EXPONENTIAL MIXING OF ALL ORDERS AND CLT FOR AUTOMORPHISMS OF COMPACT KÄHLER MANIFOLDS

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Abstract. We consider the unique measure of maximal entropy of an automorphism of a compact Kähler manifold with simple action on cohomology. We show that it is exponentially mixing of all orders with respect to Hölder observables. It follows that the Central Limit Theorem (CLT) holds for these observables. In particular, our result applies to all automorphisms of compact Kähler surfaces with positive entropy.

Notation. The pairing $\langle\cdot,\cdot\rangle$ is used for the integral of a function with respect to a measure or more generally the value of a current at a test form. By (p,p)-currents we mean currents of bi-degree (p,p). The notations $\lesssim$ and $\gtrsim$ stand for inequalities up to a multiplicative constant. If R and S are two real currents of the same bi-degree, we write $|R| \leq S$ when $S \pm R \geq 0$. Observe that this forces S to be positive.

Given a compact Kähler manifold X of dimension k, for every $0 \leq q \leq k$ we will denote by $\mathscr{D}_q(X)$ (resp. $\mathscr{D}_q^0(X)$) the real space generated by positive closed (resp. dd^c-exact) (q,q)-currents on X. For $S \in \mathscr{D}_q(X)$, we will denote by $\{S\}$ the cohomology class of S in $H^{q,q}(X,\mathbb{R})$.

1. Introduction

Let (X,ω) be a compact Kähler manifold of dimension k and f a holomorphic automorphism of X. We refer to [1, 20, 25, 26, 27, 29] for interesting examples of such maps, and to [7, 8, 13, 14] for their general properties, see also [4, 5, 6, 19, 23, 28]. We denote by f^n the iterate of order n of f. For $0 \leq q \leq k$, the dynamical degree d_q is defined as the spectral radius of the pull-back operator acting on the cohomology group $H^{q,q}(X,\mathbb{R})$. We have $d_0 = d_k = 1$ and $d_q(f^n) = d_q(f)^n$ for all $n \in \mathbb{N}$.

By a fundamental result of Khovanskii [24], Teissier [31], and Gromov [21], the sequence $q \mapsto \log d_q$ is concave, see also [10]. This implies that there exist integers $0 \leq p \leq p' \leq k$ such that

$$1 = d_0 < d_1 < \ldots < d_p = \ldots = d_{p'} > \ldots > d_k = 1.$$

We say that f has *simple action on cohomology* if $p = p'$ and if moreover the action of f^* on $H^{p,p}(X,\mathbb{R})$ admits a unique eigenvalue of maximal modulus. Such eigenvalue is then necessarily equal to d_p. We denote in this case by $\delta = \delta(f)$ the maximum between $\max_{q\neq p} d_q$ and the moduli of the other eigenvalues for the action of f^* on $H^{p,p}(X,\mathbb{R})$. We call d_p the *main dynamical degree* and δ the *auxiliary dynamical degree* of f.

From now on, we assume that f has simple action on cohomology. It admits a unique probability measure of maximal entropy μ, which is the intersection of a positive closed (p,p)-current T_+ and a positive closed $(k-p,k-p)$-current T_- (the main Green currents of f), see [13, 14]. Such measure is also called the *equilibrium measure* of f, and is mixing and hyperbolic. It was shown in [15] that such measure is exponentially mixing for Hölder observables, see also [9, 33, 35]. We consider here the following stronger property.

Definition 1.1. Let ν be an invariant measure and $(E, \|\cdot\|_E)$ a normed space of real functions on X, with $\|\cdot\|_{L^r(\nu)} \lesssim \|\cdot\|_E$ for all $1 \leq r < \infty$. We say that ν is *exponentially mixing of all*

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orders for observables in E if for all $\kappa \in \mathbb{N}^*$ there exist constants $C_\kappa > 0$ and $0 < \theta_\kappa < 1$ such that, for all observables $g_0, \ldots, g_\kappa \in E$ and integers $0 =: n_0 \leq n_1 \leq \cdots \leq n_\kappa$, we have

$$\Big|\langle \nu, g_0(g_1 \circ f^{n_1}) \ldots (g_\kappa \circ f^{n_\kappa})\rangle - \prod_{j=0}^{\kappa} \langle \nu, g_j\rangle\Big| \leq C_\kappa \cdot \Big(\prod_{j=0}^{\kappa} \|g_j\|_E\Big) \cdot \theta_\kappa^{\min_{0\leq j\leq \kappa-1}(n_{j+1}-n_j)}.$$

The exponential mixing of all orders is one of the strongest ergodic properties for a dynamical system. It implies a number of statistical properties that seem unattainable without such quantitative control, see for instance [3, 16]. It is a main open question if this is implied by the exponential mixing of order 1, see for instance [17, Question 1.5]. We established in [2] the exponential mixing of all orders for every complex Hénon map. The following is the main result of this paper.

Theorem 1.2. *Let f be a holomorphic automorphism of a compact Kähler manifold (X, ω) with simple action on cohomology. Let d_p be its main dynamical degree and δ its auxiliary dynamical degree. Then, for every $\delta < \delta' < d_p$ and $0 < \gamma \leq 2$, the equilibrium probability measure μ of f is exponentially mixing of all orders $\kappa \in \mathbb{N}^*$ for all $\mathcal{C}^\gamma$ observables, with $\theta_\kappa = (d_p/\delta')^{-(\gamma/2)^{\kappa+1}/2}$, see Definition 1.1.*

In order to prove Theorem 1.2 we rely on delicate estimates coming from pluripotential theory and on the theory of positive closed currents. We partially follow the strategy in [2], but the absence of plurisubharmonic functions on compact Kähler manifolds makes it not possible to employ a key step developed there. Instead, we rely on the theory of *super-potentials* for positive closed currents, and on quantitative estimates on the convergence of sufficiently regular positive closed currents towards the Green currents. We will give below an overview of our strategy.

As in [2], the following is a consequence of Theorem 1.2 and [3]. Recall that u *satisfies the Central Limit Theorem* (CLT) with variance $\sigma^2 \geq 0$ with respect to μ if $n^{-1/2}(S_n(u) - n\langle \mu, u\rangle) \to \mathcal{N}(0, \sigma^2)$ in law, where $\mathcal{N}(0, \sigma^2)$ denotes the (possibly degenerate, for $\sigma = 0$) Gaussian distribution with mean 0 and variance σ^2, i.e., for any interval $I \subset \mathbb{R}$ we have

$$\lim_{n\to\infty} \nu\Big\{\frac{S_n(u) - n\langle \mu, u\rangle}{\sqrt{n}} \in I\Big\} = \begin{cases} 1 \text{ when } I \text{ is of the form } I = (-\delta, \delta) & \text{if } \sigma^2 = 0, \\ \frac{1}{\sqrt{2\pi\sigma^2}} \displaystyle\int_I e^{-t^2/(2\sigma^2)} dt & \text{if } \sigma^2 > 0. \end{cases}$$

Corollary 1.3. *Let f be a holomorphic automorphism of a compact Kähler manifold (X, ω) with simple action on cohomology. Then all Hölder observables $u\colon X \to \mathbb{R}$ satisfy the Central Limit Theorem with respect to the measure of maximal entropy μ of f with*

$$\sigma^2 = \sum_{n\in\mathbb{Z}} \langle \mu, \tilde{u}(\tilde{u} \circ f^n)\rangle = \lim_{n\to\infty} \frac{1}{n} \int_X (\tilde{u} + \tilde{u} \circ f + \ldots + \tilde{u} \circ f^{n-1})^2 d\mu,$$

where $\tilde{u} := u - \langle \mu, u\rangle$.

Let now X be a compact Kähler surface and f an automorphism of positive entropy. By [22, 36], the topological entropy is equal to $\log d_1$, see also [12]. In particular, d_1 is strictly larger than 1. A result by Cantat [4] says that all the eigenvalues of the action of f^* on $H^{1,1}(X, \mathbb{R})$ have modulus 1, except for two eigenvalues d_1 and $1/d_1$, which have multiplicity 1. In particular, every automorphism of positive entropy of a Kähler surface has simple action on cohomology.

Corollary 1.4. *Let f be a holomorphic automorphism of positive entropy on a compact Kähler surface X. Then, for every $1 < d' < d_1$ and $0 < \gamma \leq 2$, the equilibrium probability measure μ of f is exponentially mixing of all orders $\kappa \in \mathbb{N}^*$ for all $\mathcal{C}^\gamma$ observables, with $\theta_\kappa = (d')^{-(\gamma/2)^{\kappa+1}/2}$, see Definition 1.1. Moreover, all Hölder observables satisfy the Central Limit Theorem with respect to the measure of maximal entropy of f.*

Strategy of the proof of Theorem 1.2. Using the classical theory of interpolation [32], it is enough to prove the theorem in the case where $\gamma = 2$. Consider the compact Kähler manifold $\mathbb{X} := X \times X$. The automorphism $F := (f, f^{-1})$ of $\mathbb{X}$ and its inverse have simple action on cohomology, with the largest dynamical degree being that of order k, which is equal to d_p^2. We can define the main Green currents $\mathbb{T}_+$ and $\mathbb{T}_-$ for F as $\mathbb{T}_+ := T_+ \otimes T_-$ and $\mathbb{T}_- := T_- \otimes T_+$. They satisfy $(F^n)^*(\mathbb{T}_+) = d_p^2 \mathbb{T}_+$ and $(F^n)_*(\mathbb{T}_-) = d_p^2 \mathbb{T}_-$. Proving the exponential mixing of order κ for the $\kappa + 1$ observables $g_0, \ldots, g_\kappa$ can be reduced to proving the estimate

$$(1.1) \qquad |\langle d_p^{-n_1} (F^{n_1/2})_*[\Delta] - \mathbb{T}_-, \Theta_{\{g_j\},\{n_j\}} \rangle| \lesssim \prod_{j=0}^{\kappa} \|g_j\|_{\mathcal{C}^2} d^{-\min_{0 \le j \le \kappa-1} (n_{j+1} - n_j)/2},$$

where $[\Delta]$ denotes the current of integration on the diagonal $\Delta \subset X \times X$, (z, w) the coordinates on $X \times X$, we set

$$\Theta_{\{g_j\},\{n_j\}} := g_0(w) g_1(z) (g_2 \circ f^{n_2 - n_1}(z)) \ldots (g_\kappa \circ f^{n_\kappa - n_1}(z)) \mathbb{T}_+,$$

and we assumed for simplicity that n_1 is even.

In the case of Hénon maps, we established in [2] the above convergence by proving that $\Theta_{\{g_j\},\{n_j\}}$ can be replaced by suitable currents $\Theta^\pm$ with $dd^c \Theta^\pm \ge 0$, for which the estimate above can be proved thanks to the properties of plurisubharmonic functions. As non-trivial plurisubharmonic functions do not exist on compact Kähler manifolds, that approach cannot work here. Instead, we use more refined estimates on the regularity of the currents involved. We prove that if $\Theta_{\{g_j\},\{n_j\}}$ is *Hölder continuous* in a precise sense (i.e., when seen as a function on the space of positive exact (k, k)-currents, endowed with a suitable metric), then the convergence (1.1) holds. This is done by exploiting the theory of *super-potentials* for positive closed currents, as developed by Sibony and the second author.

The main task becomes to prove the Hölder continuity of the current $\Theta_{\{g_j\},\{n_j\}}$. Observe that the estimate needs to be uniform in the n_j's and in the g_j's (assuming $\|g_j\|_{\mathcal{C}^2} \le 1$ for all j), in order for the implicit constant in (1.1) not to depend on such parameters. Observe also that this problem does not exist when just proving the mixing of order $\kappa = 1$, see [15]. This is the main technical point of the current paper.

By means of a general comparison principle for the super-potentials of positive closed currents [11], we show that it is enough to find a positive closed current Ξ with a Hölder continuous super-potential and such that

$$|dd^c \Theta_{\{g_j\},\{n_j\}}| \le \Xi \quad \text{for all } n_j \text{ and all } g_j \text{ with } \|g_j\|_{\mathcal{C}^2} \le 1.$$

Finding such Ξ and establishing such an estimate rely on the gap between d_p and the auxiliary dynamical degree of f and on Hölder estimates for the action on f^* on $(p+1, p+1)$-currents, that we also develop in this paper.

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2. Super-potentials of currents on compact Kähler manifolds

We fix in this section a k-dimensional compact Kähler manifold X and a Kähler form ω on X. For $0 \le q \le k$, we denote by $\mathscr{D}_q(X)$ the real space generated by positive closed (q, q)-currents on X, and by $\mathscr{D}_q^0(X)$ the subspace of $\mathscr{D}_q(X)$ given by the currents whose cohomology class in $H^{q,q}(X,\mathbb{R})$ vanishes. By the $\partial\bar\partial$-lemma, this is the set of currents in $\mathscr{D}_q(X)$ which are $\partial\bar\partial$-exact. Since X is fixed, for simplicity in this section we will drop the dependence on X for these spaces and denote them by $\mathscr{D}_q$ and $\mathscr{D}_q^0$. We set $h_q := \dim H^{q,q}(X,\mathbb{R})$ and fix a collection $\alpha_q = (\alpha_{q,1},\dots,\alpha_{q,h_q})$ of real smooth (q,q)-forms on X such that the family of cohomology classes $\{\alpha_{q,j}\}$ is a basis for $H^{q,q}(X,\mathbb{R})$. We also choose a collection of smooth real closed $(k-q,k-q)$-forms $\check\alpha_q = (\check\alpha_{q,1},\dots,\check\alpha_{q,h_q})$ which represent a dual basis of α_q in $H^{k-q,k-q}(X,\mathbb{R})$ for the Poincaré duality.

Recall that the mass of a positive closed current $S \in H^{q,q}(X,\mathbb{R})$ is defined as $\|S\| := \langle S, \omega^{k-q}\rangle$ and it depends only on the cohomology class $\{S\}$ of S in $H^{q,q}(X,\mathbb{R})$. The norm $\|\cdot\|_*$ is defined for $S \in \mathscr{D}_q$ as

$$\|S\|_* := \min\{\|S'\| \colon |S| \le S'\}.$$

When S is dd^c-exact the norm $\|S\|_*$ is equivalent to the norm given by $\min \|S^+\|$, where the minimum is taken over all decompositions $S = S^+ - S^-$ with $S^\pm$ positive closed. Observe that, for each such decomposition, we have $\|S^+\| = \|S^-\|$ as a consequence of the equality $\{S^+\} = \{S^-\}$. We say that a subset of $\mathscr{D}_q$ is $*$-*bounded* if it is bounded for $\|\cdot\|_*$.

Definition 2.1. We say that a sequence $S_n \in \mathscr{D}_q$ $*$-*converges to* $S \in \mathscr{D}_q$ if $S_n \to S$ in the sense of currents, and $\|S_n\|_*$ is bounded by a constant independent of n.

Remark 2.2. The convergence is Definition 2.1 defines a topology on $\mathscr{D}_q$, which is not the one given by $\|\cdot\|_*$. We can also define a topology on $\mathscr{D}_q^0$ with the same definition. By [12], smooth forms are dense in $\mathscr{D}_q$ and $\mathscr{D}_q^0$ with respect to the topology of Definition 2.1.

For any $0 < l < \infty$, denote by $\|\cdot\|_{\mathcal{C}^l}$ the standard $\mathcal{C}^l$ norm on the space of differential forms. We consider a norm $\|\cdot\|_{\mathcal{C}^{-l}}$ and a distance dist_l on $\mathscr{D}_q$ defined by

$$\|S\|_{\mathcal{C}^{-l}} := \sup_{\|\Phi\|_{\mathcal{C}^l}\le 1} |\langle S,\Phi\rangle| \quad \text{and} \quad \text{dist}_l(S,S') := \|S-S'\|_{\mathcal{C}^{-l}},$$

where the supremum in the first definition is on smooth $(k-q,k-q)$-forms Φ on X. Observe that, by interpolation [32], for every $0 < l < l' < \infty$ and $m > 0$, we have

$$\|S\|_{\mathcal{C}^{-l'}} \le \|S\|_{\mathcal{C}^{-l}} \le c_{l,l',m}\|S\|_{\mathcal{C}^{-l'}}^{l/l'} \quad \text{for all } S \text{ such that } \|S\|_* \le m, \tag{2.1}$$

for some positive constant $c_{l,l',m}$. In particular, the above inequalities imply that

$$\text{dist}_{l'} \le \text{dist}_l \le c_{l,l',m}(\text{dist}_{l'})^{l/l'} \quad \text{on} \quad \{S \in \mathscr{D}_p \colon \|S\|_* \le m/2\}.$$

We now recall the notion of *super-potential* for currents in $\mathscr{D}_q$, see [14]. This notion generalizes that of potentials for $(1,1)$-positive closed currents on X, which are the quasi-plurisubharmonic functions on X. Super-potentials of a current $S \in \mathscr{D}_q$ that we use here are functions, depending linearly on S, and defined on a subset of $\mathscr{D}_{k-q+1}^0$ [1].

Take $R \in \mathscr{D}_{k-q+1}^0$. As $\{R\} = 0$, we have $R = dd^c U_R$ for some $(k-q,k-q)$-current U_R, that we call a *potential* of R. By adding to U_R a suitable combination of the $\check\alpha_{q,j}$'s, one can assume that U_R is α_q-normalized, i.e., that $\langle U_R, \alpha_{q,j}\rangle = 0$ for all $1 \le j \le h_q$. The α_q-*normalized super-potential* $\mathcal{U}_S$ of S is defined as

$$\mathcal{U}_S(R) := \langle S, U_R\rangle \tag{2.2}$$

whenever the RHS of the above expression is well-defined. This is the case for instance when S is smooth or when R is smooth and we choose U_R smooth.

Lemma 2.3. *Let S, R, U_R be as above and such that either R or S is smooth.*

(i) *The α_q-normalized super-potential $\mathcal{U}_S$ of S does not depend on the choice of the α_q-normalized potential U_R; in particular, it does not depend on the choice of the $\check\alpha_{q,j}$'s;*

[1] Super-potentials can be defined on a subset of $\mathscr{D}_{k-q+1}$, but it is simpler to use $\mathscr{D}_{k-q+1}^0$, and we will only need this definition in this paper.

(ii) *If* $\{S\} = 0$*, then* $\mathcal{U}_S$ *does not depend on the choice of* α_q.

(iii) *If* S *is a linear combination of the* $\alpha_{q,j}$*'s, then the* α_q*-normalized super-potential* $\mathcal{U}_S$ *of* S *vanishes identically on* $\mathscr{D}^0_{k-q+1}$.

Proof. (i) Let U_R and U'_R be two α_q-normalized potentials of R. As $dd^c(U_R - U'_R) = R - R = 0$, by Poincaré duality and the $\partial\bar{\partial}$-lemma the class $\{U_R - U'_R\} \in H^{k-q,k-q}(X, \mathbb{R})$ is well defined and $\langle S, U_R - U_{R'}\rangle$ only depends on the cohomology class of S. Since both U_R and U'_R are α_q-normalized, the cup-products of such class with all the classes $\{\alpha_{q,j}\}$ are equal to 0. As the $\{\alpha_{q,j}\}$'s form a basis of $H^{q,q}(X, \mathbb{R})$, it follows that $\langle S, U_R - U'_R\rangle = 0$, as required.

(ii) As $\{S\} = 0$, we have $S = dd^c U_S$ for some $(q-1, q-1)$-current U_S. For any potential V_R of R, non necessarily α_q-normalized, we have

$$\langle S, V_R\rangle = \langle dd^c U_S, V_R\rangle = \langle U_S, dd^c V_R\rangle = \langle U_S, R\rangle. \tag{2.3}$$

In particular, the first term of the above expression does not depend on the choice of V_R, as this is the case for the last term. The assertion follows.

(iii) By definition, we have $\langle U_R, \alpha_{q,j}\rangle = 0$ for all j. As S is a linear combination of the $\alpha_{q,j}$'s, we have $\mathcal{U}_S(R) = \langle S, U_R\rangle = 0$. The assertion follows. □

Remark 2.4. Observe that, on the other hand, the value $\langle S, U_R\rangle$ and, by consequence, the α_q-normalized super-potential $\mathcal{U}_S$ of S, depend on α_q when S is not exact.

Definition 2.5. We say that $S \in \mathscr{D}_q$ has a *continuous super-potential* if $\mathcal{U}_S$ extends continuously to $\mathscr{D}^0_{k-q+1}$, with respect to the topology given by Definition 2.1. We also use the notation $\mathcal{U}_S$ for the extended super-potential in this case.

Proposition 2.6. *Take* q, q' *with* $q + q' \leq k$, $S \in \mathscr{D}_q$, $S' \in \mathscr{D}_{q'}$, *and* $R \in \mathscr{D}^0_{k-q+1}$.

(i) *If* S *is smooth, then it has a continuous* α_q*-normalized super-potential for every choice of* α_q*;*

(ii) *If* S *has a continuous* α_q*-normalized super-potential, it also has continuous* α'_q*-normalized super-potentials for every other normalization* $\alpha'_q = (\alpha'_{q,1}, \dots, \alpha'_{q,h_q})$.

(iii) *If* S *belongs to* $\mathscr{D}^0_q$ *and has continuous super-potentials and* R *is smooth we have*

$$\mathcal{U}_S(R) = \mathcal{U}_R(S),$$

where $\mathcal{U}_R$ *is the super-potential of* R *(which is also independent of the choice of the normalization);*

(iv) *If* S' *has continuous super-potentials, then the current* $S \wedge S'$ *is well defined and depends continuously on* S.

In particular, by the third item, for every $R \in \mathscr{D}^0_{k-q+1}$ we can define

$$\mathcal{U}_R(S) := \mathcal{U}_S(R).$$

when $S \in \mathscr{D}^0_q$ has a continuous super-potential.

Proof. (i) This is clear from the definition of the super-potential $\mathcal{U}_S$.

(ii) By (i) we can add to S a smooth closed form. This does not change the problem. So, we can assume that $\{S\} = 0$. By Lemma 2.3(ii), $\mathcal{U}_S$ is independent of the normalization. The assertion follows.

(iii) Since R is smooth, the RHS in (2.2) is well defined for every $S \in \mathscr{D}^0_p$. As $\{S\} = 0$, by Lemma 2.3(ii), $\mathcal{U}_S$ does not depend on the normalization and (2.3) holds. As the last term of that expression is equal to $\mathcal{U}_R(S)$, the assertion follows.

(iv) This is a consequence of [14, Theorem 3.3.2]. □

Definition 2.7. Take $S \in \mathscr{D}_q$. For $l > 0$, $0 < \lambda \leq 1$, and $M \geq 0$, we say that a super-potential $\mathcal{U}_S$ of S is (l, λ, M)-*Hölder continuous* if it is continuous and we have

$$|\mathcal{U}_S(R)| \leq M \|R\|^{\lambda}_{\mathcal{C}^{-l}} \text{ for every } R \in \mathscr{D}^0_{k-q+1} \text{ with } \|R\|_* \leq 1.$$

Lemma 2.8. *Take $S \in \mathscr{D}_q$ and $R, R' \in \mathscr{D}^0_{k-q+1}$ with $\|R\|_* \leq 1$ and $\|R'\|_* \leq 1$.*

(i) *If S is smooth, then the α_q-normalized super-potential $\mathcal{U}_S$ of S is $(2, 1, M)$-Hölder continuous with $M \leq c\|S\|_{\mathcal{C}^2}$ for some constant $c > 0$ independent of S (but possibly depending on α_q).*

(ii) *If S has an (l, λ, M)-Hölder continuous α_q-normalized super-potential $\mathcal{U}_S$, then*

$$|\mathcal{U}_S(R) - \mathcal{U}_S(R')| \leq 2^{1-\lambda} M \|R - R'\|^{\lambda}_{\mathcal{C}^{-l}}.$$

In particular, $\mathcal{U}_S$ can be seen as a Hölder continuous function on $\{R \in \mathscr{D}^0_{k-q+1} \colon \|R\|_* \leq 1\}$.

(iii) *If S has an (l, λ, M)-Hölder continuous α_q-normalized super-potential $\mathcal{U}_S$, then it also has (l, λ, M')-Hölder continuous α'_q-normalized super-potentials for every choice of α'_q and for some M' independent of S.*

Proof. (i) By Lemma 2.3(iii), we can add to S a combination of the $\alpha_{q,j}$'s and assume that $\{S\} = 0$. By Lemma 2.3(ii), we have $\mathcal{U}_S(R) = \langle U_S, R\rangle$, where U_S is a smooth potential of S. The assertion is now clear.

(ii) As $R - R' \in \mathscr{D}^0_{k-q+1}$ and $\|R - R'\|_* \leq 2$, the assertion follows from Definition 2.7 applied with $(R - R')/2$ instead of R.

(iii) As in Proposition 2.6(ii), by (i) we can assume that $\{S\} = 0$. The assertion now follows from Lemma 2.3(ii). □

Proposition 2.9. *Take q, q' with $q + q' \leq k$. Let $\alpha_q, \alpha_{q'}, \alpha_{q+q'}$ be as above and take $S, T \in \mathscr{D}_q$ and $S' \in \mathscr{D}_{q'}$ with $\|S\|_* \leq 1$, $\|T\|_* \leq 1$, and $\|S'\|_* \leq 1$. Assume that the α_q-normalized super-potential $\mathcal{U}_S$ of S is (l, λ, M)-Hölder continuous and that the $\alpha_{q'}$-normalized super-potential $\mathcal{U}_{S'}$ of S' is (l', λ', M')-Hölder continuous.*

(i) *For every $l_1 > 0$, $\mathcal{U}_S$ is (l_1, λ_1, M_1)-Hölder continuous, for some λ_1 and M_1 depending on l, l_1, λ, M, and α_q, but independent of S.*

(ii) *If S, T are positive and $T \leq S$, then any α_q-normalized super-potential $\mathcal{U}_T$ of T is $(2, \lambda_1, M_1)$-Hölder continuous, for some λ_1 and M_1 depending on l, λ, M, and α_q, but independent of S and T.*

(iii) *$S \wedge S'$ has a $(2, \lambda_1, M_1)$-Hölder continuous $\alpha_{q+q'}$-normalized super-potential, for some λ_1 and M_1 depending on l, l', λ, λ', M, M', α_q, $\alpha_{q'}$, and $\alpha_{q+q'}$, but independent of S and S'.*

Proof. The first assertion is a consequence of (2.1). The second one is a consequence of [11, Theorem 1.1] and the first one. The third assertion is a consequence of [14, Proposition 3.4.2] and the first one. □

3. Dynamical Green currents

We keep in this section the notations of Section 2.

3.1. Convergence towards Green currents.

We fix here an automorphism f of X with simple action on cohomology. We let p be such that d_p is the largest dynamical degree of f, and let δ be its auxiliary degree. We also fix δ' such that $\delta < \delta' < d_p$. By assumption, the eigenspace associated to the eigenvalue d_p of the action of f^* on $H^{p,p}(X, \mathbb{R})$ is a real line H^+. A *Green (p,p)-current* T_+ of f is a non-zero positive closed (p,p)-current invariant under $d_p^{-1} f^*$, i.e., satisfying $f^*(T_+) = d_p T_+$. The cohomology class $\{T_+\}$ generates H^+.

Lemma 3.1. *Let f, d_p, T_+, α_p be as above.*

(i) *T_+ is the unique positive closed (p,p)-current in $\{T_+\}$, and it has a $(2, \lambda, M)$-Hölder continuous α_p-normalized super-potential for some λ and M.*

(ii) *For every $S \in \mathscr{D}_p$ we have $d_p^{-n}(f^n)^* S \to sT_+$. Here the constant s depends linearly on $\{S\}$. More precisely, s is the constant such that that $d_p^{-n}(f^n)^*\{S\} \to s\{T_+\}$.*

Proof. (i) The first assertion is a consequence of [14, Theorem 4.2.1] and Proposition 2.9(i).

(ii) As f has simple action on cohomology, there exists a constant s such that $d_p^{-n}(f^n)^*\{S\} \to s\{T_+\}$. It is clear that the constant s depends linearly on S. When $S \geq 0$ the statement follows from (i). The general case follows by linearity. □

As the inverse f^{-1} of any automorphism with simple action on cohomology satisfies the same property, the above also holds for the automorphism f^{-1}. Since $(f^{-1})^* = f_*$, by Poincaré duality, we have $d_q(f) = d_{k-q}(f^{-1})$ for all $0 \leq q \leq k$. Hence, the main dynamical degree of f^{-1} is equal to the one of f and is the dynamical degree of order $k-p$ of f^{-1}. The eigenspace associated to this eigenvalue for the action of f_* on $H^{k-p,k-p}(X,\mathbb{R})$ is a real line H^-. A *Green* $(k-p,k-p)$*-current* T_- of f is a non-zero positive closed $(k-p,k-p)$-current invariant under $d_p^{-1}f_*$, i.e., satisfying $f_*(T_-) = d_pT_-$. The cohomology class $\{T_-\}$ generates H^-, T_- is the unique positive closed $(k-p,k-p)$-current in $\{T_-\}$, and we have $d_p^{-n}(f^n)_*S \to sT_-$ for every $S \in \mathscr{D}_{k-p}$, where the constant s depends linearly on $\{S\}$.

Note that the currents T_+ and T_- are unique up to multiplicative constants. We choose T_+ and T_- such that the positive measure $T_+ \wedge T_-$ is of mass 1. This is then the unique measure of maximal entropy of f, see [14] for details.

The following result gives a quantitative description of the convergences above, see [15, Proposition 3.1]. Note that the independence of the constant c from S is given in [15, Proposition 2.1].

Proposition 3.2. *Let f, d_p, δ' be as above and S be a current in $\mathscr{D}_p$ with $\|S\|_* \leq 1$. Let s be the constant such that $d_p^{-n}(f^n)^*(S)$ converge to sT_+. Let R be a current in $\mathscr{D}^0_{k-p+1}$ with $\|R\|_* \leq 1$ and whose α_{k-p+1}-normalized super-potential $\mathcal{U}_R$ is $(2,\lambda,1)$-Hölder continuous for some $\lambda > 0$. Let $\mathcal{U}_{T_+}$ and $\mathcal{U}_n$ be the α_p-normalized super-potentials of T_+ and $d_p^{-n}(f^n)^*(S)$, respectively. Then*

$$|\mathcal{U}_n(R) - s\mathcal{U}_{T_+}(R)| \leq c(d_p/\delta')^{-n},$$

where $c > 0$ is a constant independent of R, S, s, and n.

In particular, we will need the following consequence of the above result, see also Taflin [30, Theorem 3.7.1] for a similar result in the case where $p = 1$.

Corollary 3.3. *Let f, d_p, δ' be as above and S be a current in $\mathscr{D}_p$ with $\|S\|_* \leq 1$ and such that $d_p^{-n}(f^n)^*(S) \to 0$. Let ξ be a $(k-p,k-p)$-current with $\|dd^c\xi\|_* \leq 1$ and such that the super-potential $\mathcal{U}_{dd^c\xi}$ of $dd^c\xi$ (which is independent of the normalization) is $(2,\lambda,1)$-Hölder continuous for some $\lambda > 0$. Assume that either S or ξ is smooth. Then*

$$|\langle d_p^{-n}(f^n)^*(S), \xi\rangle| \leq c(d_p/\delta')^{-n},$$

where $c > 0$ is a constant independent of R, S, and n.

Observe that, since either S or ξ is smooth, the pairing in the statement is well defined.

Proof. We first consider the case where S is exact. Recall that, in this case, the super-potential of S is independent of the normalization. As we have $\{(f^n)^*(S)\} = (f^n)^*\{S\} = 0$, the same is true for the super-potential of $(f^n)^*(S)$ for all $n \in \mathbb{N}$. Setting $S_n := d_p^{-n}(f^n)^*(S)$, we then have

$$|\langle S_n, \xi\rangle| = |\mathcal{U}_{S_n}(dd^c\xi)|.$$

By the assumptions on ξ, we can apply Proposition 3.2 with $dd^c\xi$ instead of R and $s = 0$. The assertion in this case follows.

Let us now consider the general case. Observe that $d_p^{-n}(f^n)^*\{S\} \to 0$. So, the set of the classes $\{S\}$ of the currents $S \in \mathscr{D}_p$ with this property is an hyperplane $H \subset H^{p,p}(X,\mathbb{R})$, which is a complement of the line generated by $\{T_+\}$ and is invariant under the action of f^*. For simplicity, denoting by h the dimension of $H^{p,p}(X,\mathbb{R})$, we let $\alpha_1, \dots, \alpha_{h-1}$ be real smooth (p,p)-forms such that $\{\alpha_1\}, \dots, \{\alpha_{h-1}\}$ form a basis for H, and α_h be a smooth form in the class of

T_+. We will consider α-normalized super-potentials of currents in $\mathscr{D}_p$, with $\alpha = (\alpha_1, \ldots, \alpha_h)$. We also fix $\check\alpha = (\check\alpha_1, \ldots, \check\alpha_h)$ a dual basis of α. As $\{S\} \in H$, we have

$$S = a_1\alpha_1 + \cdots + a_{h-1}\alpha_{h-1} + S''$$

for some $S'' \in \mathscr{D}_p^0$ and $a_j \in \mathbb{R}$. By the first part of the proof applied to S'', it is enough to prove the statement for $S' := \sum_{j=1}^{h-1} a_j\alpha_j$ instead of S. Observe, in particular, that S' is smooth and that the α-normalized super-potential $\mathcal{U}_{S'}$ of S' satisfies $\mathcal{U}_{S'} = 0$ by Lemma 2.3(iii).

Denote $S'_n := d_p^{-n}(f^n)^*(S')$. We have

$$S'_n = \sum_{j=1}^{h} c_{n,j}\alpha_j + dd^c V_n,$$

where the $c_{n,j}$ are defined by $\{S'_n\} = \sum_{j=1}^{h} c_{n,j}\{\alpha_j\}$ and the $(p-1, p-1)$-current V_n is chosen so that $\langle V_n, \check\alpha_j\rangle = 0$ for all $1 \le j \le h$. Observe that $c_{n,h} = 0$ for all n because of the invariance of H. It follows that, for any ξ as in the statement, we have

$$\langle S'_n, \xi\rangle = \sum_{j=1}^{h-1} c_{n,j}\langle\alpha_j, \xi\rangle + \langle dd^c V_n, \xi\rangle = \sum_{j=1}^{h-1} c_{n,j}\langle\alpha_j, \xi\rangle + \mathcal{U}_{S'_n}(dd^c\xi),$$

where $\mathcal{U}_{S'_n}$ is the α-normalized super-potential of S'_n. By the assumptions on ξ and Proposition 3.2 applied with $dd^c\xi$ instead of R and with $s = 0$, we have $|\mathcal{U}_{S'_n}(dd^c\xi)| \lesssim (d_p/\delta')^{-n}$. Hence, it is enough to prove that, for all $1 \le j \le h-1$, we have $|c_{n,j}| \lesssim (d_p/\delta')^{-n}$.

Let A be the $(h-1)\times(h-1)$ matrix representing $f^*_{|H}$ with respect to the basis $\{\alpha_j\}_{1\le j\le h-1}$, i.e., whose j-th column is given by the coordinates of $f^*\{\alpha_j\}$ with respect to the given basis. Denoting $\underline{c}_n := (c_{n,1}, \ldots, c_{n,h-1})^t$ and $\underline{a} := (a_1, \ldots, a_{h-1})^t$, we see that

$$\underline{c}_n = d_p^{-n} A^n \underline{a}.$$

As the spectral radius of the action of f^* on H is smaller than δ', we see that $\|A^n\| = o(\delta'^n)$. It follows that $|c_{n,j}| \lesssim (\delta'/d_p)^{-n}$, as desired. The assertion follows. □

In our study we will also need the case where $q = p+1$. It follows from the definition of d_{p+1} and the fact that $d_{p+1} < d_p$ that $d_p^{-n}(f^n)^*(S) \to 0$ as $n \to \infty$ for every $S \in \mathscr{D}_{p+1}$. We will need later a more quantitative version of this convergence, for S with Hölder continuous super-potentials. This is given by the next proposition. Recall that we normalize potentials with respect to a given $\alpha_{p+1} = (\alpha_{p+1,1}, \ldots, \alpha_{p+1,h_{p+1}})$, where h_{p+1} is the dimension of $H^{p+1,p+1}(X, \mathbb{R})$.

Proposition 3.4. *Let f be as above. Take $S \in \mathscr{D}_{p+1}$ with $\|S\|_* \le 1$.*

(i) *$\|d_p^{-n}(f^n)^*S\|_* \to 0$ as $n \to \infty$; in particular, $d_p^{-n}(f^n)^*S \to 0$ as $n \to \infty$.*

(ii) *Assume that S has an (l, λ, M)-Hölder continuous α_{p+1}-normalized super-potential. Then there exist λ' and M' independent of S such that, for every $n \ge 0$, $d_p^{-n}(f^n)^*S$ has a $(2, \lambda', M')$-Hölder continuous α_{p+1}-normalized super-potential.*

Proof. The first assertion is a consequence of the inequality $d_{p+1} < d_p$ and of the fact that, for every $\epsilon > 0$, we have $\|(f^n)^*(S)\|_* \le C(d_{p+1} + \epsilon)^n \|S\|_*$, where the constant C is independent of S because the mass of a positive closed current only depends on its cohomology class. It remains to prove the second assertion. By Proposition 2.9(i) we can assume that $l = 4$.

Particular case. We first prove the third assertion assuming that S is exact. Recall that, in this case, the super-potentials $\mathcal{U}_S$ is independent of the normalization, see Lemma 2.3(ii).

Fix $R \in \mathscr{D}_{k-p}^0$ with $\|R\|_* \le 1$ and set $S_n := d_p^{-n}(f^n)^*S$. We need to show that

$$|\mathcal{U}_{S_n}(R)| \le M'\|R\|_{\mathcal{C}^{-4}}^{\lambda'}$$

for some λ' and M' independent of S and R. The assertion is then a consequence of Proposition 2.9(i).

Fix $L \geq \max(2, \sup_{\|\Phi\|_{\mathcal{C}^4}\leq 1} \|f^*\Phi\|_{\mathcal{C}^4})$ where the supremum is on smooth $(q-1,q-1)$-forms Φ on X. As S and R are both exact, by Lemma 2.3(ii) and the definition of super-potential we have

$$|\mathcal{U}_{S_n}(R)| = d_p^{-n}\big|\mathcal{U}_S\big((f^n)_*(R)\big)\big| \leq M\|d_p^{-n}(f^n)_*(R)\|^{\lambda}_{\mathcal{C}^{-4}}. \tag{3.1}$$

By Poincaré duality, $\|d_p^{-n}(f^n)_*(R)\|_*$ is bounded independently of n. We also have

$$\begin{aligned}\|d_p^{-n}(f^n)_*(R)\|_{\mathcal{C}^{-4}} &= \sup_{\|\Phi\|_{\mathcal{C}^4}\leq 1} \big|\big\langle d_p^{-n}(f^n)_*(R), \Phi\big\rangle\big| \\ &= \sup_{\|\Phi\|_{\mathcal{C}^4}\leq 1} \big|\mathcal{U}_{d_p^{-n}(f^n)_*(R)}(dd^c\Phi)\big| \\ &\lesssim \theta^n\end{aligned}$$

for some $0<\theta<1$, where the last step follows from Corollary 3.3 applied with f^{-1}, R, and Φ instead of f, S, and ξ. Observe that the assumption on $dd^c\xi$ in that corollary is satisfied by $dd^c\Phi$ by Lemma 2.8(i).

Assume first that $n \geq -(2\log L)^{-1}\log\|R\|_{\mathcal{C}^{-4}}$. In this case, we have

$$\|d_p^{-n}(f^n)_*(R)\|_{\mathcal{C}^{-4}} \lesssim \theta^n \leq \|R\|_{\mathcal{C}^{-4}}^{\frac{1}{2}\frac{|\log\theta|}{\log L}}.$$

Together with (3.1), this implies the assumption.

Assume instead that $n \leq -(2\log L)^{-1}\log\|R\|_{\mathcal{C}^{-4}}$. Observe that this implies that $L^n\|R\|_{\mathcal{C}^{-4}} \leq \|R\|^{1/2}_{\mathcal{C}^{-4}}$. Hence, for all such n, we have

$$\|d_p^{-n}(f^n)_*R\|_{\mathcal{C}^{-4}} \leq \sup_{\|\Phi\|_{\mathcal{C}^4}\leq 1} d_p^{-n}|\langle R, (f^n)^*\Phi\rangle| \leq d_p^{-n}L^n\|R\|_{\mathcal{C}^{-4}} \leq d_p^{-n}\|R\|^{1/2}_{\mathcal{C}^{-4}},$$

which also implies the assertion in this case. The proof in the particular case is complete.

General case. We now consider the general case. By the previous part of the proof, and as $H^{p+1,p+1}(X,\mathbb{R})$ is finite dimensional, it is enough to prove the assertion for any finite family of smooth forms whose classes generate $H^{p+1,p+1}(X,\mathbb{R})$. Therefore, it is enough to prove the statement for a fixed smooth form S.

Let $1 \leq m < \dim H^{p+1,p+1}(X,\mathbb{R})$ be the minimal integer such that $\{S\}$, $\{f^*(S)\}, \ldots, \{(f^m)^*(S)\}$ are linearly dependent over $\mathbb{R}$, and define $a_0,\ldots,a_{m-1}$ by the relation

$$\{(f^m)^*(S)\} = a_0\{S\} + \cdots + a_{m-1}\{(f^{m-1})^*(S)\}. \tag{3.2}$$

Let E be the subspace of $H^{p+1,p+1}(X,\mathbb{R})$ spanned by $\{S\},\ldots,\{(f^{m-1})^*(S)\}$. Then these m classes form a basis $\mathcal{B}$ of E. The action of $f^*_{|E}$ with respect to the basis $\mathcal{B}$ is represented by the $m\times m$ matrix

$$A_E := \begin{pmatrix} 0 & 0 & \ldots & 0 & a_0 \\ 1 & 0 & \ldots & 0 & a_1 \\ 0 & 1 & \ldots & 0 & a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & a_{m-1}\end{pmatrix}.$$

We denote by B_E the transpose of A_E. Fix $0<\epsilon<d_p-d_{p+1}$. By the definition of d_{p+1}, we have $\|A_E^n\| = \|B_E^n\| \lesssim (d_{p+1}+\epsilon)^n = o(d_p^n)$.

It follows from (3.2) that

$$U := (f^m)^*(S) - \sum_{j=0}^{m-1} a_j (f^j)^*(S)$$

is an exact form, i.e., it belongs to $\mathscr{D}^0_{p+1}$. Since $(f^j)^*(S)$, $0\leq j\leq m-1$, and U are smooth, they have (l,λ,N)-Hölder continuous α_{p+1}-normalized super-potentials for some constant N.

For every $n \geq 1$, set

$$\mathbb{W}_n := \begin{pmatrix} (f^n)^*(S) \\ (f^{n+1})^*(S) \\ \vdots \\ (f^{n+m-1})^*(S) \end{pmatrix} \quad \text{and} \quad \mathbb{U}_n := \begin{pmatrix} 0 \\ 0 \\ \vdots \\ (f^n)^*(U) \end{pmatrix},$$

where both $\mathbb{W}_n$ and $\mathbb{U}_n$ have m components. As $\mathbb{W}_1 = B_E \mathbb{W}_0 + \mathbb{U}_0$, we obtain by induction that

$$d_p^{-n}\mathbb{W}_n = d_p^{-n} B_E^n \mathbb{W}_0 + d_p^{-n} \sum_{j=0}^{n-1} B_E^{n-j} \mathbb{U}_j.$$

As the first component of $\mathbb{W}_n$ is equal to $(f^n)^*(S)$, we need to consider the α_{p+1}-normalized super-potential of the first component of the RHS of the above expression.

By the above, the currents $(f^j)^*(S)$, $0 \leq j \leq m-1$, have an (l, λ, N)-Hölder continuous α_{p+1}-normalized super-potential. As $\|B_E^n\| = o(d_{p+1}^n)$, it follows that the α_{p+1}-normalized super-potential of every component of $d_p^{-n} B_E^n \mathbb{W}_0$ is $(l, \lambda, M'/2)$-Hölder continuous, for some M' large enough.

In order to prove the assertion, using again that $\|B_E^{n-j}\| = o(d_p^{n-j})$, it is enough to show that there exist l', λ', M' such that, for every $n \geq 0$, $U_n := d_p^{-n}(f^n)^*(U)$ has a $(l', \lambda', M'/2)$-Hölder continuous α_{p+1}-normalized super-potential. As U is exact, this follows from the particular case considered above. This concludes the proof. □

3.2. **Further properties of the Green currents.** We prove here two lemmas that we will need in the next section in order to apply Proposition 3.2 and Corollary 3.3. As in the previous section, we let f be an automorphism of X with simple action on cohomology, and d_p its largest dynamical degree. In particular, the Green current T_+ has bi-degree (p, p). We also fix a normalization $\alpha_{p+1} := (\alpha_{p+1,1}, \dots, \alpha_{p+1,h_{p+1}})$, where h_{p+1} is the dimension of $H^{p+1,p+1}(X, \mathbb{R})$.

Lemma 3.5. *Let f be as above. Let $\kappa \geq 1$ be an integer and $g_0, \dots, g_\kappa \colon X \to \mathbb{R}$ satisfy $\|g_j\|_{\mathcal{C}^2} \leq 1$. Then there is a positive constant c independent of the g_j's such that for all $\ell_0, \dots, \ell_\kappa \geq 0$ we have*

$$\|dd^c\big((g_0 \circ f^{\ell_0}) \dots (g_\kappa \circ f^{\ell_\kappa})\big) \wedge T_+\|_* \leq c.$$

Proof. Set $\tilde{g}_j := g_j \circ f^{\ell_j}$ and notice that $i\partial \tilde{g}_j \wedge \bar{\partial} \tilde{g}_j = (f^{\ell_j})^*(i\partial g_j \wedge \bar{\partial} g_j)$ for all j. We have

$$dd^c\big(\tilde{g}_0 \dots \tilde{g}_\kappa\big) = \sum_{j=0}^{\kappa} dd^c \tilde{g}_j \prod_{l \neq j} \tilde{g}_l + \sum_{0 \leq j \neq l \leq \kappa} i\partial \tilde{g}_j \wedge \bar{\partial} \tilde{g}_l \prod_{m \neq j,l} \tilde{g}_m.$$

Since $\|g_j\|_{\mathcal{C}^2} \leq 1$ we have $|g_j| \leq 1$ and $|dd^c g_j| \lesssim \omega$. Then

$$\Big| \sum_{j=0}^{\kappa} dd^c \tilde{g}_j \prod_{l \neq j} \tilde{g}_l \Big| \lesssim \sum_{j=0}^{\kappa} (f^{\ell_j})^*(\omega)$$

and an application of Cauchy-Schwarz inequality gives

$$\Big| \sum_{0 \leq j \neq l \leq \kappa} i\partial \tilde{g}_j \wedge \overline{\partial} \tilde{g}_l \prod_{m \neq j,l} \tilde{g}_m \Big| \lesssim \sum_{j=0}^{\kappa} i\partial \tilde{g}_j \wedge \overline{\partial} \tilde{g}_j \lesssim \sum_{j=0}^{\kappa} (f^{\ell_j})^*(\omega).$$

We deduce from the above inequalities and the invariance of T_+ that

$$\begin{aligned} \big|dd^c\big((g_0\circ f^{\ell_0})\dots(g_\kappa\circ f^{\ell_\kappa})\big)\wedge T_+\big| &\lesssim \sum_{j=0}^{\kappa}\big(f^{\ell_j}\big)^*(\omega)\wedge T_+ \\ &= \sum_{j=0}^{\kappa}(f^{\ell_j})^*(\omega)\wedge d_p^{-\ell_j}(f^{\ell_j})^*(T_+) \\ &= \sum_{j=0}^{\kappa} d_p^{-\ell_j}(f^{\ell_j})^*\big(\omega\wedge T_+\big). \end{aligned} \tag{3.3}$$

We now use that the $(p+1,p+1)$-current $\omega\wedge T_+$ is positive and closed. We have

$$\|(f^{\ell_j})^*\big(\omega\wedge T_+\big)\| = \big\langle (f^{\ell_j})^*\big(\omega\wedge T_+\big),\omega^{k-p-1}\big\rangle = \big\langle \omega\wedge T_+,(f^{\ell_j})_*(\omega^{k-p-1})\big\rangle,$$

where the last form is positive closed. The last pairing only depends on the cohomology classes of ω, T_+, and $(f^{\ell_j})_*(\omega^{k-p-1})$. Hence, it is bounded by a constant times the mass of $(f^{\ell_j})_*(\omega^{k-p-1})$. Since

$$\|(f^{\ell_j})_*(\omega^{k-p-1})\| = \langle \omega^{p+1},(f^{\ell_j})_*(\omega^{k-p-1})\rangle = \langle (f^{\ell_j})^*(\omega^{p+1}),\omega^{k-p-1}\rangle,$$

for all $\epsilon>0$ this number is also equal to $\|(f^{\ell_j})^*(\omega^{p+1})\|\lesssim(d_{p+1}+\epsilon)^n$. As $d_{p+1}<d_p$ by assumption, it follows that each term in the last sum in (3.3) is bounded, which implies that the sum is also bounded. The lemma follows. □

Lemma 3.6. *Let f be as above. Let $\kappa\geq 1$ be an integer and $g_0,\dots,g_\kappa\colon X\to\mathbb{R}$ satisfy $\|g_j\|_{\mathcal{C}^2}\leq 1$. Then there exist constants $0<\lambda\leq 1$ and $M>0$, independent of the g_j's, such that for all $\ell_0,\dots,\ell_\kappa\geq 0$ the current*

$$dd^c\big((g_0\circ f^{\ell_0})\dots(g_\kappa\circ f^{\ell_\kappa})\big)\wedge T_+$$

has a $(2,\lambda,M)$-Hölder continuous super-potential.

Observe that the current in the statement is exact, hence its super-potential is independent of the normalization.

Proof. We denote the current in the statement by R. It is a real exact $(p+1,p+1)$-current. Recall that, by Proposition 2.9(i), it is enough to show that R has an (l,λ,M)-Hölder continuous super-potential, for some $l>0$, $0<\lambda\leq 1$, and $M>0$ independent of $g_0,\dots,g_\kappa$ and $\ell_0,\dots,\ell_\kappa$.

By Proposition 2.9(ii), it is enough to show that there exists a positive closed current V with a $(2,\lambda',M')$-Hölder continuous α_{p+1}-normalized super-potential such that $|R|\leq V$. Indeed, this implies that both $V+R$ and $V-R$ are positive, closed, and bounded by $2V$. Hence, they have $(2,\lambda'',M'')$-Hölder continuous α_{p+1}-normalized super-potentials for some constants $0\leq\lambda''\leq 1$ and $M''>0$. It follows that R has a $(2,\lambda'',M'+M'')$-Hölder continuous super-potential.

We have already seen in (3.3) that

$$|dd^c\big(g_0\circ f^{\ell_0}\big)\dots\big(g_\kappa\circ f^{\ell_\kappa}\big)\big)\wedge T_+|\lesssim\sum_{j=0}^{\kappa} d_p^{-\ell_j}(f^{\ell_j})^*\big(\omega\wedge T_+\big).$$

In order to prove the assertion, it is then enough to prove that there exist λ' and M' such that, for every $j\geq 0$, the current $d_p^{-j}(f^j)^*\big(\omega\wedge T_+\big)$ has a $(2,\lambda',M')$-Hölder continuous α_{p+1}-normalized super-potential. For $j=0$, this follows from Lemma 3.1(i) and Proposition 2.9(iii). Proposition 3.4(ii) implies that that same holds for all $j\in\mathbb{N}$. The proof is complete. □

4. Exponential mixing of all orders and Central Limit Theorem

We prove now Theorem 1.2. By interpolation techniques [32] as in [9, pp. 262-263], it is enough to prove the statement in the case $\gamma = 2$. The statement is clear for $\kappa = 0$. By induction, we can assume that the statement holds for up to κ functions and prove it for $\kappa + 1 \geq 2$ functions. In particular, by the induction assumption, we are allowed to add to the g_j's a constant and assume that $\langle \mu, g_j \rangle = 0$ for all $0 \leq j \leq \kappa$. We can also assume that $\|g_j\|_{\mathcal{C}^2} \leq 1$ for all $0 \leq j \leq \kappa$. Then, we need to show that

$$\Big| \langle \mu, g_0 (g_1 \circ f^{n_1}) \dots (g_\kappa \circ f^{n_\kappa}) \rangle \Big| \lesssim d^{-\min_{0 \leq j \leq \kappa - 1} (n_{j+1} - n_j)/2}.$$

We can also assume that n_1 is even. Indeed, by the invariance of μ, the case where n_1 is odd can be treated similarly by replacing n_j with $n_j - 1$ for $1 \leq j \leq \kappa$ and g_0 with $g_0 \circ f^{-1}$.

Consider the automorphism F of the compact Kähler manifold $\mathbb{X} := X \times X$ given by $F(z, w) := (f(z), f^{-1}(w))$. We first show that $F^{-1} = (f^{-1}, f)$ satisfies the assumptions of Proposition 3.2 and Corollary 3.3. Recall that δ is the auxiliary degree of f and that by assumption we have $\delta < d_p$.

Lemma 4.1. *The automorphisms F and F^{-1} of $\mathbb{X}$ have simple action on cohomology. More precisely, the dynamical degrees of order k of F and F^{-1} are equal to $d_p(f)^2$, they are eigenvalues of multiplicity one of both F^* and F_* acting on $H^{k,k}(\mathbb{X}, \mathbb{C})$, and all the other dynamical degrees of F and F^{-1}, as well as the other eigenvalues of F^* and F_* on $H^{k,k}(\mathbb{X}, \mathbb{C})$, have modulus smaller than or equal to $d_p \cdot \delta$.*

Proof. By Künneth formula [34, Theorem 11.38], for every $0 \leq q \leq k$ we have a canonical isomorphism

$$H^{q,q}(X \times X, \mathbb{C}) = \bigoplus_{s+r=q} H^{s,r}(X, \mathbb{C}) \otimes H^{r,s}(X, \mathbb{C}).$$

The operators F^* and F_* preserve the above decomposition. By [8], the spectral radius of f^* on $H^{r,s}(X, \mathbb{C})$ is (equal to the spectral radius of f_* on $H^{k-r,k-s}(X, \mathbb{C})$ and) smaller than or equal to $\sqrt{d_r d_s}$. The assertion follows from the fact that f and f^{-1} have simple actions on cohomology and the definitions of p and δ. □

In this section, we will denote by $T_\pm$ the Green currents of f and by $\mathbb{T}_\pm$ the Green currents of F. More precisely, we fix Green currents $T_\pm$ for f (they are unique up to a multiplicative constant) such that $\mu = T_+ \wedge T_-$ and we set

$$\mathbb{T}_+ = T_+ \otimes T_- \quad \text{and} \quad \mathbb{T}_- = T_- \otimes T_+,$$

see [18, Section 4.1.8] for the tensor (or cartesian) product of currents. Since $f^*(T_+) = d_p T_+$ and $f_*(T_-) = d_p T_-$, we have $F^*(\mathbb{T}_+) = d_p^2 \mathbb{T}_+$ and $F_*(\mathbb{T}_-) = d_p^2 \mathbb{T}_-$.

Set $h := g_1 (g_2 \circ f^{n_2 - n_1}) \dots (g_\kappa \circ f^{n_\kappa - n_1})$. Recalling that $\langle \mu, g_0 \rangle = 0$ and that $\|g_j\|_{\mathcal{C}^2} \leq 1$, by the induction hypothesis it is enough to show that

$$\big| \langle \mu, g_0 (h \circ f^{n_1}) \rangle \big| \lesssim d_p^{-n_1/2}.$$

We denote by (z, w) the coordinates on $\mathbb{X} = X \times X$ and we set

$$\Psi(z, w) := g_0(w) h(z).$$

Lemma 4.2. *The following assertions hold:*

(i) $\|dd^c \Psi \wedge \mathbb{T}_+\|_* \leq c$;

(ii) $dd^c \Psi \wedge \mathbb{T}_+$ *has a $(2, \lambda, M)$-Hölder continuous super-potential,*

where c, λ, and M are positive constants depending on κ, but not on the g_j's and the n_j's.

Proof. The two assertions are consequences of Lemmas 3.5 and 3.6 applied to $F \colon \mathbb{X} \to \mathbb{X}$, respectively. Here, we use the functions $\tilde{g}_0(z, w) := g_0(w)$ and $\tilde{g}_j(z, w) := g_j(z)$ for $j \geq 1$, and the integers $\ell_0 := 0$, $\ell_1 := 0$, and $\ell_j := n_{j+1} - n_j$ for $j \geq 2$. □

End of the proof of Theorem 1.2. Recall that we are assuming that n_1 is even. By the invariance of μ, we see that

$$\langle \mu, g_0(h \circ f^{n_1})\rangle = \langle \mu, (g_0 \circ f^{-n_1/2})(h \circ f^{n_1/2})\rangle.$$

We first transform the above integral on X to an integral on $\mathbb{X}$ by means of the map F. Namely, using the invariance of $\mathbb{T}_+$, we have

$$\begin{aligned}
\big\langle \mu, (g_0 \circ f^{-n_1/2})(h \circ f^{n_1/2})\big\rangle &= \big\langle T_+ \wedge T_-, (g_0 \circ f^{-n_1/2})(h \circ f^{n_1/2})\big\rangle \\
&= \big\langle (T_+ \otimes T_-) \wedge [\Delta], (g_0 \circ f^{-n_1/2}(w))(h \circ f^{n_1/2}(z))\big\rangle \\
&= \big\langle d_p^{-n_1} (F^{n_1/2})^*(\mathbb{T}_+) \wedge [\Delta], (F^{n_1/2})^* \Psi\big\rangle \\
&= \big\langle \mathbb{T}_+ \wedge (d_p^{-n_1} (F^{n_1/2})_*[\Delta]), \Psi\big\rangle \\
&= \big\langle d_p^{-n_1} (F^{n_1/2})_*[\Delta], \Psi \mathbb{T}_+\big\rangle.
\end{aligned} \tag{4.1}$$

Observe, in particular, that the wedge product $[\Delta] \wedge \mathbb{T}_+$ is well defined by Proposition 2.6(iv).

Lemma 4.3. *We have $d_p^{-2n}(F^n)_*[\Delta] \to \mathbb{T}_-$ as $n \to \infty$.*

Proof. By Lemma 4.1, F^{-1} has simple action on cohomology and its main dynamical degree is that of order k, which is equal to d_p^2. As $[\Delta]$ has bi-degree (k, k), there exists $s \in \mathbb{R}$ such that $d_p^{-2n}(F^n)_*[\Delta] \to s\mathbb{T}_-$ as $n \to \infty$. Hence, we only need to show that $s = 1$. We have

$$\big\langle d_p^{-2n}(F^n)_*[\Delta], \mathbb{T}_+\big\rangle \to \langle s\mathbb{T}_-, \mathbb{T}_+\rangle = s\langle T_- \otimes T_+, T_+ \otimes T_-\rangle = s\|\mu\|^2 = s.$$

On the other hand, by the invariance of $\mathbb{T}_+$, for every $n \in \mathbb{N}$ we also have

$$\big\langle d_p^{-2n}(F^n)_*[\Delta], \mathbb{T}_+\big\rangle = \big\langle [\Delta], d_p^{-2n}(F^n)^*\mathbb{T}_+\big\rangle = \langle [\Delta], T_+ \otimes T_-\rangle = \|\mu\| = 1.$$

The assertion follows. □

We will apply the results of Sections 2 and 3 to the automorphism F^{-1} of $\mathbb{X}$. In order to do this, for simplicity, we let h be the dimension of $H^{k,k}(\mathbb{X}, \mathbb{R})$ and we fix a collection $\alpha := (\alpha_1, \ldots, \alpha_h)$ of real smooth (k, k)-forms on $\mathbb{X}$ with the property that $\{\alpha_1\} = \{\mathbb{T}_-\}$ and $\{\alpha_2\}, \ldots, \{\alpha_h\}$ are a basis for the F_*-invariant hyperplane in $H^{k,k}(\mathbb{X}, \mathbb{R})$ transversal to the line generated by $\{T_-\}$. In the following, we only consider α-normalized super-potentials for currents in $\mathscr{D}_k(\mathbb{X})$.

By Lemma 4.2, the current $R := dd^c\Psi \wedge \mathbb{T}_+$ satisfies $\|R\|_* \lesssim 1$ and has a $(2, \lambda, M)$-Hölder continuous potential $\mathcal{U}_R$, for some $0 < \lambda \le 1$ and $M > 0$ independent of the g_j's and n_j's. By linearity, up to multiplying g_0 by a constant, we can assume that $\|R\|_* \le 1$ and that $\mathcal{U}_R$ is $(2, \lambda, 1)$-Hölder continuous, for some $0 < \lambda \le 1$. Recall that $\delta < \delta' < d_p$. By Lemmas 4.1 and 4.3, we can apply Corollary 3.3 with F^{-1}, $\Psi\mathbb{T}_+$, and $n_1/2$ instead of f, ξ, and n, and get that, for all smooth $S \in \mathscr{D}_k(\mathbb{X})$ such that $d_p^{-2n}(F^n)_*(S) \to 0$, we have

$$\big|\big\langle d_p^{-n_1}(F^{n_1/2})_*(S), \Psi\mathbb{T}_+\big\rangle\big| \lesssim (d_p/\delta')^{-n_1/2}.$$

Let now S_ϵ be a regularization of $[\Delta] - \mathbb{T}_-$, with $S_\epsilon \to [\Delta] - \mathbb{T}_-$ as $\epsilon \to 0$ and $\|S_\epsilon\|_*$ bounded [12]. We can also add to S_ϵ a small smooth form so that $\{S_\epsilon\} = \{[\Delta] - \mathbb{T}_-\}$. We apply the above inequality with S_ϵ instead of S and take $\epsilon \to 0$. It follows from Proposition 2.6(iv) and Lemma 4.2 that

$$\big|\big\langle d_p^{-n_1}(F^{n_1/2})_*([\Delta] - \mathbb{T}_-), \Psi\mathbb{T}_+\big\rangle\big| \lesssim (d_p/\delta')^{-n_1/2}. \tag{4.2}$$

Combining (4.1) and (4.2) and using the invariance of $\mathbb{T}_-$ we obtain that

$$\big|\big\langle \mu, (g_0 \circ f^{-n_1/2})(h \circ f^{n_1/2})\big\rangle - \langle \mathbb{T}_-, \Psi\mathbb{T}_+\rangle\big| \lesssim (d_p/\delta')^{-n}.$$

To conclude, it is enough to show that $\langle \mathbb{T}_-, \Psi\mathbb{T}_+\rangle = 0$. As $\langle \mu, g_0\rangle = 0$, we have

$$\langle \mathbb{T}_-, \Psi\mathbb{T}_+\rangle = \langle T_- \otimes T_+, g_0(w)h(z)T_+ \otimes T_-\rangle = \langle \mu, g_0\rangle \cdot \langle \mu, h\rangle = 0.$$

The proof is complete. □

Remark 4.4. The rate of mixing in Theorem 1.2 can be improved by considering $F' := (f^l, f^{-m})$ instead of F, for suitable positive integers l and m, see for instance [15, Remark 4.1].

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