic varieties, force plurisubharmonic functions (and hence holomorphic functions) to be constant on them; therefore, no strictly plurisubharmonic function can exist in a neighbourhood of a compact local maximum set. This consideration was the core of the investigation of weakly complete spaces, started by Slodkowski and Tomassini in [39], with the definition of the kernel of a weakly complete space (which is the set of points where no plurisubharmonic exhaustion can be strictly plurisubharmonic) and continued by them and the second author in a series of papers [15–20].

At the same time, Sibony introduced and studied ([33–35]) the idea of Levi current: a positive current of bidimension $(1, 1)$ which is $\partial\bar{\partial}$ -closed and vanishes when wedged with $\partial\bar{\partial}u$, for u a plurisubharmonic function; in a previous paper [2], we investigated the relation between Levi currents, local maximum sets and the kernel of a weakly complete space [39]. It is quite clear that local maximum sets are, as the name suggests, just sets, whereas currents imply more structure; in fact, while it is clear that the support of a Levi current is a local maximum set and while it is possible, given a local maximum set, to produce a Levi current with support contained in it, we do not know yet whether it is true that every local maximum set *is* the support of a Levi current.

It is interesting to note that, already in [1, Remark 1.5 - (i)], the authors noted the link between local maximum sets and pluriharmonic positive currents.

Again in the work [34], Sibony also briefly mentioned Liouville currents, whose definition is similar to the one of Levi current, but in the last property: one asks that $T \wedge \partial\bar{\partial}u = 0$ for every u plurisubharmonic and bounded; as Levi currents are related to the kernel of a weakly complete space, Liouville currents could be linked to the core of a complex space, as defined and investigated by Harz, Shcherbina, and Tomassini in [11–13], which is the set of points where no bounded plurisubharmonic function can be strictly plurisubharmonic.

In a recent paper [38], Slodkowski showed that the core is a union of “primitive” sets, which are 1-pseudoconcave and enjoy a Liouville property, namely every bounded plurisubharmonic function is constant on them; this result had been previously obtained for complex surfaces in [12] and was also independently proved in the general case by Poletski and Shcherbina in [26]. Slodkowski proved this by tackling a more

general problem, with respect to a family of plurisubharmonic functions, satisfying some given properties, called admissible class.

In the present paper, we intend to deepen the investigation of the relations between plurisubharmonic functions (and local maximum sets) and positive currents of bidimension $(1, 1)$ (as generalizations of complex analytic varieties or Levi-flat sets), by expanding the results of [2]: we introduce a generalization of Levi currents (the \mathcal{F} -currents, see Definition 3.1), where the condition $T \wedge \partial\bar{\partial}u$ should hold for u in an *admissible class* \mathcal{F} , as defined by Slodkowski in [38], see Definition 2.1. To any such class, we can also associate a *singular locus*, as the set where no element of the class can be strictly plurisubharmonic. Examples of admissible classes, and of the corresponding singular sets, are given by all plurisubharmonic function (and the *minimal kernels*, as introduced in [39]), or by all bounded plurisubharmonic functions (and the *cores*, as introduced in [11]).

The following is our main result.

Theorem 1.1. *Let X be a complex manifold and let \mathcal{F} be an admissible class.*

- (1) *All \mathcal{F} -currents are supported in the singular locus of \mathcal{F} . If \mathcal{F} contains an exhaustion function, the singular locus is empty if and only if there are no \mathcal{F} -currents.*
- (2) *All elements of \mathcal{F} are constant on the supports of extremal \mathcal{F} -currents.*
- (3) *There exists an \mathcal{F} -current whose support is equal to the union of the supports of all \mathcal{F} -currents.*
- (4) *Assume that $\mathcal{F}(X)$ contains an exhaustion function. Let T be a \mathcal{F} -current. If $\text{spt } T$ is compact, then it is a local maximum set.*
- (5) *Assume that $K \subset X$ is an \mathcal{F} -component, or a compact local maximum set. Then there exists $T \in \hat{\mathcal{F}}$ such that $\text{spt } T \subseteq K$.*

The paper is organized as follows. In Section 2 we define admissible classes and their associated singular loci, and we state the properties that we need in the sequel. In Section 3 we introduce the notion of \mathcal{F} -currents, and study such currents and their supports. The proof of Theorem 1.1 is then given in Section 4. We conclude the paper with some remarks about the decomposition of \mathcal{F} on the levels sets of elements of \mathcal{F} , and on the localization of the definitions in the paper to the case of compact subsets, see Section 5.

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2. ADMISSIBLE FAMILIES AND SINGULAR LOCI

In this section, following [38], we define admissible class and the associated singular loci, and we recall their properties that we will need in the sequel. We fix a complex manifold X . We say that an open subset $U \subset X$ is *allowable* if it is either relatively compact in X or cocompact. Notice in particular that X is allowable.

Definition 2.1. An *admissible class* \mathcal{F} is the datum, for every allowable open set $U \subset X$, of a family $\mathcal{F}(U)$ of continuous plurisubharmonic functions satisfying the following properties for every allowable sets W, U, U_1, \dots, U_m :

- (A1) for every sequence $\phi_n \in \mathcal{F}(X)$, there exists a sequence of positive $\epsilon_n \in \mathbb{R}$ such that the series $\sum_{n=1}^{\infty} \epsilon_n \phi_n$ converges locally uniformly to an element of $\mathcal{F}(X)$;
- (A2) whenever $\phi \in \mathcal{F}(U)$ and $W \subset U$, then $\phi|_W \in \mathcal{F}(W)$;
- (A3) if $\{U_1, \dots, U_m\}$ is a finite cover of X and $\phi: X \rightarrow \mathbb{R}$ is such that $\phi|_{U_i} \in \mathcal{F}(U_i)$ for all $1 \leq i \leq m$, then $\phi \in \mathcal{F}(X)$;
- (A4) the set $\mathcal{F}(U)$ is a convex cone and contains all bounded \mathcal{C}^∞ plurisubharmonic functions on U ;
- (A5) for every $\phi_1, \dots, \phi_m \in \mathcal{F}(U)$, and every \mathcal{C}^∞ convex function $v: \mathbb{R}^m \rightarrow \mathbb{R}$ of at most linear growth and such that $\frac{\partial v}{\partial t_i} \geq 0, i = 1, \dots, m$ on the joint range of (ϕ_1, \dots, ϕ_m) , the function $\phi := v(\phi_1, \dots, \phi_m)$ belongs to $\mathcal{F}(U)$;
- (A6) for every $\phi \in \mathcal{F}(U)$ which is strongly plurisubharmonic on U and $\rho \in \mathcal{C}^\infty(U)$ with $\overline{\text{spt } \rho} \subset U$, there exists $t > 0$ such that $\phi + t\rho \in \mathcal{F}(U)$.

Remark 2.2. Observe that condition (A4) implies the following property:

- (A7) every point $p \in X$ admits an open neighbourhood Ω such that, for all allowable open subset $V \subset \Omega$, $\mathcal{F}(V) \cap \mathcal{C}^\infty$ is dense (for the topology of locally uniform convergence) in $\mathcal{F}(V)$

We will use this property a number of times in the following, hence we prefer to add it to the list of properties of an admissible class.

Remark 2.3. In general, an allowable class \mathcal{F} is not closed by maximum, i.e., the maximum of two elements in a given class \mathcal{F} is not necessarily in \mathcal{F} . On the other hand, by the condition (A5), the class is closed by \max_ϵ for all positive ϵ , where $\max_\epsilon(x, y)$ is any smooth approximation of the function $\max(x, y)$.

Definition 2.4. The *singular locus* of an admissible class \mathcal{F} is the complement $\Sigma^\mathcal{F} = \Sigma_X^\mathcal{F}$ of the set of points $x \in X$ such that there exists $\phi \in \mathcal{F}(X)$ which is strongly plurisubharmonic at x .

Definition 2.5. An \mathcal{F} -component is an equivalence class of the relation \sim defined as follows: $x \sim y$ if $\phi(x) = \phi(y)$ for all $\phi \in \mathcal{F}$.

Observe that this relation is meaningful mostly on $\Sigma^\mathcal{F}$. Indeed, the component of a point outside of $\Sigma^\mathcal{F}$ is given by that single point.

The following are examples of allowable families, giving rise to well-studied singular sets, see [38, Section 5] for more details.

Example 2.6. Let us consider, for a given $k \in \mathbb{N} \cup \{\infty\}$, the class \mathcal{F} defined as follows:

- the set of all lower-bounded, \mathcal{C}^k psh functions on U , for all relatively compact open set U , and
- the set of all lower-bounded, \mathcal{C}^k psh functions ϕ on U , such that the sub-levels sets $\{\phi \leq c\}$ are relatively compact in X for all $c \in \mathbb{R}$, for all cocompact open set U .

The singular loci associated to \mathcal{F} as above are the *minimal kernels*, as introduced and studied in [18, 39]; in particular, in the series of paper [15–20], the authors considered

the case of complex surfaces such that $\mathcal{F}(X)$ contains a real analytic exhaustion. As a consequence of this detailed study, one notices that, in such a case, the singular locus does not depend on the regularity k .

Example 2.7. Let us now consider, for a given $k \in \mathbb{N} \cup \{\infty\}$ the class \mathcal{F} given, on any admissible open set U , by all uniformly bounded \mathcal{C}^k plurisubharmonic functions.

The singular loci in this case correspond to the *cores*, as introduced and studied in [11–13]. It is known that, in the case of cores, regularity plays a role (see [10]).

In the remaining part of this section, we summarize the main properties of singular loci, mainly from [38], that we will need in the sequel. We first need to recall the following further definitions.

Definition 2.8. Given an admissible class \mathcal{F} , an element $\phi \in \mathcal{F}(X)$ is a \mathcal{F} -*minimal function* if it is strongly plurisubharmonic on $X \setminus \Sigma_X^{\mathcal{F}}$.

Definition 2.9. Let Z be a locally closed set. We say that Z is a local maximum set if every $x \in Z$ admits a neighbourhood V with the following property: for every compact set $K \subset V$ and every function ϕ which is psh in a neighbourhood of K , we have

$$(1) \quad \max_{Z \cap K} \psi = \max_{Z \cap \text{ob}K} \psi.$$

Remark 2.10. More generally, given an admissible class \mathcal{F} as in Definition 2.1, one may say that Z is a local maximum set for \mathcal{F} if the condition in Definition 2.9 is satisfied for all $\phi \in \mathcal{F}(V)$, where V is an allowable open neighbourhood of Z . However, it is not difficult to see that this is actually equivalent to be a local maximum set. It is clear that any local maximum set is a local maximum set for \mathcal{F} . On the other hand, take let Z be a local maximum set for \mathcal{F} . Take any point of Z , an open neighbourhood U and a compact set $K \subset U$. We can, without loss of generality, reduce U and assume that it is relatively compact, hence admissible. Take a psh function ϕ on a neighbourhood of K . Again, without loss of generality, we can assume that ϕ is defined on U . If ϕ is smooth, by condition **(A4)** in Definition 2.1, ϕ belongs to $\mathcal{F}(U)$. Hence (1) holds for ϕ . The statement for a general upper semicontinuous plurisubharmonic ϕ now follows by approximation.

Theorem 2.11 (Theorem 4.2 of [37]). *Let X be a n -dimensional complex manifold. A closed set Z is a local maximum set if and only if it is 1-pseudoconcave: it can be covered by open sets V_i such that $V_i \setminus Z$ admits a $(n-2)$ -plurisubharmonic exhaustion function.*

Recall that, for a \mathcal{C}^2 function, to be $(n-2)$ -plurisubharmonic means that its complex Hessian has at least 2 non-negative eigenvalues.

The next result gives a characterization of local maximum sets in terms of the local behaviour of admissible functions. Although, by Remark 2.10, such result is implied by [37, Proposition 2.3], we give here a proof of this, to show how the definition of admissible classes precisely allows one to work as if doing so in the algebra of psh functions.

Proposition 2.12. *Let Z be a locally closed set and \mathcal{F} be an admissible class. The following conditions are equivalent.*

- (1) Z is a local maximum set for the class \mathcal{F} ;
- (2) there do not exist $z^* \in Z$, $r > 0$, $\epsilon > 0$ and a strictly psh function u in $\mathcal{F}(B(z^*, r))$ such that $u(z^*) = 0$ and $u(z) \leq -\epsilon|z - z^*|^2$ for $z \in Z \cap B(z^*, r)$.

Proof. First, it is clear that the existence of a function u as in the second item contradicts the fact that Z is a local maximum set for the class \mathcal{F} .

For the other implication, suppose that Z is not a local maximum set for the class \mathcal{F} . Then there exists a compact subset $K \subset Z$, an allowable open neighbourhood U of K , and an element $u_0 \in \mathcal{F}(U)$ such that

$$\max_{K \cap Z} u_0 > \max_{\partial K \cap Z} u_0.$$

By the density of smooth elements in $\mathcal{F}(U)$, we can assume that $u_0 \in \mathcal{C}^2(U)$. Then, [37, Lemma 2.2] gives the existence of a strictly convex function $f: U \rightarrow \mathbb{R}$ and a point $x^* \in K \setminus \partial K$ such that

$$(u_0 + f)(x^*) = 0 \quad \text{and} \quad (u_0 + f)(x) \leq -\epsilon|x - x^*|^2 \text{ for } x \in K.$$

(the lemma is stated in \mathbb{C}^n – and actually \mathbb{R}^n , just for upper semicontinuous functions – but the construction is local). By taking r sufficiently small, the function $u := u_0 + f$ is strictly psh on $B(x^*, r)$ and satisfies the requirements in the statement. \square

Theorem 2.13 ([38]). *Let \mathcal{F} be an admissible class.*

- (1) there exists a minimal function in \mathcal{F} ;
- (2) the singular locus $\Sigma^{\mathcal{F}}$ is a local maximum set (hence 1-pseudoconcave), or empty;
- (3) all \mathcal{F} -components of points in $\Sigma^{\mathcal{F}}$ are 1-pseudoconcave;
- (4) if $x \notin \Sigma^{\mathcal{F}}$, then the \mathcal{F} -component containing x is $\{x\}$.

Remark 2.14. Items (1) and (2) in Theorem 2.13 were proved in [39] in the case of minimal kernels (see Example 2.6) and in [11] in the case of cores (see Example 2.7); the decomposition in \mathcal{F} -components for cores was already proved in the 2-dimensional case in [12] and extended to every dimension by Poletski and Shcherbina in [26].

3. GENERALIZED LEVI CURRENTS

In this section we fix a complex manifold X . Following the definition of Levi currents by Sibony [34], we define a natural generalization of this notion adapted to any admissible class as in Definition 2.1.

Definition 3.1. Let \mathcal{F} be an admissible class. An \mathcal{F} -current is a current T on X satisfying the following properties:

- (C1) T is non-zero;
- (C2) T is of bidimension $(1, 1)$;
- (C3) T is positive;
- (C4) $i\partial\bar{\partial}T = 0$;
- (C5) $T \wedge i\partial\bar{\partial}u = 0$ for all $u \in \mathcal{F}(U)$ for U an allowable neighbourhood of the support of T .

We denote by $\hat{\mathcal{F}}$ the set of all \mathcal{F} -currents. We say that an \mathcal{F} -current is *extremal* if $T = T_1 = T_2$ whenever $T = (T_1 + T_2)/2$ for T_1, T_2 \mathcal{F} -currents.

Example 3.2. When \mathcal{F} is as in Example 2.6 (resp. Example 2.7), we recover the definition of Levi (resp. Liouville) currents, as in [34].

The following lemma permits to extend the definition of the intersections between \mathcal{F} -currents and some exact forms. The arguments of the proof are given in [34] and are based on a method developed in [4]. We use here **(A7)** in order to (locally) approximate continuous elements of \mathcal{F} with smooth ones.

Lemma 3.3. *Let \mathcal{F} be an admissible class. Take $u, u_n \in \mathcal{F}$, with u_n smooth and such that $u_n \rightarrow u$, and let T be a positive closed current on X of bidimension $(1, 1)$. Then the current $T \wedge \partial u$ is well defined and*

$$T \wedge \partial u_n \rightarrow T \wedge \partial u.$$

Similar assertions hold for $T \wedge \bar{\partial} u, T \wedge \partial u \wedge \bar{\partial} u$, and $T \wedge \partial \bar{\partial} u$.

The following properties are a consequence of the previous lemma. They are stated in [34, Section 4], see also [2, Lemma 2.3], in the case of Levi currents. Since a similar proof works also in this more general settings, we will omit it here.

Lemma 3.4. *Let \mathcal{F} be an admissible class. Take $u \in \mathcal{F}$ and let T be a \mathcal{F} -current. Then the currents*

$$T \wedge \partial u, \quad T \wedge \bar{\partial} u, \quad \text{and} \quad T \wedge \partial u \wedge \bar{\partial} u$$

are well defined and vanish identically on X .

Corollary 3.5. *Let \mathcal{F} be an admissible class and take $T \in \hat{\mathcal{F}}$. If $u \in \mathcal{F} \cap \mathcal{C}^1$, then the 2-vector field associated to T belongs to the kernel of $i\partial u \wedge \bar{\partial} u$ ($\|T\|$ -almost everywhere), whenever the latter is non-zero.*

In the statement above, $\|T\|$ denotes the mass measure associated to T , see for instance [6, p. 310].

Proof. The statement is equivalent to $T \wedge i\partial u \wedge \bar{\partial} u$, hence follows from Lemma 3.4. \square

The next lemma gives a first indication of the relation between the supports of \mathcal{F} -currents and the points where elements of \mathcal{F} are strictly psh. The case of Levi currents is given in [2, Corollary 2.6 and Lemma 2.7].

Lemma 3.6. *Let \mathcal{F} be an admissible class.*

- (1) *If there exists $u \in \mathcal{F}$ and $x \in X$ such that u is strictly psh at x , then $x \notin \text{spt } T$ for any $T \in \hat{\mathcal{F}}$;*
- (2) *Assume $T \in \hat{\mathcal{F}}$ has compact support. If u belongs to $\mathcal{F}(U)$ for some allowable open neighbourhood of $\text{spt } T$, and is strictly psh at some $x \in U$, then $x \notin \text{spt } T$.*

Proof. We prove the two assertion separately.

- (1) By **(A7)**, we can assume that u is smooth, and strictly psh near x . Let U be a small neighbourhood of x , where u is strictly psh. Then U is allowable, and $u \in \mathcal{F}(U)$. Take a smooth function ρ supported on U . By **(A4)** in Definition 2.1, we have that $u + \rho \in \mathcal{F}(U)$. If now T is an element of $\hat{\mathcal{F}}$, by Lemma 3.4 we must have $T \wedge i\partial(u + \rho) \wedge \bar{\partial}(u + \rho) = 0$. Since ρ is arbitrary, this implies that $T = 0$. Hence, there are no elements of $\hat{\mathcal{F}}$ having x in their support.

- (2) As above, by **(A7)**, we can assume that u is smooth. Since u is strictly psh at x , the same is true in a neighbourhood. In order to prove that $x \notin \text{spt } T$, it is then enough to prove that $T \wedge i\partial\bar{\partial}u = 0$ near x . We prove that this is true in the allowable open set U .

First observe that $T \wedge i\partial\bar{\partial}u$ is a positive measure on U . Consider a second open neighbourhood U' of $\text{spt } T$ with $\text{spt } T \subset U' \Subset U$ and let χ be a smooth function, compactly supported on U and equal to 1 on U' . By replacing u with χu , we can assume that u is defined on X , psh near $\text{spt } T$, and equal to zero near the boundary of U . Since $i\partial\bar{\partial}T = 0$ (by **(C4)** in Definition 3.1) an application of Stokes Theorem gives that $\langle T \wedge i\partial\bar{\partial}u, 1 \rangle = 0$. Since $T \wedge i\partial\bar{\partial}u$ is a positive measure, this completes the proof. \square

A similar application of Stokes Theorem also gives the following result.

Lemma 3.7. *Let \mathcal{F} be an admissible class. Suppose that a current T satisfies the properties **(C1)**-**(C4)** in Definition 3.1. If T has compact support, or if T is supported on a \mathcal{F} -component, then T satisfies the property **(C5)**. In particular, $T \in \hat{\mathcal{F}}$.*

Proof. Assume first that T has compact support. Take U an allowable relatively compact open neighbourhood of the support of T and fix $u \in \mathcal{F}(U)$. The current $T \wedge i\partial\bar{\partial}u$ is well defined by Lemma 3.3 (and conditions **(C2)** and **(C3)**). It is also compactly supported in U and, since T is positive (by **(C3)**), is a positive measure. We show that $\langle T \wedge i\partial\bar{\partial}u, 1 \rangle = 0$. Again by Lemma 3.3 (and conditions **(C2)** and **(C3)**), the currents $T \wedge i\partial u$ and $T \wedge i\bar{\partial}u$ are also well defined. By Stokes theorem, we have that $\langle i\partial\bar{\partial}(uT), 1 \rangle = \langle i\partial(\bar{\partial}u \wedge T), 1 \rangle = \langle i\bar{\partial}(\partial u \wedge T), 1 \rangle = 0$. The assertion follows (again by Stokes theorem and **(C4)**) in this case.

In the case where the support of T is contained in a \mathcal{F} -component, condition **(C5)** in Definition 3.1 is trivially satisfied since any $u \in \mathcal{F}$ is constant on the support of T . This completes the proof. \square

We conclude this section with the following lemma, giving a relation between the existence of \mathcal{F} -currents and the absence of strictly psh functions.

Lemma 3.8. *Let K be a compact set, or a \mathcal{F} -component. If there are no \mathcal{F} -currents supported on K , there exists an allowable open neighbourhood U of K and an element of $\mathcal{F}(U)$ which is strictly psh (on U).*

Proof. Let S be a positive, $\partial\bar{\partial}$ -closed current of bidimension (1,1) supported on K . By Lemma 3.7, S is a \mathcal{F} -current or $S = 0$. By assumption, we have that $S = 0$. Hence, we can assume that there are no non-zero positive $\partial\bar{\partial}$ -closed currents of bidimension (1,1) supported on K . To conclude, we use a duality argument as in [34, Proposition 2.1], see also [23, 40]. Consider the topological vector space of currents of bidimension (1,1) with the topology of weak convergence; denote by C the set of positive currents of bidimension (1,1) of mass 1 (with respect to some Hermitian metric on X) and supported on K and by Y the set of $\partial\bar{\partial}$ -closed currents on X . By the arguments above, we have $C \cap Y = \emptyset$. By Hahn-Banach theorem, C and Y are separated, i.e., there exist $\delta > 0$ and a continuous linear functional L such that $Y \subseteq \ker L$ and $L(T) > \delta$ for all $T \in C$.

By definition, Y is the annihilator of the space of $\partial\bar{\partial}$ -exact $(1, 1)$ -forms with compact support, i.e., $Y = \{i\partial\bar{\partial}u : u \in \mathcal{C}_c^\infty(X)\}^\perp$.

As the spaces of test forms are reflexive, we have that the continuous linear functional L can be represented as $L(T) = \langle T, i\partial\bar{\partial}u \rangle$ for some $u \in \mathcal{C}_c^\infty(X)$.

The separation condition implies that $\langle T, i\partial\bar{\partial}u \rangle \geq \delta$ for all $T \in C$; if we test this condition against the current $T = \delta_x i\xi \wedge \bar{\xi}$ with $x \in K$ (where δ_x is the Dirac mass at x and ξ is any $(1, 0)$ tangent vector at x), we obtain that u is strictly subharmonic on the disc through x with complex direction ξ . The function u is then strictly psh on a neighbourhood of K , hence it belongs to $\mathcal{F}(U)$ for every allowable neighbourhood of its support, by the property **(A4)** of admissible classes. The proof is complete. \square

4. PROOF OF THEOREM 1.1

In this section we prove the assertions in Theorem 1.1.

Proposition 4.1. *Let X be a complex manifold and \mathcal{F} an admissible class. All \mathcal{F} -currents are supported in the singular locus $\Sigma^\mathcal{F}$ of \mathcal{F} . In particular, if $\Sigma^\mathcal{F}$ is empty, there are no \mathcal{F} -currents.*

Proof. Take $x \in X$ and assume that there is a function $\phi \in \mathcal{F}$ which is strictly psh at x . In particular, $\phi \in (U)$ for some small allowable neighbourhood of x . By **(A7)** (and up to possibly restricting U), we can approximate ϕ by smooth elements ϕ_n of \mathcal{F} , which are then strictly psh on some neighbourhood of x . We can then assume that ϕ is smooth and strictly psh near x . We then use the property **(A6)** in Definition 2.1, applied to a family of smooth functions $\rho_i, 1 \leq j \leq 2 \dim X$ such that the kernels of $\alpha_i := \partial(\phi + t_j \rho_j) \wedge \bar{\partial}(\phi + t_j \rho_j)$ are independent over \mathbb{R} , where the t_j 's are given by that property. This property stays true in a neighbourhood of x .

If T is a Levi current whose support contains x , by Lemma 3.4 we should have $T \wedge \alpha_i = 0$ for all i . This gives a contradiction, and concludes the proof. \square

Proposition 4.2. *Let X be a complex manifold and \mathcal{F} an admissible class. Suppose that \mathcal{F} contains an exhaustion function and that there are no \mathcal{F} -currents. Then there exists an element of $\mathcal{F}(X)$ which is an exhaustion function and everywhere strictly psh. In particular, the singular locus $\Sigma^\mathcal{F}$ is empty.*

Proof. We can follow the arguments of the proof of [34, Theorem 4.4]. We will construct a strictly psh exhaustion function on X by applying inductively Lemma 3.8.

By assumption, X admits an exhaustion function ϕ in $\mathcal{F}(X)$. We can then construct a sequence of compact sets K_n such that $K_n \Subset K_{n+1}$, $\cup_n K_n = X$, and with the property that, for all n ,

$$K_n = \{x \in X : u(x) \leq \max_{y \in K_n} u(y) \quad \forall u \in \mathcal{F}\}.$$

By applying Lemma 3.8, we can find a sequence of functions v_n which are strictly psh on \mathcal{V}_n on some open allowable neighbourhood V_n of K_n . Choose for every n a convex increasing function $\chi_n : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\begin{cases} \chi_n(\phi) < \inf_{K_n} v_n \text{ on } K_{n-1} \\ \chi_n(\phi) \geq \sup v_n \text{ near } \partial K_n \end{cases}$$

and define

$$u_n := \widetilde{\max}(\chi_n(\phi), v_n).$$

where $\widetilde{\max}$ is some smooth function sufficiently close to \max . This function is then equal to v_n on K_{n-1} and to $\chi_n(\phi)$ on $X \setminus K_n$. It is smooth strictly psh on a neighbourhood V'_n of K_{n-1} , and it is a psh and continuous exhaustion function for X . It belongs to \mathcal{F} by Property **(A3)** of Definition 2.1.

By the first condition in Definition 2.1, there exist ϵ_n such that the sequence $\sum_n \epsilon_n u_n \in \mathcal{F}(X)$. This function satisfies the required properties. \square

Proposition 4.3. *Let T be an extremal \mathcal{F} -current. All elements of \mathcal{F} are constant on the support of T .*

Proof. Let U be an admissible neighbourhood of the support of T and fix $v \in \mathcal{F}(U)$; we have that vT is again an \mathcal{F} -current. Indeed, by Lemmas 3.3 and 3.4, we have that the currents

$$\partial v \wedge T, \quad \bar{\partial} v \wedge T, \quad \text{and} \quad \partial \bar{\partial} v \wedge T$$

are all well-defined and vanish identically. Therefore, $\partial \bar{\partial}(vT)$ is well defined and vanishes as well. Hence vT is a \mathcal{F} -current.

Now, suppose that $u \in \mathcal{F}(U)$ is not constant on the support of T ; then, without loss of generality, we can suppose that $\{u < 0\}$ and $\{u > 0\}$ both intersect the support of T in a proper subset with non-empty interior. Consider a convex increasing function $h : \mathbb{R} \rightarrow \mathbb{R}$ such that $h(t) = 0$ if and only if $t \leq 0$ and $h(t) > 0$ otherwise. Then, by the first part of the proof, $h(u)T$ is a \mathcal{F} -current, which is a contradiction with the extremality of T . \square

Definition 4.4. Let \mathcal{F} be an admissible class. The *support* $S^{\mathcal{F}} = S_X^{\mathcal{F}}$ of \mathcal{F} is the union of the supports of all the \mathcal{F} -currents.

Proposition 4.5. *Let \mathcal{F} be an admissible class. The set $\hat{\mathcal{F}}$ is closed for the weak topology of currents. Moreover, there exists $T \in \hat{\mathcal{F}}$ such that $\text{spt } T = S^{\mathcal{F}}$.*

In particular, observe that $S^{\mathcal{F}}$ is closed in X .

Proof. Let T_n be a sequence of elements in $\hat{\mathcal{F}}$ and assume that there exists a current T on X such that $T_n \rightarrow T$ (in the sense of currents). We can assume that T is non-zero. Clearly T is positive and of bidimension (1,1). Since $i\partial\bar{\partial}T_n = 0$ for all n , we deduce that $i\partial\bar{\partial}T = 0$. We need to prove that $T \wedge i\partial\bar{\partial}u = 0$ for all $u \in \mathcal{F}$. We follow the argument given in [34] for the case of Levi currents.

Assume first that u is smooth. In this case, for any smooth function χ on X , we have $\chi T_n \wedge i\partial\bar{\partial}u \rightarrow \chi T \wedge i\partial\bar{\partial}u$. Since the left hand side of this expression vanishes for all n , we deduce that the same is true for the right hand side. Since χ is arbitrary, we obtain that $T \wedge i\partial\bar{\partial}u = 0$, as desired.

Let now u be any element of \mathcal{F} and let χ a smooth function with compact support. It is enough to work locally near the support of χ , and we can assume that this support is arbitrarily small. Recall that any element in \mathcal{F} is a continuous psh function on X . By **(A7)**, there exists a sequence u_n of smooth psh functions, converging to u in a neighbourhood of χ . By the arguments above, we have $\chi T \wedge \partial\bar{\partial}u_n = 0$ for all n . The first assertion now follows from Lemma 3.3.

Let us now prove the second statement. Since $\hat{\mathcal{F}}$ is closed, it is separable. Let us consider a countable dense subset T_j of $\hat{\mathcal{F}}$. For ϵ_n small enough, consider the current $T = \sum_n \epsilon_n T_n$ (which is in $\hat{\mathcal{F}}$) and denote by S its support. By the density of the T_j in $\hat{\mathcal{F}}$, we obtain that any element of $\hat{\mathcal{F}}$ is supported on S . Hence, $S = S^{\mathcal{F}}$, and the proof is complete. \square

The following proposition gives the relation between \mathcal{F} -currents and local maximum sets, and concludes the proof of Theorem 1.1.

Proposition 4.6. *Let \mathcal{F} be an admissible class.*

- (1) *Assume that $\mathcal{F}(X)$ contains an exhaustion function. Let T be a \mathcal{F} -current. If $\text{spt } T$ is compact, then it is a local maximum set.*
- (2) *Assume that $K \subset X$ is an \mathcal{F} -component, or a compact local maximum set. Then there exists $T \in \hat{\mathcal{F}}$ such that $\text{spt } T \subseteq K$.*

Proof. We prove the two assertions separately.

- (1) Denote $K := \text{spt } T$, and assume it is not a local maximum set. We are going to construct a psh function in neighbourhood of K , which is strictly psh at a point of $\text{spt } T$. This will contradict Lemma 3.6.

In order to construct such a function, we apply Proposition 2.12: there exist $x \in K$, $\epsilon > 0$, a neighbourhood B of x (which we can assume to be the unit ball centered at $x = 0$ in local coordinates y), and a smooth strictly psh function u on B such that $u(0) = 0$ and $-\epsilon|y|^2 - \epsilon/8 \leq u(y) \leq -\epsilon|y|^2$ for all $y \in K \cap B$ (by **(A7)**, we can assume that u is smooth), where the first inequality follows by possibly reducing the ball B .

The function u is only defined near x . In order to apply Lemma 3.6, we need to extend it, as a psh function, on a neighbourhood of K . By assumption, X admits an exhaustion function $\phi \in \mathcal{F}(X)$. We can also assume that

$$\phi(x) = -\epsilon/4 \quad \text{and} \quad |\phi| \leq \epsilon/4 \quad \text{on } B.$$

By considering a smooth function χ which is 1 on a small neighbourhood of K , and also compactly supported in a small neighbourhood of k , we can then consider the function v defined by

$$v = \begin{cases} \chi \max_{\epsilon}(u\phi) & \text{on } B, \\ \chi u & \text{on } X \setminus B. \end{cases}$$

where \max_{ϵ} is smooth approximation of the max function. As in [2, Proposition 3.2], one can verify that this function is indeed psh in a neighbourhood of K , and coincides with u in a neighbourhood of x . This concludes the proof.

- (2) Assume that no \mathcal{F} current is supported on K . Then, by Lemma 3.8, there exists a strictly psh function in $\mathcal{F}(U)$, where U is an open allowable neighbourhood of K . Fix $x_0 \in K$. First observe that $du(x_0) \neq 0$. For $j = 1, \dots, 2n-1$, choose a smooth function ρ_j compactly supported in U , with the property that $du_0, d\rho_1, \dots, d\rho_{2n-1}$ are linearly independent at x_0 . Property **(A6)** in Definition 2.1 gives positive numbers t_j such that $u_0 + t_j \rho_j \in \mathcal{F}(U)$ for all j . Set $v_j := u_0 + t_j \rho_j$ for all $1 \leq j \leq 2n-1$ and $v_0 := u_0$. As $du_0, d\rho_1, \dots, d\rho_{2n-1}$ are linearly independent at x_0 , the same holds true for $dv_0, dv_1, \dots, dv_{2n-1}$. This

implies that, in a neighbourhood of x_0 , we have $\bigcap_{j=0}^{2n-1} \{v_j(x) = u(x_0)\} = \{x_0\}$. Since, by [38, Corollary 1.11], there is a compact 1-pseudoconcave subset of K where all the v_j 's are constant, this gives the desired contradiction. \square

In particular, the following is then a consequence of Propositions 4.5 and 4.6.

Corollary 4.7. *Let \mathcal{F} be an admissible class. If the set $S^{\mathcal{F}}$ is compact, it is a local maximum set.*

5. LOCALIZATION RESULTS

We end this note with some results about the localization of \mathcal{F} -currents and singular loci to compact (or closed) subsets; first of all, by a standard disintegration procedure, we can decompose any \mathcal{F} -current on the levels of any admissible function, see [2, Corollary 2.4] for the case of Levi currents.

Proposition 5.1. *Let \mathcal{F} be an admissible class. Take $u \in \mathcal{F}$ and let T be a \mathcal{F} -current. There exists a measure μ on \mathbb{R} and a collection of currents T_c , $c \in \mathbb{R}$ such that*

- T_c is supported on $Y_c = \{x \in X : u(x) = c\}$ for all $c \in \mathbb{R}$;
- T_c is non zero for μ -almost every $c \in \mathbb{R}$;
- whenever $T_c \neq 0$, T_c is an \mathcal{F} -current;
- for every 2-dimensional form α on X we have

$$\langle T, \alpha \rangle = \int_{\mathbb{R}} \langle T_c, \alpha \rangle d\mu(c).$$

Moreover, if $u \in \mathcal{C}^1 \cap \mathcal{F}$ and c is a regular value for u , then $T_c = j_* S_c$, where j is the inclusion of Y_c in X and S_c a current on the real manifold Y_c .

Let now K be a compact set inside the complex manifold X and \mathcal{U} be the set of all relatively compact neighbourhoods of K in X . We can define the *singular locus of \mathcal{F} in K* as

$$\Sigma_K^{\mathcal{F}} = \bigcap_{U \in \mathcal{U}} \Sigma_U^{\mathcal{F}}.$$

Theorem 5.2. *Let \mathcal{F} be an admissible class, $K \subset X$ a compact set, and \mathcal{U} be defined as above.*

- (1) *The set $\Sigma_K^{\mathcal{F}}$ can be partitioned in subsets $\{F_\alpha\}_{\alpha \in A}$ such that, for every $U \in \mathcal{U}$ and $\phi \in \mathcal{F}(U)$, ϕ is constant on F_α for all $\alpha \in A$.*
- (2) *Every extremal \mathcal{F} -current supported in K is supported in some F_α .*
- (3) *There exists $T_K \in \hat{\mathcal{F}}$, supported in K , such that its support is maximal, and $\text{spt } T_K \subseteq \Sigma_K^{\mathcal{F}}$.*

Proof. (1) Consider, for each $U \in \mathcal{U}$ and $x \in \Sigma_K^{\mathcal{F}}$, the \mathcal{F} -component $F_{x,U}$ containing x in U ; we have that $\{F_{x,U} \cap K\}_{U \in \mathcal{U}}$ is a net of subsets whose intersection is a set $F_\alpha \subseteq \Sigma_K^{\mathcal{F}}$ which has the desired property.

- (2) If T is an extremal \mathcal{F} -current supported in K , functions in $\mathcal{F}(U)$ are constant on $\text{spt } T$ for all $U \in \mathcal{U}$; therefore, we have $\text{spt } T \subseteq F_\alpha$ for some α .

- (3) By Lemma 3.7, \mathcal{F} -currents supported on K are non-zero positive pluriharmonic currents of bidimension $(1, 1)$. Consider the compact convex set of $(1, 1)$ -bidimension, positive, $\partial\bar{\partial}$ -closed currents supported in K and with mass 1. By Krein-Milman theorem, this set is the closure of the convex hull of its extremal elements. Therefore, taking a dense sequence of extremal currents T_j , we can build the current

$$T_K = \sum 2^{-j} T_j,$$

which is again a \mathcal{F} -current. Then, $\text{spt } T_K$ contains the support of every \mathcal{F} -current supported in K . By the previous point, we have that $\text{spt } T_K \subseteq \Sigma_K^{\mathcal{F}}$. \square

- Remark 5.3.** (1) The sets F_α as above satisfy a local maximum property, outside a suitably defined “boundary”, i.e., the Hausdorff limit of $\overline{F_{x,U}} \cap bU$ for $U \in \mathcal{U}$.
- (2) In general, currents T_j may not have disjoint supports: consider $K \simeq \mathbb{P}^2$ as the exceptional divisor in the blow-up of \mathbb{C}^3 at the origin; for any allowable class \mathcal{F} , $\Sigma_K^{\mathcal{F}} = K$ and we can pick T_j as the current of integration on some projective line \mathbb{P}^1 in K . Obviously, all the supports of the T_j 's will intersect and in fact the whole K is one unique \mathcal{F} -component.
- (3) The sets $\Sigma_K^{\mathcal{F}}$ and $\text{spt } T_K$ can be different. Consider in \mathbb{P}^2 the set $C := \overline{K}_+$ for a given Hénon map f (i.e., a polynomial diffeomorphism of \mathbb{C}^2). Recall that K^+ is the set of points in \mathbb{C}^2 with bounded forward orbit. Then C is compact in the projective plane and, by the main result of [5], it supports only one positive, $\partial\bar{\partial}$ -closed, $(1, 1)$ -current of mass 1, which is the Green current T_+ . So, $T_+ = T_C$ and $\text{spt } T_C$ is equal to the closure in \mathbb{P}^2 of $J_+ = \partial K_+$. If f is appropriately chosen, some connected component Ω of $K^+ \setminus J_+$ is a Fatou-Bieberbach domain, biholomorphic to \mathbb{C}^2 , and we have $\partial\Omega = J^+$ [3, 27]. Therefore, every psh function which is continuous on $C' := \Omega \cup J^+$ needs to be constant on C' (since it restricts to a bounded psh function on $\Omega \cong \mathbb{C}^2$). Hence, $\Sigma_{C'}^{\mathcal{F}} = C' \supsetneq \text{spt } T_{C'} = \text{spt } T_C$. (We used in this remark the notations C and C' to avoid confusion with the set K already defined in the dynamical setting as the set of points with bounded both forward and backward orbits).
- (4) In the case of all continuous plurisubharmonic functions, the set $\Sigma_K^{\mathcal{F}}$ is the same as the psh kernel defined in [20], or the core of a compact set as defined in [30].

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