EVERY COMPLEX HÉNON MAP IS EXPONENTIALLY MIXING OF ALL ORDERS AND SATISFIES THE CLT

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ABSTRACT. We show that the measure of maximal entropy of every complex Hénon map is exponentially mixing of all orders for Hölder observables. As a consequence, the Central Limit Theorem holds for all Hölder observables.

Notation. The pairing $\langle \cdot, \cdot \rangle$ is used for the integral of a function with respect to a measure or more generally the value of a current at a test form. By (p, p)-currents we mean currents of bi-degree (p, p). Given $k \geq 1$, we denote by ω_{FS} the Fubini-Study form on $\mathbb{P}^k = \mathbb{P}^k(\mathbb{C})$. The mass of a positive closed (p, p)-current R on $\mathbb{P}^k$ is equal to $\langle R, \omega_{\mathrm{FS}}^{k-p} \rangle$ and is denoted by $\|R\|$. The notations $\lesssim$ and $\gtrsim$ stand for inequalities up to a multiplicative constant. If R and S are two real currents of the same bi-degree, we write $|R| \leq S$ when $S \pm R \geq 0$. Observe that this forces S to be positive.

1. INTRODUCTION

Hénon maps are among the most studied dynamical systems that exhibit interesting chaotic behaviour. They were introduced by Michel Hénon in the real setting as a simplified model of the Poincaré section for the Lorenz model, see, e.g., [5, 27]. Hénon maps are also actively studied in the complex setting, where complex analysis offers additional powerful tools. The reader can find in the work of Bedford, Fornaess, Lyubich, Sibony, Smillie, and the second author fundamental dynamical properties of these systems, see [1, 2, 3, 17, 25, 26, 30] and the references therein. It is shown in [1] that the measure of maximal entropy of such systems is Bernoulli. In particular, it is mixing of all orders. On the other hand, the control of the speed of mixing for general dynamical systems is a challenging problem, and usually one can obtain it only under strong hyperbolicity assumptions on the system. The main goal of this work is prove that the measure of maximal entropy of any complex Hénon map is *exponentially mixing of all orders* with respect to Hölder observables. As a consequence, we also solve a long-standing question proving the Central Limit Theorem for all Hölder observables with respect to the maximal entropy measures of complex Hénon maps.

Let us first recall the following general definition.

Definition 1.1. Let (X, f) be a dynamical system and ν an f-invariant measure. Let $(E, \| \cdot \|_E)$ be a normed space of real functions on X with $\| \cdot \|_{L^p(\nu)} \lesssim \| \cdot \|_E$ for all $1 \leq p < \infty$. We say that ν is *exponentially mixing of order* $\kappa \in \mathbb{N}^*$ for observables in E if there exist constants $C_\kappa > 0$ and $0 < \theta_\kappa < 1$ such that, for all $g_0, \ldots, g_\kappa$ in E and integers $0 =: n_0 \leq n_1 \leq \cdots \leq n_\kappa$, we have

$$\left| \langle \nu, g_0 (g_1 \circ f^{n_1}) \ldots (g_\kappa \circ f^{n_\kappa}) \rangle - \prod_{j=0}^{\kappa} \langle \nu, g_j \rangle \right| \leq C_\kappa \cdot \Big(\prod_{j=0}^{\kappa} \|g_j\|_E \Big) \cdot \theta_\kappa^{\min_{0 \leq j \leq \kappa - 1}(n_{j+1} - n_j)}.$$

We say that ν is *exponentially mixing of all orders* for observables in E if it is exponentially mixing of order κ for every $\kappa \in \mathbb{N}$.

A recent major result by Dolgopyat, Kanigowski, and Rodriguez-Hertz [20] ensures that, under suitable assumptions on the system, the exponential mixing of order 1 implies that the

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system is Bernoulli. In particular, it implies the mixing of all orders. (with no control on the rate of decay of correlation). It is a main open question whether the exponential mixing of order 1 implies the exponential mixing of all orders, see for instance [20, Question 1.5].

Let now f be a complex Hénon map on $\mathbb{C}^2$. It is a polynomial diffeomorphism of $\mathbb{C}^2$. We can associate to f its unique measure of maximal entropy μ [1, 2, 3, 30], see Section 2 for details. It was established by the second author in [12] that such measure if exponential mixing of order 1 for Hölder observables, see also Vigny [34] and Wu [35]. Similar results were obtained by Liverani [28] in the case of uniformly hyperbolic diffeomorphisms and Dolgopyat [18] for Anosov flows.

Theorem 1.2. *Let f be a complex Hénon map and μ its measure of maximal entropy. Then, for every $\kappa \in \mathbb{N}^*$, μ is exponential mixing of order κ as in Definition 1.1 for $\mathcal{C}^\gamma$ observables ($0 < \gamma \leq 2$), with $\theta_\kappa = d^{-(\gamma/2)^{\kappa+1}/2}$.*

For endomorphisms of $\mathbb{P}^k(\mathbb{C})$, the exponential mixing for all orders for the measure of maximal entropy and Hölder observables was established in [14]. We recently proved such property for a large class of invariant measures with strictly positive Lyapunov exponents [6]. This was done by constructing a suitable (semi-)norm on functions that turns the so-called Ruelle-Perron-Frobenius operator (suitably normalized) into a contraction. As far as we know, the present paper gives the first instance where the exponential mixing of all orders is established for general dynamical systems with both positive and negative Lyapunov exponents.

The exponential mixing of all orders is one of the strongest properties in dynamics. It was recently shown to imply a number of statistical properties, see for instance [8, 19]. As an example, a consequence of Theorem 1.2 is the following result. Take $u \in L^1(\mu)$. As μ is ergodic, Birkhoff's ergodic theorem states that

$$n^{-1} S_n(u) := n^{-1}\big(u(x) + u \circ f(x) + \cdots + u \circ f^{n-1}(x) \big) \to \langle \mu, u \rangle \quad \text{for } \mu - \text{a.e. } x \in X.$$

We say that *u satisfies the Central Limit Theorem* (CLT) with variance $\sigma^2 \geq 0$ with respect to μ if $n^{-1/2}(S_n(u) - n\langle \mu, u \rangle) \to \mathcal{N}(0, \sigma^2)$ in law, where $\mathcal{N}(0, \sigma^2)$ denotes the (possibly degenerate, for $\sigma = 0$) Gaussian distribution with mean 0 and variance σ^2, i.e., for any interval $I \subset \mathbb{R}$ we have

$$\lim_{n \to \infty} \nu\left\{ \frac{S_n(u) - n\langle \mu, u \rangle}{\sqrt{n}} \in I \right\} = \begin{cases} 1 \text{ when } I \text{ is of the form } I = (-\delta, \delta) & \text{if } \sigma^2 = 0, \\ \frac{1}{\sqrt{2\pi\sigma^2}} \int_I e^{-t^2/(2\sigma^2)} dt & \text{if } \sigma^2 > 0. \end{cases}$$

By a result of Björklund and Gorodnik [8], the following is then a consequence of Theorem 1.2. We refer to [6, 11, 15, 22, 29, 31, 32] for other cases where the CLT for Hölder observables was established in holomorphic dynamics. As is the case for Theorem 1.2, this is the first time that this is done for systems with both positive and negative Lyapunov exponents.

Corollary 1.3. *Let f be a complex Hénon map and μ its measure of maximal entropy. Then all Hölder observables u satisfy the Central Limit Theorem with respect to μ with*

$$\sigma^2 = \sum_{n \in \mathbb{Z}} \langle \mu, \tilde{u}(\tilde{u} \circ f^n) \rangle = \lim_{n \to \infty} \frac{1}{n} \int_X (\tilde{u} + \tilde{u} \circ f + \ldots + \tilde{u} \circ f^{n-1})^2 d\mu,$$

where $\tilde{u} := u - \langle \mu, u \rangle$.

Theorem 1.2 and Corollary 1.3 in particular apply to any real Hénon map of maximal entropy [4], i.e., complex Hénon maps with real coefficients and whose measure of maximal entropy is supported by $\mathbb{R}^2$. They hold also in the larger settings of regular automorphisms of $\mathbb{C}^k$ in any dimension [30], see Definition 2.1 and Remark 3.3, and invertible horizontal-like maps in any dimension [13, 16], see Remark 3.4. We postpone the case of automorphisms of compact Kähler manifolds to the forthcoming paper [7], see Remark 3.5.

Our method to prove Theorem 1.2 relies on pluripotential theory and on the theory of positive closed currents. The idea is as follows. Using the classical theory of interpolation [33], we can reduce the problem to the case $\gamma = 2$. For simplicity, assume that $\|g_j\|_{\mathcal{C}^2} \leq 1$ for all j. The measure of maximal entropy μ of a Hénon map f of $\mathbb{C}^2$ of algebraic degree $d \geq 2$ is the intersection $\mu = T_+ \wedge T_-$ of the two Green currents T_+ and T_- of f [2, 30]. If we identify $\mathbb{C}^2$ to an affine chart of $\mathbb{P}^2$ in the standard way, these currents are the unique positive closed $(1,1)$-currents of mass 1 on $\mathbb{P}^2$, without mass at infinity, satisfying $f^*T_+ = dT_+$ and $f_*T_- = dT_-$.

Consider the automorphism F of $\mathbb{C}^4$ given by $F := (f, f^{-1})$. Such automorphism also admits Green currents $\mathbb{T}_+ = T_+ \otimes T_-$ and $\mathbb{T}_- = T_- \otimes T_+$. These currents satisfy $(F^n)^*\mathbb{T}_+ = d^2\mathbb{T}_+$ and $(F^n)_*\mathbb{T}_- = d^2\mathbb{T}_-$. Under mild assumptions on their support, other positive closed $(2,2)$-currents S of mass 1 of $\mathbb{P}^4$ satisfy the estimate

$$(1.1) \qquad |\langle d^{-2n}(F^n)_*(S) - \mathbb{T}_-, \Phi \rangle| \leq c_{S,\Phi} d^{-n}$$

when Φ is a sufficiently smooth test form. Here, $c_{S,\Phi}$ is a constant depending on S and Φ.

We show that proving the exponential mixing for $\kappa + 1$ observables $g_0, \ldots, g_\kappa$ with $\|g_j\|_{\mathcal{C}^2} \leq 1$ can be reduced to proving the convergence (we assume that n_1 is even for simplicity)

$$(1.2) \qquad |\langle d^{-n_1}(F^{n_1/2})_*[\Delta] - \mathbb{T}_-, \Theta \rangle| \lesssim d^{-\min_{0 \leq j \leq \kappa-1}(n_{j+1}-n_j)/2},$$

where $[\Delta]$ denotes the current of integration on the diagonal Δ of $\mathbb{C}^2 \times \mathbb{C}^2$, (z, w) denote the coordinates on $\mathbb{C}^2 \times \mathbb{C}^2$ and

$$\Theta := g_0(w)g_1(z)(g_2 \circ f^{n_2-n_1}(z)) \ldots (g_\kappa \circ f^{n_\kappa-n_1}(z))\mathbb{T}_+.$$

A crucial point here is that the estimate should not only be uniform in the g_j's, but also in the n_j's. Note also that the current $[\Delta]$ is singular and the dependence of the constant $c_{S,\Phi}$ in (1.1) from S makes it difficult to employ regularization techniques to deduce the convergence (1.2) from (1.1).

The key point here is to notice that, when $dd^c\Phi \geq 0$ (on a suitable open set), one can also get the following variation of (1.1):

$$(1.3) \qquad \langle d^{-2n}(F^n)_*(S) - \mathbb{T}_-, \Phi \rangle \leq c_\Phi d^{-n}.$$

With respect to (1.1), only the bound from above is present, but the constant c_Φ is now independent of S. This permits to regularize Δ and work as if this current were smooth. Note also that, although Θ is not smooth, we can handle it using a similar regularization.

Working by induction, we show that it is possible to replace both Θ and $-\Theta$ in (1.2) with currents $\Theta^\pm$ satisfying $dd^c\Theta^\pm \geq 0$. This permits to deduce the estimate (1.2) from two upper bounds given by (1.3) for $\Theta^\pm$, completing the proof.

2. Regular automorphisms of $\mathbb{C}^k$ and convergence towards Green currents

Let F be a polynomial automorphism of $\mathbb{C}^k$. We still denote by F its extension as a birational map of $\mathbb{P}^k$. Denote by $\mathbb{H}_\infty := \mathbb{P}^k \setminus \mathbb{C}^k$ the hyperplane at infinity and by $\mathbb{I}^+, \mathbb{I}^-$ the indeterminacy sets of F and F^{-1} respectively. They are analytic sets strictly contained in $\mathbb{H}_\infty$. If $\mathbb{I}^+ = \varnothing$ or $\mathbb{I}^- = \varnothing$, then both of them are empty and F is given by a linear map and its dynamics is easy to describe. Hence, we assume that $\mathbb{I}^\pm \neq \varnothing$. The following definition is due to Sibony [30].

Definition 2.1. We say that F is a *regular automorphism of* $\mathbb{C}^k$ if $\mathbb{I}^\pm \neq \varnothing$ and $\mathbb{I}^+ \cap \mathbb{I}^- = \varnothing$.

Given F a regular automorphism of $\mathbb{C}^k$, it is clear that F^{-1} is also regular. We denote by $d_+(F)$ and $d_-(F)$ the algebraic degrees of F and F^{-1} respectively. Observe that $d_\pm(F) \geq 2$, $d_+(F) = d_-(F^{-1})$ and $d_-(F) = d_+(F^{-1})$. Later, we will drop the letter F and just write $d_\pm$ instead of $d_\pm(F)$ for simplicity. We will recall here some basic properties of F and refer the reader to [2, 17, 25, 26, 30] for details.

Proposition 2.2. *Let F be a regular automorphism of $\mathbb{C}^k$ as above.*

   (i) *There exists an integer $1 \leq p \leq k-1$ such that $\dim \mathbb{I}^+ = k - p - 1$, $\dim \mathbb{I}^- = p - 1$, and $d_+(F)^p = d_-(F)^{k-p}$.*

   (ii) *The analytic sets $\mathbb{I}^\pm$ are irreducible and we have*

$$F(\mathbb{H}_\infty \setminus \mathbb{I}^+) = F(\mathbb{I}^-) = \mathbb{I}^- \quad and \quad F^{-1}(\mathbb{H}_\infty \setminus \mathbb{I}^-) = F^{-1}(\mathbb{I}^+) = \mathbb{I}^+.$$

   (iii) *For every $n \geq 1$, both F^n and F^{-n} are regular automorphisms of $\mathbb{C}^k$, of algebraic degrees $d_+(F)^n$ and $d_-(F)^n$, and indeterminacy sets $\mathbb{I}^+$ and $\mathbb{I}^-$, respectively.*

Example 2.3. (Generalized) Hénon maps on $\mathbb{C}^2$ correspond to the case $k = 2$ in Definition 2.1. In this case, we have $p = k - p = 1$ and $d_+ = d_- = d$, the algebraic degree of the map, see [2, 24, 30].

The set $\mathbb{I}^+$ (resp. $\mathbb{I}^-$) is attracting for F^{-1} (resp. F). Let $\widetilde{W}^\pm$ be the basin of attraction of $\mathbb{I}^\pm$. Set $W^\pm := \widetilde{W}^\pm \cap \mathbb{C}^k$. Then the sets $\mathbb{K}^+ := \mathbb{C}^k \setminus W^-$ and $\mathbb{K}^- := \mathbb{C}^k \setminus W^+$ are the sets of points (in $\mathbb{C}^k$) with bounded orbit for F and F^{-1}, respectively. We have $\overline{\mathbb{K}^+} = \mathbb{K}^+ \cup \mathbb{I}^+$ and $\overline{\mathbb{K}^-} = \mathbb{K}^- \cup \mathbb{I}^-$ where the closures are taken in $\mathbb{P}^k$. We also define $\mathbb{K} := \mathbb{K}^+ \cap \mathbb{K}^-$ which is a compact subset of $\mathbb{C}^k$.

In the terminology of [17], the set $\overline{\mathbb{K}^+}$ (resp. $\overline{\mathbb{K}^-}$) is *p-rigid* (resp. *$(k-p)$-rigid*): it supports a unique positive closed (p, p)-current (resp. $(k - p, k - p)$-current) of mass 1, that we denote by $\mathbb{T}_+$ (resp. $\mathbb{T}_-$). The currents $\mathbb{T}_\pm$ have no mass on $\mathbb{H}_\infty$ and satisfy the invariance relations

$$F^*(\mathbb{T}_+) = d_+^p \mathbb{T}_+ \quad and \quad F_*(\mathbb{T}_-) = d_-^{k-p} \mathbb{T}_-$$

as currents on $\mathbb{C}^k$ or $\mathbb{P}^k$. We call them *the main Green currents of F*. They can be obtained as intersections of positive closed $(1, 1)$-currents with local Hölder continuous potentials in $\mathbb{C}^k$. Therefore, the measure $\mathbb{T}_+ \wedge \mathbb{T}_-$ is well-defined and supported by the compact set $\mathbb{K}$. This is the unique invariant probability measure of maximal entropy [9, 30], see also [1, 2, 3, 21] for the case of dimension $k = 2$.

Using the above description of the dynamics of F, we can fix neighbourhoods U_1, U_2 of $\overline{\mathbb{K}^+}$ and V_1, V_2 of $\overline{\mathbb{K}^-}$ such that $F^{-1}(U_i) \Subset U_i$, $U_1 \Subset U_2 \Subset \mathbb{P}^k \setminus \mathbb{I}^-$, $F(V_i) \Subset V_i$, $V_1 \Subset V_2 \Subset \mathbb{P}^k \setminus \mathbb{I}^+$, and $U_2 \cap V_2 \Subset \mathbb{C}^k$. Let Ω be a real $(p + 1, p + 1)$-current with compact support in U_1. Assume that there exists a positive closed $(p + 1, p + 1)$-current Ω' with compact support in U_1 such that $|\Omega| \leq \Omega'$. Define the norm $\|\Omega\|_{*, U_1}$ of Ω as

$$\|\Omega\|_{*, U_1} := \inf\{\|\Omega'\| : |\Omega| \leq \Omega'\}.$$

Observe that when Ω is a d-exact current we can write $\Omega = \Omega' - (\Omega' - \Omega)$, which is the difference of two positive closed current in the same cohomology class in $H^{p+1, p+1}(\mathbb{P}^k, \mathbb{R})$. Therefore, the norm $\|\cdot\|_{*, U_1}$ is equivalent to the norm given by $\inf \|\Omega^\pm\|$, where $\Omega^\pm$ are positive closed currents with compact support in U_1 such that $\Omega = \Omega^+ - \Omega^-$. Note that Ω^+ and Ω^- have the same mass as they belong to the same cohomology class.

The following property was obtained by the second author, see [12, Proposition 2.1].

Proposition 2.4. *Let R be a positive closed $(k-p, k-p)$-current of mass 1 with compact support in V_1 and smooth on $\mathbb{C}^k$. Let Φ be a real-valued (p, p)-form of class $\mathcal{C}^2$ with compact support in $U_1 \cap \mathbb{C}^k$. Assume that $dd^c\Phi \geq 0$ on V_2. Then there exists a constant $c > 0$ independent of R and Φ such that*

$$\left\langle d_-^{-(k-p)n}(F^n)_*(R) - \mathbb{T}_-, \Phi \right\rangle \leq c\, d_-^{-n} \|dd^c\Phi\|_{*, U_1} \quad for\ all\ n \geq 0.$$

Note that in what follows, since $\mathbb{T}_-$ is an intersection of positive closed $(1,1)$-currents with local continuous potentials [30], the intersections $R \wedge \mathbb{T}_-$ and $\mathbb{T}_+ \wedge \mathbb{T}_-$ are well-defined and the former depends continuously on R. In particular, the pairing in the next statement is meaningful and depends continuously on R.

Corollary 2.5. *Let R be a positive closed $(k-p, k-p)$-current of mass 1 supported in V_1. Let ϕ be a $\mathcal{C}^2$ function with compact support on $\mathbb{C}^k$ such that $dd^c\phi \geq 0$ in a neighbourhood of $\mathbb{K}_+ \cap V_2$. Then there exists a constant $c > 0$ independent of R and ϕ such that*

$$(2.1) \qquad \left\langle d_-^{-(k-p)n}(F^n)_*(R) - \mathbb{T}_-, \phi\mathbb{T}_+ \right\rangle \leq c\, d_-^{-n}\|dd^c\phi \wedge \mathbb{T}_+\|_{*,U_1} \quad \text{for all } n \geq 0.$$

Proof. As $\mathbb{P}^k$ is homogeneous, we will use the group $\mathrm{PGL}(k+1, \mathbb{C})$ of automorphisms of $\mathbb{P}^k$ and suitable convolutions in order regularize the currents R and $\phi\mathbb{T}_+$ and deduce the result from Proposition 2.4. Choose local coordinates centered at the identity $\mathrm{id} \in \mathrm{PGL}(k+1, \mathbb{C})$ so that a small neighbourhood of id in $\mathrm{PGL}(k+1, \mathbb{C})$ is identified to the unit ball $\mathbb{B}$ of $\mathbb{C}^{k^2+2k}$. Here, a point of coordinates ϵ represents an automorphism of $\mathbb{P}^k$ that we denote by τ_ϵ. Thus, $\tau_0 = \mathrm{id}$.

Consider a smooth non-negative function ρ with compact support on $\mathbb{B}$ and of integral 1 with respect to the Lebesgue measure and, for $0 < r \leq 1$, define $\rho_r(\epsilon) := r^{-2k^2-4k}\rho(r^{-1}\epsilon)$, which is supported by $\{|\epsilon| \leq r\}$. This function allows us to define an approximation of the Dirac mass at $0 \in \mathbb{B}$ when $r \to 0$. We define $\Psi := \phi\mathbb{T}_+$ and consider the following regularized currents

$$R_r := \int \rho_r(\epsilon)(\tau_\epsilon)^*(R) \quad \text{and} \quad \Psi_r := \int \rho_r(\epsilon)(\tau_\epsilon)^*(\Psi) = \int \rho_r(\epsilon)(\phi \circ \tau_\epsilon)(\tau_\epsilon)^*(\mathbb{T}_+),$$

where the integrals are with respect to the Lebesgue measure on $\epsilon \in \mathbb{B}$.

When r is small enough and goes to 0, the current R_r is smooth, positive, closed, with compact support in V_1, and converges to R. Since the RHS of (2.1) depends continuously on R, we can replace R by R_r and assume that R is smooth. When ϵ goes to 0, $\phi \circ \tau_\epsilon$ converges uniformly to ϕ and $(\tau_\epsilon)^*(\mathbb{T}_+)$ converges to $\mathbb{T}_+$. Using that R is smooth and $\mathbb{T}_-$ is a product of $(1,1)$-currents with continuous potentials, we deduce that the LHS of (2.1) is equal to

$$\lim_{r \to 0} \left\langle d_-^{-(k-p)n}(F^n)_*(R) - \mathbb{T}_-, \Psi_r \right\rangle.$$

Since $\mathbb{T}_+$ is supported by $\overline{\mathbb{K}_+}$ and we have $dd^c\phi \geq 0$ on a neighbourhood of $\mathbb{K}_+ \cap V_2$, we deduce that $dd^c\Psi \geq 0$ on V_2. By reducing slightly V_2, we still have $dd^c\Psi_r \geq 0$ on V_2 for r small enough. We will use the last limit and Proposition 2.4 for Ψ_r instead of Φ and U_2 instead of U_1. Observe that for ϵ small enough, since $U_1 \Subset U_2$, we have $\|(\tau_\epsilon)^*(dd^c\Psi)\|_{*,U_2} \leq \|dd^c\Psi\|_{*,U_1}$. We deduce that the LHS of (2.1) is smaller than or equal to

$$\lim_{r \to 0} c\, d_-^{-n}\|dd^c\Psi_r\|_{*,U_2} \leq c\, d_-^{-n}\|dd^c\Psi\|_{*,U_1} = c\, d_-^{-n}\|dd^c\phi \wedge \mathbb{T}_+\|_{*,U_1}.$$

This completes the proof of the corollary. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

In order to use the above corollary, we will need the following lemmas.

Lemma 2.6. *Let $\kappa \geq 1$ be an integer and $g_0, \ldots, g_\kappa$ compactly supported functions on $\mathbb{C}^k$ with $\|g_j\|_{\mathcal{C}^2} \leq 1$. Then there is a constant $c_\kappa > 0$ independent of the g_j's such that for all $\ell_0, \ldots, \ell_\kappa \geq 0$ we have*

$$\|dd^c\big((g_0 \circ F^{\ell_0}) \ldots (g_\kappa \circ F^{\ell_\kappa})\big) \wedge \mathbb{T}_+\|_{*,U_1} \leq c_\kappa.$$

Proof. Set $\tilde{g}_j := g_j \circ f^{\ell_j}$ for simplicity. We have

$$dd^c\big(\tilde{g}_0 \ldots \tilde{g}_\kappa\big) = \sum_{j=0}^{\kappa} dd^c\tilde{g}_j \prod_{l \neq j} \tilde{g}_l + \sum_{0 \leq j \neq l \leq \kappa} i\partial\tilde{g}_j \wedge \bar{\partial}\tilde{g}_l \prod_{m \neq j,l} \tilde{g}_m.$$

Since $\|g_j\|_{\mathcal{C}^2} \leq 1$ we have $|g_j| \leq 1$. Denote by ω_{FS} the Fubini-Study form on $\mathbb{P}^k$. Then

$$\left|\sum_{j=0}^{\kappa} dd^c\tilde{g}_j \prod_{l \neq j} \tilde{g}_l\right| \lesssim \sum_{j=0}^{\kappa} (F^{\ell_j})^*\omega_{\mathrm{FS}}$$

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and an application of Cauchy-Schwarz inequality gives

$$\Big| \sum_{0 \leq j \neq l \leq \kappa} i\partial \tilde{g}_j \wedge \overline{\partial} \tilde{g}_l \prod_{m \neq j,l} \tilde{g}_m \Big| \lesssim \sum_{j=0}^{\kappa} i\partial \tilde{g}_j \wedge \overline{\partial} \tilde{g}_j$$

$$= \sum_{j=0}^{\kappa} (F^{\ell_j})^*(i\partial g_j \wedge \overline{\partial} g_j)$$

$$\lesssim \sum_{j=0}^{\kappa} (F^{\ell_j})^*(\omega_{\mathrm{FS}}).$$

As we have $dd^c(\tilde{g}_0 \ldots \tilde{g}_\kappa) = 0$ near $\mathbb{H}_\infty$, its intersection with $\mathbb{T}_+$ can be computed on $\mathbb{C}^k$. We deduce from the above inequalities and $d_-^{k-p} = d_+^p$ that

$$\Big| dd^c \big((g_0 \circ F^{\ell_0}) \ldots (g_\kappa \circ F^{\ell_\kappa}) \big) \wedge \mathbb{T}_+ \Big| \lesssim \sum_{j=0}^{\kappa} (F^{\ell_j})^*(\omega_{\mathrm{FS}}) \wedge \mathbb{T}_+$$

(2.2)
$$= \sum_{j=0}^{\kappa} (F^{\ell_j})^*(\omega_{\mathrm{FS}}) \wedge d_+^{-p\ell_j}(F^{\ell_j})^* \mathbb{T}_+$$

$$= \sum_{j=0}^{\kappa} d_-^{-(k-p)\ell_j}(F^{\ell_j})^* \big(\omega_{\mathrm{FS}} \wedge \mathbb{T}_+ \big).$$

We will use that the $(p+1, p+1)$-current $\omega_{\mathrm{FS}} \wedge \mathbb{T}_+$ is positive, closed, of mass 1, and its support is contained in $\overline{\mathbb{K}^+} \subset U_1$. We have

$$\|(F^{\ell_j})^* \big(\omega_{\mathrm{FS}} \wedge \mathbb{T}_+ \big)\| = \big\langle (F^{\ell_j})^* \big(\omega_{\mathrm{FS}} \wedge \mathbb{T}_+ \big), \omega_{\mathrm{FS}}^{k-p-1} \big\rangle = \big\langle \omega_{\mathrm{FS}} \wedge \mathbb{T}_+, (F^{-\ell_j})^* (\omega_{\mathrm{FS}}^{k-p-1}) \big\rangle,$$

where the last form is positive closed and smooth outside $\mathbb{I}^-$. The last pairing only depends on the cohomology classes of ω_{FS}, $\mathbb{T}_+$, and $(F^{-\ell_j})^*(\omega_{\mathrm{FS}}^{k-p-1})$. Hence, it is equal to the mass of $(F^{-\ell_j})^*(\omega_{\mathrm{FS}}^{k-p-1})$, which is equal to $d_-^{(k-p-1)\ell_j}$, see [30]. It follows that each term in the last sum in (2.2) is bounded by 1, which implies that the sum is bounded by $\kappa + 1$. The lemma follows. $\qquad \square$

Lemma 2.7. *Let $D \Subset D'$ be two bounded domains in $\mathbb{C}^k$. Let g be a function with compact support in D and such that $\|g\|_{\mathcal{C}^2} \leq 1$. Then there are a constant $A > 0$ independent of g and functions $g^\pm$ with compact supports in D' and $\|g^\pm\|_{\mathcal{C}^2} \leq 1$ such that*

$$g = A(g^+ - g^-), \qquad i\partial g^+ \wedge \overline{\partial} g^+ \leq dd^c g^+ \ \text{on} \ D \qquad \text{and} \qquad i\partial g^- \wedge \overline{\partial} g^- \leq dd^c g^- \ \text{on} \ D.$$

Proof. Let ρ be a smooth non-negative function, compactly supported on D' and equal to 1 in a neighbourhood of $\overline{D}$. Observe that $\rho g = g$. We denote by z the coordinates of $\mathbb{C}^k$. Since $\|g\|_{\mathcal{C}^2} \leq 1$ and g has compact support in D, there exists a constant $A_1 > 0$ independent of g such that $|dd^c g| \leq A_1 dd^c(\|z\|^2)$. Set $g^+ := A^{-1}\rho(g + 2A_1\|z\|^2)$ and $g^- := 2A^{-1}A_1\rho\|z\|^2$ for some constant $A > 0$. It is not difficult to check that we have $dd^c g^\pm \geq A^{-1}A_1 dd^c(\|z\|^2)$ on D, $\|i\partial g^\pm \wedge \overline{\partial} g^\pm\|_\infty = O(A^{-2})$ on D, $g = A(g^+ - g^-)$, and $\|g^\pm\|_{\mathcal{C}^2} = O(A^{-1})$. Taking A large enough gives the lemma. $\qquad \square$

3. Exponential mixing of all orders for Hénon maps and further remarks

Throughout this section (except for Remarks 3.2, 3.3, and 3.4), f denotes a Hénon map on $\mathbb{C}^2$ of algebraic degree $d = d_+ = d_- \geq 2$. Define $F := (f, f^{-1})$. It is not difficult to check that F is a regular automorphism of $\mathbb{C}^4 = \mathbb{C}^2 \times \mathbb{C}^2$. We will use the notations and the results of Section 2 with $k = 4$ and $p = 2$. We denote in this section by $T_\pm$ the Green $(1,1)$-currents of f, and reserve the notation $\mathbb{T}_\pm$ for the main Green currents of F. Observe that $\mathbb{T}_+ = T_+ \otimes T_-$ and $\mathbb{T}_- = T_- \otimes T_+$, see [23, Section 4.1.8] for the tensor (or cartesian) product of currents. We denote by $K^\pm$ the sets of points of bounded orbit for $f^{\pm 1}$. The wedge product $\mu := T_+ \wedge T_-$ is

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well defined, and is the measure of maximal entropy of f [1, 2, 3, 30]. Its support is contained in the compact set $K = K^+ \cap K^-$. We have $\mathbb{K}^+ = K^+ \times K^-$ and $\mathbb{K}^- = K^- \times K^+$. Note also that the diagonal Δ of $\mathbb{C}^2 \times \mathbb{C}^2$ satisfies $\overline{\Delta} \cap \mathbb{I}^+ = \varnothing$ and $\overline{\Delta} \cap \mathbb{I}^- = \varnothing$ in $\mathbb{P}^4$, see also [12].

We now prove Theorem 1.2. By a standard interpolation [33] (see for instance [12, pp. 262-263] and [18, Corollary 1] for similar occurrences) it is enough to prove the statement for $\gamma = 2$, i.e., in the case where all the functions g_j are of class $\mathcal{C}^2$. The statement is clear for $\kappa = 0$, i.e., for one test function. By induction, we can assume that the statement holds for up to κ test functions and prove it for $\kappa + 1 \geq 1$ test functions, i.e., show that

$$\left| \langle \mu, g_0 (g_1 \circ f^{n_1}) \dots (g_\kappa \circ f^{n_\kappa}) \rangle - \prod_{j=0}^{\kappa} \langle \mu, g_j \rangle \right| \lesssim \left(\prod_{j=0}^{\kappa} \|g_j\|_{\mathcal{C}^2} \right) \cdot d^{-\min_{0 \leq j \leq \kappa-1}(n_{j+1}-n_j)/2}.$$

Recall that $n_0 = 0$. The induction assumption implies that we are allowed to modify each g_j by adding a constant. Moreover, using the invariance of ν, the desired estimate does not change if we replace n_j by $n_j - 1$ for $1 \leq j \leq \kappa$ and g_0 by $g_0 \circ f^{-1}$. Therefore, we can for convenience assume that n_1 is even.

We fix a large bounded domain $B \subset \mathbb{C}^2$ satisfying

$$K \subset B, \quad K^- \cap B \subset f(B), \quad \text{and} \quad K^+ \cap B \subset f^{-1}(B).$$

By induction, the inclusions above imply that

(3.1) $\qquad K \subset B, \quad K^- \cap B \subset f^n(B), \quad \text{and} \quad K^+ \cap B \subset f^{-n}(B) \quad \text{for all } n \geq 1.$

Because of Lemma 2.7 and the fact that we are only interested in the values of the g_j's on the support of μ, we can assume that all the g_j's are compactly supported in $\mathbb{C}^2$ and satisfy

(3.2) $\qquad \|g_j\|_{\mathcal{C}^2} \leq 1 \text{ on } \mathbb{C}^2 \quad \text{and} \quad i\partial g_j \wedge \overline{\partial} g_j \leq dd^c g_j \text{ on } B.$

For simplicity, write $h := g_1 (g_2 \circ f^{n_2-n_1}) \dots (g_\kappa \circ f^{n_\kappa-n_1})$. We need to prove that

$$|\langle \mu, g_0 (h \circ f^{n_1}) \rangle - \langle \mu, g_0 \rangle \cdot \langle \mu, h \rangle| \lesssim d^{-\min_{0 \leq j \leq \kappa-1}(n_{j+1}-n_j)/2}$$

since this estimate, together with the induction assumption applied to $\langle \mu, h \rangle$, would imply the desired statement. In order to obtain the result, we will prove separately the two estimates

(3.3) $\qquad \langle \mu, g_0 (h \circ f^{n_1}) \rangle - \langle \mu, g_0 \rangle \cdot \langle \mu, h \rangle \lesssim d^{-\min_{0 \leq j \leq \kappa-1}(n_{j+1}-n_j)/2}$

and

(3.4) $\qquad -\langle \mu, g_0 (h \circ f^{n_1}) \rangle + \langle \mu, g_0 \rangle \cdot \langle \mu, h \rangle \lesssim d^{-\min_{0 \leq j \leq \kappa-1}(n_{j+1}-n_j)/2}.$

Set $M := 10\kappa$ and fix a smooth function χ with compact support in $\mathbb{C}^2$ and equal to 1 in a neighbourhood of $\overline{B}$. Consider the following four functions, which will later allow us to produce some p.s.h. test functions:

$$g_0^+ := \chi \cdot (g_0 + M) \quad \text{and} \quad h^+ := \chi \cdot (g_1 + M)(g_2 \circ f^{n_2-n_1} + M) \dots (g_\kappa \circ f^{n_\kappa-n_1} + M)$$

and

$$g_0^- := \chi \cdot (M - g_0) \quad \text{and} \quad h^- := \chi \cdot \left((g_1+M)(g_2 \circ f^{n_2-n_1}+M) \dots (g_\kappa \circ f^{n_\kappa-n_1}+M) - 2(M+1)^\kappa \right).$$

Recall that $n_0 = 0$. To prove (3.3) and (3.4), it is enough to show that

(3.5) $\qquad \langle \mu, g_0^+ (h^+ \circ f^{n_1}) \rangle - \langle \mu, g_0^+ \rangle \cdot \langle \mu, h^+ \rangle \lesssim d^{-n_1/2}$

and

(3.6) $\qquad \langle \mu, g_0^- (h^- \circ f^{n_1}) \rangle - \langle \mu, g_0^- \rangle \cdot \langle \mu, h^- \rangle \lesssim d^{-n_1/2}.$

Indeed, we observe that χ does not play any role in (3.5) and (3.6). Hence, the difference between the LHS of (3.5) and the one of (3.3) (resp. of (3.6) and of (3.4)) is a finite combination of expressions involving no more than κ functions among $g_0, \dots, g_\kappa$, that we can estimate using the induction hypothesis on the mixing of order up to $\kappa - 1$. It remains to prove the two inequalities (3.5) and (3.6).

Denote by (z, w) the coordinates on $\mathbb{C}^4 = \mathbb{C}^2 \times \mathbb{C}^2$ and define

$$\phi^{\pm}(z, w) := g_0^{\pm}(w) \, h^{\pm}(z).$$

We have the following lemma for a fixed domain U_1 as in Section 2.

Lemma 3.1. *The functions $\phi^{\pm}$ satisfy*

(i) $dd^c \phi^{\pm} \wedge \mathbb{T}_+ \geq 0$ *on* $B \times B$;

(ii) $\|dd^c \phi^{\pm} \wedge \mathbb{T}_+\|_{*, U_1} \leq c_\kappa,$

where c_κ is a positive constant depending on κ, but not on the g_j's and the n_j's.

Proof. (i) For simplicity, we set $\ell_0 = \ell_1 := 0$ and $\ell_j := n_j - n_1$. Define also $\tilde{g}_j := g_j \circ f^{\ell_j}$. In what follows $\tilde{g}_0$ depends on w and $\tilde{g}_j$ depends on z when $j \geq 1$. Observe that by the invariance property of $K^+ \cap B$ in (3.1) and the constraints in (3.2), the following inequalities hold in a neighbourhood of $K^+ \cap B$:

$$(3.7) \qquad i\partial \tilde{g}_j \wedge \overline{\partial} \tilde{g}_j = (f^{\ell_j})^*(i\partial g_j \wedge \overline{\partial} g_j) \leq (f^{\ell_j})^*(dd^c g_j) = dd^c \tilde{g}_j.$$

In particular, we have $dd^c \tilde{g}_j \geq 0$ in a neighbourhood of $K^+ \cap B$. Note that for $\tilde{g}_0 = g_0$ the properties hold on B, which contains $K^- \cap B$.

Now, since $\mathbb{T}_+$ is closed, positive, and supported by $\overline{\mathbb{K}^+} = \overline{K^+ \times K^-}$, in order to prove the first assertion it is enough to show that $dd^c \phi^{\pm} \geq 0$ on a neighbourhood of $(K^+ \cap B) \times (K^- \cap B)$ in $\mathbb{C}^4$ where $\chi = 1$. In what follows, we only work on such a neighbourhood. We have

$$dd^c \phi^+ = \sum_{j=0}^{\kappa} dd^c \tilde{g}_j \prod_{l \neq j} (\tilde{g}_l + M) + \sum_{0 \leq j \neq l \leq \kappa} i\partial \tilde{g}_j \wedge \overline{\partial} \tilde{g}_l \prod_{m \neq j,l} (\tilde{g}_m + M),$$

where we recall that $\tilde{g}_0$ is $\tilde{g}_0(w)$ and the other $\tilde{g}_j$'s are $\tilde{g}_j(z)$ for $1 \leq j \leq \kappa$. For the first term in the RHS of the last expression, we have

$$\sum_{j=0}^{\kappa} dd^c \tilde{g}_j \prod_{l \neq j} (\tilde{g}_l + M) \geq (M-1)^{\kappa} \sum_{j=0}^{\kappa} dd^c \tilde{g}_j.$$

For the second term, an application of Cauchy-Schwarz inequality and (3.7) give

$$\Big| \sum_{0 \leq j \neq l \leq \kappa} i\partial \tilde{g}_j \wedge \overline{\partial} \tilde{g}_l \prod_{m \neq j,l} (\tilde{g}_m + M) \Big| \leq (\kappa + 1)(M+1)^{\kappa-1} \sum_{j=0}^{\kappa} i\partial \tilde{g}_j \wedge \overline{\partial} \tilde{g}_j$$

$$\leq (\kappa + 1)(M+1)^{\kappa-1} \sum_{j=0}^{\kappa} dd^c \tilde{g}_j.$$

It follows that

$$\begin{aligned} dd^c \phi^+ &\geq (M-1)^{\kappa} \sum_{j=0}^{\kappa} dd^c \tilde{g}_j - (\kappa + 1)(M+1)^{\kappa-1} \sum_{j=0}^{\kappa} dd^c \tilde{g}_j \\ &= (M-1)^{\kappa} \Big[1 - \frac{(\kappa+1)}{M+1} \Big(1 + \frac{2}{M-1}\Big)^{\kappa} \Big] \sum_{j=0}^{\kappa} dd^c \tilde{g}_j, \end{aligned}$$

which gives $dd^c \phi^+ \geq 0$ since the choice $M = 10\kappa$ implies that $\left(1 + \frac{2}{M-1}\right)^{\kappa} < \left(1 + \frac{1}{\kappa}\right)^{\kappa} < 3$. Similarly, in the same way, we also have

$$dd^c \phi^- \geq (M-1)^{\kappa} \sum_{j=0}^{\kappa} dd^c \tilde{g}_j - (M+1)^{\kappa-1} \sum_{0 \leq j \neq l \leq \kappa} \big| i\partial \tilde{g}_j \wedge \overline{\partial} \tilde{g}_l \big|,$$

which gives $dd^c \phi^- \geq 0$. This concludes the proof of the first assertion of the lemma.

(ii) The second assertion of the lemma is a consequence of Lemma 2.6. $\qquad \square$

End of the proof of Theorem 1.2. Recall that it remains to prove (3.5) and (3.6). Since μ is invariant and n_1 is even, we have

$$\langle \mu, g_0^\pm \cdot (h^\pm \circ f^{n_1}) \rangle = \langle \mu, (g_0^\pm \circ f^{-n_1/2})(h^\pm \circ f^{n_1/2}) \rangle.$$

We now transform the last integral on $\mathbb{C}^2$ to an integral on $\mathbb{C}^4$ in order to use the dynamical system $F = (f, f^{-1})$ on $\mathbb{C}^4$ introduced above. We are using the coordinates (z, w) on $\mathbb{C}^4 = \mathbb{C}^2 \times \mathbb{C}^2$. We will also use the diagonal of $\mathbb{C}^2 \times \mathbb{C}^2$, which is given by $\Delta = \{(z, w) \colon z = w\}$.

Recall that we have $\mu = T_+ \wedge T_-$ and that the currents $T_\pm$ have local continuous potentials in $\mathbb{C}^2$. It follows that the intersections of $\mathbb{T}_\pm$ with positive closed currents on $\mathbb{C}^4$ are meaningful. Moreover, the invariance of $T_\pm$ implies that $(F^{n_1/2})_*(\mathbb{T}_+) = d^{-n_1}\mathbb{T}_+$ on $\mathbb{C}^4$. Thanks to the above identities, we have

$$
\begin{aligned}
(3.8) \qquad \langle \mu, g_0^\pm \cdot (h^\pm \circ f^{n_1}) \rangle &= \langle T_+ \wedge T_-, (g_0^\pm \circ f^{-n_1/2})(h^\pm \circ f^{n_1/2}) \rangle \\
&= \langle (T_+ \otimes T_-) \wedge [\Delta], (g_0^\pm \circ f^{-n_1/2}(w))(h^\pm \circ f^{n_1/2}(z)) \rangle \\
&= \langle \mathbb{T}_+ \wedge [\Delta], (F^{n_1/2})^*(\phi^\pm) \rangle \\
&= \langle d^{-n_1}\mathbb{T}_+ \wedge (F^{n_1/2})_*[\Delta], \phi^\pm \rangle \\
&= \langle d^{-n_1}(F^{n_1/2})_*[\Delta], \phi^\pm \mathbb{T}_+ \rangle.
\end{aligned}
$$

We apply Corollary 2.5 with the functions $\phi^\pm = g_0^\pm(w) \cdot h^\pm(z)$ instead of ϕ and the current $[\Delta]$ instead of R. For this purpose, since $\overline{\Delta} \cap \mathbb{I}^+ = \varnothing$, we can choose a suitable open set V_1 containing Δ. We also fix an open set V_2 as in Section 2. Since B is large enough, Lemma 3.1 implies that $dd^c\phi^\pm \geq 0$ on a neighbourhood of $\mathbb{K}^+ \cap V_2$. Thus, we obtain from Corollary 2.5 that

$$\langle d^{-n_1}(F^{n_1/2})_*[\Delta] - \mathbb{T}_-, \phi^\pm \mathbb{T}_+ \rangle \lesssim d^{-n_1/2}$$

or equivalently

$$(3.9) \qquad \langle d^{-n_1}(F^{n_1/2})_*[\Delta], \phi^\pm \mathbb{T}_+ \rangle - \langle \mathbb{T}_-, \phi^\pm \mathbb{T}_+ \rangle \lesssim d^{-n_1/2}.$$

Together, (3.8), (3.9), and the fact that

$$\langle \mathbb{T}_-, \phi^\pm \mathbb{T}_+ \rangle = \langle \mathbb{T}_+ \wedge \mathbb{T}_-, \phi^\pm \rangle = \langle \mu \otimes \mu, g_0^\pm(w) \cdot h^\pm(z) \rangle = \langle \mu, g_0^\pm \rangle \cdot \langle \mu, h^\pm \rangle$$

give the desired estimates (3.5) and (3.6). The proof of Theorem 1.2 is complete. $\qquad \square$

Remark 3.2. Friedland and Milnor [24] proved that any polynomial automorphism of $\mathbb{C}^2$ is either conjugate to an elementary automorphism which preserves a fibration by parallel complex lines or to a Hénon map as above. So, our results apply to all automorphisms of $\mathbb{C}^2$ which are not conjugated to an elementary one.

Remark 3.3. When f is a regular automorphism of $\mathbb{C}^k$ with k even and $p = k/2$, the map (f, f^{-1}) is regular on $\mathbb{C}^{2k}$. The same proof as above gives us the exponential mixing of all orders and the CLT for f. The results still hold for every regular automorphism but the proof requires some extra technical arguments that we choose to do not present here for simplicity, see for instance de Thélin and Vigny [10, 34].

Remark 3.4. When f is a horizontal-like map such that the main dynamical degree is larger than the other dynamical degrees, the same strategy gives the exponential mixing of all orders and the CLT, see the papers [13, 16] by Nguyen, Sibony and the second author, and in particular [13] for the necessary estimates. In particular, these results hold for all Hénon-like maps in dimension 2, see also Dujardin [21].

Remark 3.5. In the companion paper [7], we explain how to adapt our strategy to get the exponential mixing of all orders and the Central Limit Theorem for automorphisms of compact Kähler manifolds with simple action on cohomology. As the proof in that case requires the theory of super-potentials, which is not needed for Hénon maps, we choose to do not present it here.

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