STRONG PROBABILISTIC STABILITY IN HOLOMORPHIC FAMILIES OF ENDOMORPHISMS OF $\mathbb{P}^k(\mathbb{C})$ AND POLYNOMIAL-LIKE MAPS

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ABSTRACT. We prove that, in stable families of endomorphisms of $\mathbb{P}^k(\mathbb{C})$, all invariant measures whose measure-theoretic entropy is strictly larger than $(k-1)\log d$ at a given parameter can be followed holomorphically with the parameter in all the parameter space. As a consequence, almost all points (with respect to any such measure at any parameter) in the Julia set can be followed holomorphically without intersections. This generalizes previous results by Berteloot, Dupont, and the first author for the measure of maximal entropy, and provides a parallel in this setting to the probabilistic stability of Hénon maps by Berger-Dujardin-Lyubich. Our proof relies both on techniques from the theory of stability/bifurcation in any dimension and on an explicit lower bound for the Lyapunov exponents for an ergodic measure in terms of its measure-theoretic entropy, due to de Thélin and Dupont.

A local version of our result holds also for all measures supported on the Julia set with just strictly positive Lyapunov exponents and not charging the post-critical set. Analogous results hold in families of polynomial-like maps of large topological degree. In this case, as part of our proof, we also give a sufficient condition for the positivity of the Lyapunov exponents of an ergodic measure for a polynomial-like map in any dimension in term of its measure-theoretic entropy, generalizing to this setting the analogous result by de Thélin and Dupont valid on $\mathbb{P}^k(\mathbb{C})$.

Notation. We denote by $\mathbb{P}^k = \mathbb{P}^k(\mathbb{C})$ the complex projective space of dimension k. A (p,p)-current is a current of bidegree (p,p). If $W \subset \mathbb{C}^k$ is an open set and S a positive closed (p,p)-current on W, we denote its mass by $\|S\|_W := \int_W S \wedge \omega^{k-p}$, where ω is the standard Kähler form of $\mathbb{C}^k$. Given a holomorphic map f we denote by $\mathrm{Jac}\, f$ the Jacobian of f. If ν is an ergodic f-invariant probability measure we denote by $h_\nu(f)$ the measure-theoretic entropy of ν with respect to f.

1. INTRODUCTION

By a classical result by Mané-Sad-Sullivan [42] and Lyubich [40], the stability of a family of rational maps of degree $d \geq 2$ is determined by the stability of its repelling cycles. More precisely, the so-called λ-lemma ensures that, once one can follow holomorphically with the parameter a dense subset of the Julia set J_{λ_0} (the support of the unique measure of maximal entropy of f_{λ_0} [34, 41]) at a given parameter λ_0, then *every* point of J_{λ_0} can be followed holomorphically. The important point here is that the motions corresponding to different points (that exist thanks to Montel theorem) do not intersect. This is a consequence of Hurwitz theorem. The *stability locus* is the (open dense) subset of the parameter space where the above condition of stability holds true. The complement of such set is the *bifurcation locus.* By a result of DeMarco [21], such locus is the support of a naturally defined *bifurcation current*, see also [45, 47] for the polynomial case.

A generalization of the theory by Mané-Sad-Sullivan, Lyubich, and DeMarco to families of endomorphisms of $\mathbb{P}^k$ of algebraic degree $d \geq 2$ in any dimension $k \geq 1$ has been recently carried out by Berteloot, Dupont, and the first author in [8, 11], see also the presentation in [7]. As most of the one-dimensional techniques do not apply anymore, the main tools were ergodic and (pluri)potential theory. Very roughly speaking, compactness of suitable spaces of currents and plurisubharmonic functions played the role of Montel theorem. And precise statistical properties of the measure of maximal entropy [18, 24, 32] (among them, an explicit lower bound for its Lyapunov exponents [17]) played somehow the role of an asymptotic Hurwitz theorem. As a consequence, dynamical stability in such families (defined for instance by the vanishing of a natural bifurcation current [3, 44], or by a condition on the stability of the repelling points) is equivalent to the existence of a holomorphic motion for a *full measure subset* of the Julia set, with respect to the measure of maximal entropy. We refer to [1, 5, 6, 9, 28] for further developments of the theory of stability/bifurcation in any dimension, and in particular to [2, 15, 16, 27, 35, 49] for new phenomena with respect to the one-dimensional case.

The main goal of this paper is to strengthen the main result in [8] by showing that, in stable families of endomorphisms of $\mathbb{P}^k$ of algebraic degree $d \geq 2$, dynamical stability implies the existence of a well-defined local motion for *almost all points* with respect to *all measures on the Julia set with strictly positive Lyapunov exponents and not charging the post-critical set.* By a result of de Thélin and Dupont [20, 30], this in particular applies to all ergodic measures whose measure-theoretic entropy is strictly larger than $(k-1)\log d$. In this case, the motion is well-defined on all the parameter space. Observe that the topological entropy of an endomorphism of $\mathbb{P}^k$ of algebraic degree d is equal to $k \log d$ [36] (which is then equal to the measure-theoretic entropy of the unique measure of maximal entropy), and that $(k-1)\log d$ is a natural threshold for many dynamical phenomena. See [12, 13, 30, 48, 50] for large classes of examples of such measures in any dimension, and their statistical properties.

Let us notice that an analogous result is already known in the setting of polynomial diffeomorphisms of $\mathbb{C}^2$ (usually called Hénon maps). Indeed, a parallel theory to that of [8, 11] has been developed in this setting by Dujardin and Lyubich, see [29, 28]. Stability is defined also in this setting by means of a number of equivalent conditions, among them the existence of a *branched* holomorphic motion for the Julia sets, meaning that natural motions of distinct points can a priori intersect. In [4] Berger and Dujardin proved that such motion is *unbranched* at almost every point with respect to all measures of positive entropy. Since the topological entropy is the logarithm of the algebraic degree in this context, this corresponds to our threshold. The current work was inspired by [4] and provides a parallel to that result for families of endomorphisms of $\mathbb{P}^k$. As was already the case for [29] and [8], our approach is completely different from the one in [4], since the key point here is to understand the relation between the Julia sets and the dynamics of the critical set (which does not exist in the case of diffeomorphims of $\mathbb{C}^2$). This is achieved with techniques from pluripotential theory.

1.1. **Definitions and results.** We will consider in this paper a more general setting than that of the endomorphisms of $\mathbb{P}^k$, that is that of *polynomial-like maps of large topological degree*, see Definition 2.5. However, we restrict to the family $\mathcal{H}_d(\mathbb{P}^k)$ of all the endomorphisms of $\mathbb{P}^k$ of a given algebraic degree $d \geq 2$ in this introduction for simplicity.

The main result of [8] in this setting can be stated as follows. Given a holomorphic family $(f_\lambda)_{\lambda \in M}$ of endomorphisms of $\mathbb{P}^k$, we denote by μ_λ the equilibrium measure of f_λ (i.e., the unique measure of maximal entropy $k \log d$ of f_λ [18, 24, 32]), and recall that the Julia set J_λ of f_λ is by definition the support of μ_λ.

Theorem-Definition 1.1 (Berteloot-B.-Dupont [8])**.** Let M be an open connected and simply connected subset of the family $\mathcal{H}_d(\mathbb{P}^k)$ of the endomorphisms of $\mathbb{P}^k$ of a given algebraic degree $d \geq 2$. The following conditions are equivalent:

1. the repelling points in the Julia sets move holomorphically with λ (see Definition 3.5);
2. the sum $L(\lambda) = \int \log|\mathrm{Jac}\, f_\lambda|\mu_\lambda$ of the Lyapunov exponents of μ_λ satisfies $dd^c L \equiv 0$;
3. there exists an equilibrium lamination.

We say that a family is *stable* if any of the equivalent conditions above holds.

An equilibrium lamination is defined as follows. Denote by $\mathcal{J}$ the set of all holomorphic maps $\gamma\colon M \to \mathbb{P}^k$ such that $\gamma(\lambda)$ belongs to J_λ for all $\lambda \in M$. We often identify a map γ with its graph Γ_γ in the product space $M \times \mathbb{P}^k$. The family $(f_\lambda)_{\lambda\in M}$ induces a dynamical system $\mathcal{F}$ on the space $\mathcal{J}$ by $\mathcal{F}\gamma(\lambda) := f_\lambda(\gamma(\lambda))$. Observe also that the maps f_λ can be seen as fibers of a single holomorphic map $f\colon M \times \mathbb{P}^k \to M \times \mathbb{P}^k$. We denote by C_f the critical set of such f, and by $GO(C_f)$ the grand orbit of C_f, i.e., $GO(f) := \cup_{n,m\geq 0} f^{-m}(f^n(C_f))$. We also denote by d_t the topological degree of any element of $\mathcal{H}_d(\mathbb{P}^k)$. Observe that in this case we have $d_t = d^k$.

Definition 1.2. A *dynamical lamination* for the family f is an $\mathcal{F}$-invariant subset $\mathcal{L}$ of $\mathcal{J}$ such that

(1) $\Gamma_\gamma \cap \Gamma_{\gamma'} = \emptyset$ for all $\gamma \neq \gamma' \in \mathcal{L}$;
(2) $\Gamma_\gamma \cap GO(C_f) = \emptyset$ for all $\gamma \in \mathcal{L}$;
(3) $\mathcal{F} : \mathcal{L} \to \mathcal{L}$ is d_t-to-1.

An *equilibrium lamination* or *μ_λ-measurable holomorphic motion of J_λ* is a dynamical lamination satisfying the following further property:

(4) $\mu_\lambda(\{\gamma(\lambda) : \gamma \in \mathcal{L}\}) = 1$ for all $\lambda \in M$;

Our main result in the case of the family $\mathcal{H}_d(\mathbb{P}^k)$ can be stated as follows.

Theorem 1.3. *Let M be an open connected and simply connected subset of the family $\mathcal{H}_d(\mathbb{P}^k)$ of the endomorphisms of $\mathbb{P}^k$ of a given algebraic degree $d \geq 2$ and assume that the family $(f_\lambda)_{\lambda\in M}$ is stable. Then there exists a dynamical lamination $\mathcal{L}$ satisfying*

(4') *$\nu(\{\gamma(\lambda) : \gamma \in \mathcal{L}\}) = 1$ for every $\lambda \in M$ and every ergodic f_λ-invariant measure ν whose measure-theoretic entropy is strictly larger than $(k-1)\log d$.*

In particular, given any $\lambda_0 \in M$ and any ergodic f_{λ_0}-invariant measure ν_0 whose measure-theoretic entropy is strictly larger than $(k-1)\log d$, it is possible to define a measurable holomorphic motion of J_λ associated to the measure ν_0 on all of M.

An analogous result holds in the much more general setting of families of polynomial like maps of large topological degree (see Definition 2.5), or of arbitrary subfamilies of $\mathcal{H}_d(\mathbb{P}^k)$. A version of Theorem-Definition 1.1 in this more general setting is the main result of [11], see also [10]. We refer to Theorem 3.12 and Corollary 3.13 for precise statements of our results in these cases and to Theorem 3.11 for a weaker (local) version of it holding for all measures supported on the Julia set with strictly positive Lyapunov exponents and not charging the postcritical set. As part of our proof, we also prove a generalization of de Thélin and Dupont theorem above [20, 30] in this setting, giving the strict positivity of the Lyapunov exponents of measures of sufficiently large measure-theoretic entropy for polynomial like maps of large topological degree, see Theorem 2.7. This result is crucial to get a uniform domain of existence for the laminations associated with different measures, depending only on their measure-theoretic entropy.

1.2. **Organization of the paper.** In Section 2 we recall some definitions and results on polynomial-like maps of large topological degrees. We also study the Lyapunov exponents of measures with sufficiently large measure-theoretic entropy and prove their strict positivity, with an explicit lower bound depending on the measure-theoretic entropy of the measure, see Theorem 2.7. In Section 3 we recall definitions and results on holomorphic families of polynomial-like maps and we state the precise versions of our main results, see Theorems 3.11 and 3.12 and Corollary 3.13. Section 4 is dedicated to the proof of these results. In Appendix A we record an intermediate contraction estimate along generic inverse orbits (with respect to naturally defined measures) in the space $(\mathcal{J}, \mathcal{F})$, for later reference.

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2. Polynomial-like maps

2.1. **Preliminary notions.** We recall here the main notions and results about polynomial-like maps that we will use in the sequel. We refer to [22, 24] for more details.

Definition 2.1. A *polynomial-like map* is a proper holomorphic map $f: U \to V$, where $U \Subset V$ are open subsets of $\mathbb{C}^k$ and V is convex.

In particular, a polynomial-like map f is a ramified covering $U \to V$, and the *topological degree* d_t (i.e., the number of preimages of any point in V, counting the multiplicity) of f is well-defined. We will always assume that $d_t \geq 2$. The (compact) set $K := \bigcap_{n=1}^{\infty} f^{-n}(U)$ is the *filled-in Julia set* of f. It consists of the set of points whose orbit is well-defined. The system (K, f) is a well-defined dynamical system.

Notice in particular that homogeneous lifts to $\mathbb{C}^{k+1}$ of endomorphisms of $\mathbb{P}^k$ give polynomial-like maps. On the other hand, while in dimension $k = 1$ any polynomial-like map is conjugated to an actual polynomial on the Julia set [25], in higher dimensions this class is known to be much larger than that of regular polynomial endomorphisms of $\mathbb{P}^k$ (i.e., those extending holomorphically to $\mathbb{P}^k$), see for instance [24, Example 2.25].

Definition 2.2. Let $f: U \to V$ be a polynomial-like map. For $0 \leq p \leq k$ we set

$$d_p = d_p(f) := \limsup_{n\to\infty} \|(f^n)_*(\omega^{k-p})\|_W^{1/n}$$

and

$$d_p^* = d_p^*(f) := \limsup_{n\to\infty} \sup_S \|(f^n)_*(S)\|_W^{1/n} \tag{2.1}$$

where $W \Subset V$ is a neighbourhood of K, ω is the standard Kähler form on $\mathbb{C}^k$, and the supremum in (2.1) is taken over all positive closed $(k-p, k-p)$-currents of mass less than or equal to 1 on a fixed neighbourhood $W' \Subset V$ of K. We say that d_p and d_p^* are the *dynamical* and $*$-*dynamical degrees of order* p of f, respectively. Similarly, we define

$$\delta_p = \delta_p(f) := \limsup_{n\to\infty} \sup_X \|(f^n)_*[X]\|_W^{1/n} \tag{2.2}$$

where the supremum in (2.2) is taken over all p-dimensional complex analytic sets $X \subset V$.

These definitions are independent of the open neighbourhoods W, W'. Moreover, we have $d_p \leq d_p^*$ and $\delta_p \leq d_p^*$ for all $0 \leq p \leq k$, $d_0^* = 1$, and $d_k = \delta_k = d_k^* = d_t$. In the case of endomorphisms of $\mathbb{P}^k$ of algebraic degree d, for every $0 \leq p \leq k$ the above definitions reduce to $d_p^* = \delta_p = d_p = d^p$. The next lemma in particular implies that $d_p \leq \delta_p$.

Lemma 2.3 (see [11, Lemma 4.7])**.** *Let $f: U \to V$ be a polynomial-like map. Let X be an analytic subset of V of pure dimension $p \leq k-1$. There exists a function $C: \mathbb{N} \to \mathbb{R}$ with $\limsup_{n\to\infty} C(n)^{1/n} = 1$ and independent of X such that*

$$\int_{f^{-n}(V)} [X] \wedge (f^{n_1})^*\omega \wedge \ldots \wedge (f^{n_p})^*\omega \leq C(n)\,\delta_p^n \tag{2.3}$$

and

$$\int_{f^{-n}(V)} \omega^{k-p} \wedge (f^{n_1})^*\omega \wedge \ldots \wedge (f^{n_p})^*\omega \leq C(n)\,\delta_p^n \tag{2.4}$$

where $0 \leq n_j \leq n-1$ for all $1 \leq j \leq p$. In particular, the topological entropy of the restriction of f to X is bounded by $\log \delta_p$.

Proof. The proof of (2.3) follows the strategy used by Gromov to estimate the topological entropy of endomorphisms of $\mathbb{P}^k$, see for instance [36], and adapted by Dinh and Sibony [22, 24] to the setting of polynomial-like maps. Since only minor modifications are needed, we refer to [10, Lemma A.2.6] for a complete proof. The second expression is deduced from the first (by possibly increasing $C(n)$ by a bounded factor) as ω^{k-p} can be written as an average of currents of integration on p-dimensional analytic sets. ∎

Corollary 2.4 (see [10, Lemma A.2.9]). *Let $f: U \to V$ be a polynomial-like map. Let ν be an ergodic f-invariant probability measure such that $h_\nu(f) > \log \delta_p$. Then ν gives no mass to analytic sets of dimension less than or equal to p.*

Proof. Let X be an analytic set of dimension at most p, and assume by contradiction that $m := \nu(X) > 0$. We can assume that X is irreducible and has pure dimension $q \le p$. Since ν is invariant, for all $n \in \mathbb{N}$ we have $\nu(f^n(X)) = \nu(f^{-n}(f^n(X))) \ge \nu(X) = m$. Hence, we must have $\nu(f^{n_1}(X) \cap f^{n_2}(X)) > 0$ for some $n_1 \neq n_2 \in \mathbb{N}$. By the minimality of X, we deduce that $f^{n_1}(X) = f^{n_2}(X)$. Up to replacing f with an iterate, we can then assume that X is invariant and such that $\nu(X) > 0$. The ergodicity of ν implies that $\nu(X) = 1$. This implies that $h_\nu(f)$ is smaller than or equal to the topological entropy of the restriction of f to X. Since this is bounded by $\log \delta_p$ by Lemma 2.3, the assertion follows. □

We will focus in this paper on maps satisfying the following condition [24]. Observe that the condition is always satisfied by lifts of endomorphisms, and moreover it is stable by small perturbation of the coefficients (since $d^*_{k-1}(f)$ depends upper semicontinuously from the map f). This gives large families of examples.

Definition 2.5. We say that a polynomial-like map f has *large topological degree* if $d^*_{k-1} < d_t$.

Polynomial-like maps of large topological degree enjoy many of the dynamical properties of endomorphisms (however, their study is usually technically more involved, because of the lack of a naturally defined Green function). For instance, they admit a unique measure of maximal entropy $\log d_t$, whose Lyapunov exponents (see below) are strictly positive. We denote this measure by μ and define the Julia set as the support of f. Observe that J is a subset of the boundary of K.

In the current paper, we will often need that f satisfies the (a priori) stronger property that $\max\{d^*_0, \dots, d^*_{k-1}\} < d_t$. The following result by Dinh and the authors implies that this is always the case for maps of large topological degree.

Proposition 2.6 (see [14, Theorem 1.3]). *Let f be a polynomial-like map. Then the sequence $\{d^*_j\}_{0 \le j \le k}$ is non-decreasing. In particular, if f has large topological degree then*

$$\max\{d^*_0, \dots, d^*_{k-1}\} = d^*_{k-1} < d_t.$$

2.2. **Strict positivity of the Lyapunov exponents.** Let $f: U \to V$ be a polynomial-like map and ν an ergodic f-invariant probability measure. By Oseledets Theorem [43], ν admits k *Lyapunov exponents* (counting multiplicity, and admitting the possible value of $-\infty$). We will denote them by

$$-\infty \le \chi_l < \chi_{l-1} < \ldots < \chi_1,$$

with multiplicity $m_j \ge 1$, $j = 1, \dots, l$ respectively. Note that $m_1 + m_2 + \ldots + m_l = k$. As soon as the measure-theoretic entropy of ν is positive, its largest Lyapunov exponent is strictly positive by Ruelle inequality [46] (the result is stated for compact manifolds, but the proof applies here too, by triangulating a neighbourhood of the filled Julia set K).

The following is the main result of this section.

Theorem 2.7. *Let $f: U \to V$ be a polynomial-like map of large topological degree. Let ν be an ergodic f-invariant probability measure satisfying $h_\nu(f) > \log d^*_{k-1}(f)$. Then*

1. *all the Lyapunov exponents of f with respect to ν are larger than or equal to $(h_\nu(f) - \log d^*_{k-1})/(2m_l) > 0$ (where m_l is the multiplicity of the smallest Lyapunov exponent of ν);*
2. *the function $\log|\mathrm{Jac}\, f|$ is integrable with respect to ν.*

The second assertion follows from the first, since the sum of the Lyapunov exponents is equal to $\langle \nu, \log|\mathrm{Jac}\, f| \rangle$ by Birkhoff Theorem, so this last integral is finite as soon as all Lyapunov exponents are finite. Hence, we only need to prove the first assertion. This is known in the case of endomorphisms of $\mathbb{P}^k$, see de Thélin [20] and Dupont [30]. When ν is the measure of maximal entropy, it is a result of Dinh-Sibony [22], see [17] for the case of endomorphisms of $\mathbb{P}^k$. Although we will follow the strategy of the proof de Thélin and Dupont, because of the lack of a Hodge theory in this setting, we will need to replace some cohomological arguments when working with polynomial-like maps. We also cannot exploit the linearity of the sequence $\{d_p^*\}_{1\leq p\leq k}$ of the dynamical degrees, see also Remark 2.10.

As in [30], the assertion will be deduced from the following estimate. Recall that $l \geq 1$ is the number of distinct Lyapunov exponents of ν.

Theorem 2.8. *Let $f: U \to V$ be a polynomial-like map and let ν be an ergodic f-invariant probability measure. Then, if $l \geq 2$, for every $2 \leq j \leq l$, we have*

$$h_\nu(f) \leq \log \max_{1\leq i\leq k-l_j} \delta_i + 2m_j\chi_j^+ + \ldots + 2m_l\chi_l^+ \tag{2.5}$$

where we set $\chi^+ := \max\{\chi, 0\}$, and $l_j = m_j + \ldots + m_l$.

If $h_\nu(f) = 0$, the assertion is clear. So we can assume that $h_\nu(f) > 0$. As we mentioned above we have $\chi_1 > 0$ by Ruelle inequality. Let $1 \leq q \leq l$ be such that $q := \max\{i : \chi_i > 0\}$. Since the same proof below works for the case $l = q$, without loss of generality we can assume that $l > q$. It is not difficult to see that the right-hand side of (2.5), seen as a function of j, is non-decreasing for $j \geq q+1$, hence it is enough to prove Theorem 2.8 for $2 \leq j \leq q+1$.

Proposition 2.9 (see [30, Proposition 6.3]). *Let $f: U \to V$ be a polynomial-like map. Fix $0 < \beta_0 \leq 1$. For every $\varepsilon > 0$ there exists $r_0 > 0$ and $n_0 \in \mathbb{N}$ and, for any $n \geq n_0$, a maximal (n, r_0)-separated set $\mathcal{E}_n$ with* $\mathrm{Card}(\mathcal{E}_n) \geq e^{n(h_\nu(f)-2\varepsilon)}$ *and such that, for any $z \in \mathcal{E}_n$ and every $2 \leq j \leq q+1$, there exists a neighbourhood U_z^j of the origin of $\mathbb{D}^{l_j}$ and an injective mapping $\Psi_z^j : U_z^j \to f^{-n}(V)$ which satisfies the following properties:*

1. $\Psi_z^j(0) = z$ *and* $\mathrm{Lip}\Psi_z^j \leq \beta_0$*;*
2. $\mathrm{diam}\, f^i(\Psi_z^j(U_z^j)) \leq e^{-n\varepsilon}$*, for* $0 \leq i \leq n-1$*;*
3. $\mathrm{Vol}(\Psi_z^j(U_z^j)) \geq e^{-n(2m_j\chi_j^+ + \ldots + 2m_q\chi_q^+)}e^{-8kn\varepsilon}$ *for $j \leq q$ and* $\mathrm{Vol}(\Psi_z^{q+1}(U_z^{q+1})) \geq e^{-8kn\varepsilon}$.

Recall that a set A is (n, r_0)-separated, if for all $x, y \in A$ we have

$$\max_{0\leq i\leq n-1} |f^i(x) - f^i(y)| > r_0$$

(we assume here that $|f^i(x) - f^i(y)| > r_0$ if at least one among $f^i(x)$ and $f^i(y)$ is not defined).

Proof. The statement being local, the same proof as in [30] applies here. □

Proof of Theorem 2.8. We keep the notations of Proposition 2.9 and set $\Psi_n^j := \cup_{z\in\mathcal{E}_n}\Psi_z^j(U_z^j)$. Up to taking suitable local charts, we can assume that the $\Psi_z^j(U_z^j)$'s are graphs above $\sigma(f^{-n}(V))$ where $\sigma : \mathbb{C}^k \to \mathbb{C}^{l_j}$ is the orthogonal projection. Let $\omega := dd^c\|z\|^2$ be the standard Kähler $(1,1)$-form on V. For $a \in \sigma(f^{-n}(V))$, set

$$\Gamma_n(a) := \{(z, f(z), \ldots, f^{n-1}(z)), z \in \sigma^{-1}(a) \cap f^{-n+1}(V)\} \subset V^n.$$

Then $\mathrm{Vol}(\Gamma_n(a)) = \int_{\Gamma_n(a)} \omega_n^{k-l_j}$, where $\omega_n = \sum_{i=1}^n \Pi_i^*\omega$ and, for every $1 \leq i \leq n$, Π_i denotes the projection of V^n onto its i-th factor. Since $\mathcal{E}_n$ is (n, r_0)-separated and $\mathrm{diam}\, f^i(\Psi_z^j(U_z^j)) \leq e^{-n\varepsilon}$ the set $\Psi_n^j \cap \sigma^{-1}(a)$ is $(n, r_0/2)$-separated. Lelong inequality implies that $\mathrm{Vol}(\Gamma_n(a)) \geq \mathrm{Card}\left(\Psi_n^j \cap \sigma^{-1}(a)\right)$, see for instance [36]. By using Proposition 2.9 we get

$$\begin{aligned} \int_{a\in\sigma(f^{-n}(V))} \mathrm{Vol}(\Gamma_n(a))da &\geq \int_{a\in\sigma(f^{-n}(V))} \mathrm{Card}\left(\Psi_n^j \cap \sigma^{-1}(a)\right) da \\ &= \mathrm{Vol}(\sigma(\Psi_n^j)) \geq e^{n(h_\nu-2\varepsilon)}e^{-n(2m_j\chi_j^+ + \ldots + 2m_q\chi_q^+)}e^{-8kn\varepsilon}. \end{aligned} \tag{2.6}$$

On the other hand, we also have

$$\mathrm{Vol}(\Gamma_n(a)) = \sum_{0\le n_i\le n-1} \int_{f^{-n}(V)} [\sigma^{-1}(a)] \wedge (f^{n_1})^*\omega \wedge \ldots \wedge (f^{n_{k-l_j}})^*\omega.$$

By Lemma 2.3, and since the sum in the last expression contains n^{k-l_j} terms, there exists a function $C\colon \mathbb{N} \to \mathbb{R}$ independent of a such that $\limsup_{n\to\infty} C(n)^{1/n} = 1$ and, for all $n \in \mathbb{N}$,

$$\mathrm{Vol}(\Gamma_n(a)) \le C(n)\, \delta_{k-l_j}^n.$$

Hence, for all $n \in \mathbb{N}$, we have

$$\int_{a\in\sigma(f^{-n}(V))} \mathrm{Vol}(\Gamma_n(a))da \le C(n)\, \delta_{k-l_j}^n \int_{a\in\sigma(f^{-n}(V))} da \le \alpha(n)\, \delta_{k-l_j}^n \tag{2.7}$$

where $\alpha(n) := C(n) \int_{a\in\sigma(V)} da = C(n)\,\mathrm{Vol}(\sigma(V))$ (notice that $\sigma(f^{-n}(V)) \subset \sigma(V)$). In particular, we have $\limsup_{n\to\infty} \alpha(n)^{1/n} = 1$.

Combining the two inequalities (2.6) and (2.7), we deduce that for any $\varepsilon > 0$ there exists an integer n_0 such that for any $n > n_0$ we have

$$\log \delta_{k-l_j} + \frac{1}{n} \log \alpha(n) \ge h_\nu(f) - (2m_j\chi_j^+ + \ldots + 2m_q\chi_q^+) - (8k+2)\varepsilon.$$

By letting $n \to \infty$ and $\varepsilon \to 0$, we have

$$\begin{aligned} h_\nu(f) &\le \log \delta_{k-l_j} + 2m_j\chi_j^+ + \ldots + 2m_q\chi_q^+ \\ &\le \log \max_{1\le i\le k-l_j} \delta_i + 2m_j\chi_j^+ + \ldots + 2m_q\chi_q^+. \end{aligned}$$

The proof is complete. □

End of the proof of Theorem 2.7. Let us first assume that f admits $l \ge 2$ distinct Lyapunov exponents. By using (2.5) and Proposition 2.6 we have

$$\chi_l^+ \ge \frac{h_\nu(f) - \log \max_{1\le i\le k-m_l} \delta_i}{2m_l} \ge \frac{h_\nu(f) - \log d_{k-1}^*}{2m_l}, \tag{2.8}$$

which concludes the proof in this case.

Assume now that all the Lyapunov exponents are equal to χ. In particular, we have $l = 1$ and $m_l = m_1 = k$. By Ruelle inequality, we have $h_\nu(f) \le 2k\chi$, hence the desired estimate in this case also follows. □

Remark 2.10. In the case of endomorphisms of $\mathbb{P}^k$ (see [20] and [30]) the denominator in Theorem 2.7 (1) can be taken to be equal to 2 instead of $2m_l$, even when $m_l > 1$. This is a consequence of the log-concavity of the sequence $\{d_p^*\}_{0\le p\le k}$ (which in that case is actually linear).

Let f be a polynomial-like map as in Theorem 2.7 and assume that the sequence $\{d_p^*\}_{0\le p\le k}$ is log-concave, i.e., that we have $d_{p-1}^* \cdot d_{p+1}^* \le (d_p^*)^2$ for $1 \le p \le k-1$. Then, the denominators $2m_l$ can also be replaced by 2 in Theorem 2.7 (1). Indeed, the log-concavity implies that $(d_k^*)^{m-1} \cdot d_{k-m}^* \le (d_{k-1}^*)^m$ for all $1 \le m \le k$. By the last inequality and the fact that $h_\nu(f) \le \log d_t = \log d_k^*$ we obtain

$$(m-1)h_\nu(f) \le (m-1)\log d_k^* \le \log \frac{(d_{k-1}^*)^m}{d_{k-m}^*} = m \log d_{k-1}^* - \log d_{k-m}^*$$

and hence

$$\frac{h_\nu(f) - \log d_{k-m}^*}{2m} \ge \frac{h_\nu(f) - \log d_{k-1}^*}{2}. \tag{2.9}$$

Let us first assume that f admits $l \geq 2$ distinct Lyapunov exponents, and let m_l be the multiplicity of the smallest Lyapunov exponent, as above. Since $\max_{1\leq i\leq k-m_l} \delta_i \leq d^*_{k-m_l}$ by Proposition 2.6, thanks to (2.9) we have

$$\frac{h_\nu(f) - \log \max_{1\leq i\leq k-m_l} \delta_i}{2m_l} \geq \frac{h_\nu(f) - \log d^*_{k-m_l}}{2m_l} \geq \frac{h_\nu(f) - \log d^*_{k-1}}{2}.$$

This permits to improve the second inequality in (2.8), and proves the assertion in this case.

Assume now that all the Lyapunov exponents are equal to χ. Then, thanks to the arguments in the last lines of the proof of Theorem 2.7, it is enough to prove the inequality $\frac{h_\nu(f)-\log d^*_{k-1}}{2} \leq \frac{h_\nu(f)}{2k}$. Since this is a consequence of (2.9) applied with $m = k$ (recall that $d^*_0 = 1$), the assertion follows in this case, too.

3. Holomorphic families of polynomial-like maps

3.1. **General definitions.** We will consider in all this paper holomorphic families of polynomial-like maps. These are defined as follows, see [44, 24].

Definition 3.1. Let M be a complex manifold and $\mathcal{U}, \mathcal{V}$ be connected open subsets of $M \times \mathbb{C}^k$ such that $\mathcal{U} \subset \mathcal{V}$. Let $\pi_M : M \times \mathbb{C}^k \to M$ be the standard projection. Assume that for every $\lambda \in M$ we have $\emptyset \neq U_\lambda \Subset V_\lambda \Subset \mathbb{C}^k$ with U_λ connected and V_λ convex, where $U_\lambda := \mathcal{U} \cap \pi_M^{-1}(\lambda)$ and $V_\lambda := \mathcal{V} \cap \pi_M^{-1}(\lambda)$. Assume also that U_λ and V_λ depend continuously on λ. A *holomorphic family of polynomial-like maps* is a proper holomorphic map $f : \mathcal{U} \to \mathcal{V}$ of the form $(\lambda, z) \mapsto (\lambda, f_\lambda(z))$.

From the definition, f has a well-defined topological degree d_t. We will always assume that $d_t \geq 2$. All the maps $f : U_\lambda \to V_\lambda$ are polynomial-like with the same topological degree d_t. We denote by μ_λ, J_λ, and K_λ the equilibrium measure, the Julia set, and the filled Julia set of f_λ, respectively. Since the function $\lambda \to K_\lambda$ is upper semicontinuous for the Hausdorff topology, when working locally near a given $\lambda_0 \in M$ we can assume without loss of generality that $\mathcal{V}$ is equal to $M \times V$ for some open and convex subset $V \Subset \mathbb{C}^k$. We denote by C_f the critical set of f. Observe that the current of integration $[C_f]$ is given by $dd^c \log|\mathrm{Jac}\, f|$, where $\mathrm{Jac}\, f$ is the Jacobian of f.

The following result is due to Pham [44] in the case of $\nu_\lambda = \mu_\lambda$, see also [24].

Proposition 3.2. *Let $(f_\lambda)_{\lambda\in M}$ be a holomorphic family of polynomial like maps of large topological degree. Let $\mathcal{R}$ be a horizontal positive closed current on $\mathcal{V} = M \times V$ of bidegree (k, k). Assume that, for all $\lambda \in M$, the slice measure $\nu_\lambda := \mathcal{R}_\lambda$ is an ergodic f_λ-invariant measure. Let $L_1(\lambda) \geq \ldots \geq L_k(\lambda)$ denote the k Lyapunov exponents of ν_λ, counting multiplicities, and, for every $1 \leq \ell \leq k$ let*

$$\Sigma_\ell(\lambda) := \sum_{j=1}^{\ell} L_j(\lambda)$$

be the sum of the ℓ largest exponents of ν_λ. If there exists $\lambda_0 \in M$ such that $L_k(\lambda_0)$ is finite, then, for all $1 \leq l \leq k$, the function $\Sigma_\ell(\lambda)$ is plurisubharmonic (psh) on M.

Recall that a current $\mathcal{R}$ in $M \times V$ is *horizontal* if $\pi_V(\mathrm{Supp}\, \mathcal{R}) \Subset V$ [26, 23]. For a horizontal positive closed current of bidegree (k, k), the *slice* $\mathcal{R}_\lambda$ (which, for smooth $\mathcal{R}$, coincides with the intersection $\mathcal{R} \wedge \pi_M^* \delta_\lambda = \mathcal{R} \wedge [\pi_M^{-1}(\{\lambda\})]$) is well-defined for all $\lambda \in M$ and can be seen as a positive measure on V, whose mass does not depend on λ, see [23].

Proof of Proposition 3.2. The proof is essentially the same as for the case $\nu_\lambda = \mu_\lambda$ (see [44, 24]) hence we only sketch it.

The differential $D_z f_\lambda$ depends holomorphically on (λ, z). For every $1 \leq \ell \leq k$, it induces the linear map

$$\bigwedge^{\ell} D_z f_\lambda \quad : \quad \bigwedge^{\ell} T_z \mathbb{C}^k \to \bigwedge^{\ell} T_{f_\lambda(z)} \mathbb{C}^k$$

which is defined as

$$\bigwedge^{\ell} D_z f_\lambda(e_1 \wedge \ldots \wedge e_\ell) := D_z f_\lambda(e_1) \wedge \ldots \wedge D_z f_\lambda(e_\ell).$$

This map still depends holomorphically on (λ, z). Hence, the map $(\lambda, z) \mapsto \log\|\wedge^\ell D_z f_\lambda\|$ is psh. For every $n \geq 0$, set $\Psi_n(\lambda) := \langle \nu_\lambda, \log\|\wedge^\ell D_z f_\lambda^n\|\rangle$. By [44, Proposition A.1] (see also [10, Appendix A.1]), Ψ_n is psh or equal to $-\infty$ on M. For all $n, m \geq 0$, $z \in U$, and $\lambda \in M$ we also have

$$\|\wedge^\ell D_z f^{n+m}\| \leq \|\wedge^\ell D_{f_\lambda^n(z)} f^m\| \cdot \|\wedge^\ell D_z f^n\|,$$

which implies that $\Psi_{n+m}(\lambda) \leq \Psi_n(\lambda) + \Psi_m(\lambda)$. Hence, the sequence $n^{-1}\Psi_n$ decreases to $\Psi := \inf n^{-1}\Psi_n$. By Oseledets theorem, we have $\Psi(\lambda) = \Sigma_\ell(\lambda)$. We deduce that Σ_ℓ is psh or identically $-\infty$. The latter possibility is excluded since the assumption on λ_0 implies that $\Sigma_\ell(\lambda_0) > -\infty$. The assertion follows. □

3.2. **Stability notions.** We fix in this section a connected and simply connected complex manifold M and a holomorphic family of polynomial-like maps of large topological degree $f\colon \mathcal{U} \to \mathcal{V} = M \times V$. We first consider the space of maps

$$\mathcal{O}(M, \mathbb{C}^k) := \{\gamma : M \to \mathbb{C}^k : \gamma \text{ holomorphic}\}$$

with the topology of local uniform convergence. $\mathcal{O}(M, \mathbb{C}^k)$ is a metric space. We then define the subspace

$$\mathcal{J} := \{\gamma \in \mathcal{O}(M, \mathbb{C}^k) : \gamma(\lambda) \in J_\lambda \, \forall \lambda \in M\} \tag{3.1}$$

and the two natural maps:

(1) $\mathcal{F} : \mathcal{J} \to \mathcal{J}$, which is defined by $\mathcal{F}(\gamma)(\lambda) = f_\lambda(\gamma(\lambda))$;
(2) $p_\lambda : \mathcal{J} \to J_\lambda$, which is defined by $p_\lambda(\gamma) = \gamma(\lambda)$.

Observe that $\mathcal{J}$ is a compact metric space, and that $(\mathcal{J}, \mathcal{F})$ is a well-defined dynamical system.

Definition 3.3. A *web* for the family f is an $\mathcal{F}$-invariant probability measure on $\mathcal{J}$.

Given $\lambda_0 \in M$ and an f_{λ_0}-invariant probability measure ν supported on J_{λ_0}, a (λ_0, ν)-*web* (or ν-*web* for brevity) is a web $\mathcal{M}$ such that $(p_{\lambda_0})_*\mathcal{M} = \nu$.

If a web $\mathcal{M}$ satisfies $(p_\lambda)_*\mathcal{M} = \mu_\lambda$ (the equilibrium measure for f_λ) for all $\lambda \in M$, $\mathcal{M}$ is an *equilibrium web* for the family f.

Definition 3.4. Given $\lambda_0 \in M$ and an f_{λ_0}-invariant probability measure ν supported on J_{λ_0}, a dynamical lamination $\mathcal{L}$ (see Definition 1.2) is said to be a ν-*lamination* if $\nu(\{\gamma(\lambda_0) : \gamma \in \mathcal{L}\}) = 1$.

The *repelling J-cycles* of f_λ are the repelling cycles of f_λ which belong to J_λ. In dimension $k = 1$, all repelling cycles are automatically repelling J-cycles. When $k \geq 2$ there are examples of repelling points outside of the Julia set (see for instance [33, 37]). Recall, however, that repelling J-cycles are dense in the Julia set [17, 22].

Definition 3.5. We say that the repelling J-cycles of f_λ *move holomorphically* over an open subset $\Omega \subseteq M$ if for every $n \geq 1$ there exists a set of holomorphic maps $\rho_{j,n} \in \mathcal{J}$ such that $\mathcal{R}_n(\lambda) = \{\rho_{j,n}(\lambda) : 1 \leq j \leq N_d(n)\}$ for all $\lambda \in \Omega$, where $\mathcal{R}_n(\lambda) := \{$repelling $n-$ periodic points of f_λ in $J_\lambda\}$ and $N_d(n) = \mathrm{Card}(\mathcal{R}_n(\lambda))$ for all $\lambda \in M$.

The following, a priori weaker, condition on the repelling cycles was introduced in [11].

Definition 3.6. We say that *asymptotically all* repelling cycles move holomorphically on Ω if there exists a set $\mathcal{P} = \cup \mathcal{P}_n \subset \mathcal{J}$ with the following properties:

(1) $\mathrm{Card}(\mathcal{P}_n) = d_t^n + o(d_t^n)$;
(2) every $\gamma \in \mathcal{P}_n$ is n-periodic;
(3) for all open $\Omega' \Subset \Omega$, we have

$$\frac{\mathrm{Card}\{\gamma \in \mathcal{P}_n \colon \gamma(\lambda) \text{ is repelling for all } \lambda \in \Omega'\}}{d_t^n} \to 1.$$

Definition 3.7. We say that $\lambda_0 \in M$ is a *Misiurewicz* parameter if there exist integers $p_0, n_0 \geq 1$ and a holomorphic map σ defined on some neighbourhood of λ_0 such that $\sigma(\lambda) \in \mathcal{R}_{p_0}(\lambda)$ and $\Gamma_\sigma \cap W \neq \emptyset$ but $\Gamma_\sigma \not\subseteq W$ for some irreducible component W of $f^{n_0}(C_f)$, where Γ_σ denotes the graph of σ.

The following result is a generalization of Theorem-Definition 1.1 to the setting of polynomial-like maps of large topological degree. Observe in particular that it applies to any subfamily of $\mathcal{H}_d(\mathbb{P}^k)$, for any $k \geq 1$ and $d \geq 2$. In particular, it permits to extend the definition of stability to these settings.

Theorem 3.8 ([11, Theorem C]). *Let M be a connected and simply connected complex manifold and let $(f_\lambda)_{\lambda \in M}$ be a holomorphic family of polynomial-like maps of large topological degree. Then the following assertions are equivalent:*

- **(S1)** *asymptotically all repelling J-cycles of f_λ move holomorphically over M;*
- **(S2)** $dd^c L_\mu(\lambda) \equiv 0$, *where* $L_\mu(\lambda) := \int_{U_\lambda} \log|\text{Jac}\, f_\lambda(z)|\, \mu_\lambda(z)$ *is the sum of the Lyapunov exponents of the unique measure of maximal entropy μ_λ of f_λ;*
- **(S3)** *there are no Misiurewicz parameters;*
- **(S4)** *the family admits an equilibrium lamination.*

Condition **(S3)** is already present in [8] (and is in particular equivalent to the conditions in Theorem-Definition 1.1), see also [39, 40] for the analogous equivalence in dimension 1 and [9] for further characterization of stability.

Remark 3.9. By [5], condition **(S1)** can actually be weakened to the following, a priori weaker, condition: there exists a function $N : \mathbb{N} \to \mathbb{N}$ with $\limsup_{n\to\infty} d_t^{-n} N(n) > 0$ such that, for every n, $N(n)$ repelling periodic points move holomorphically (as repelling periodic points).

A crucial role in the proof of both Theorems 1.1 and 3.8 is the concept of *acritical web*. While the definition in [8] and [11] is given only for equilibrium webs, we can give it for general webs.

Definition 3.10. A web $\mathcal{M}$ is said to be *acritical* if $\mathcal{M}(\mathcal{J}_s) = 0$, where

$$\mathcal{J}_s := \{\gamma \in \mathcal{J} : \Gamma_\gamma \cap GO(C_f) \neq \emptyset\} \quad \text{and} \quad GO(C_f) := \cup_{n,m \geq 0} f^{-m}(f^n(C_f)).$$

In particular (see for instance [11, Theorem 4.11]), the conditions **(S1)**-**(S4)** in Theorem 3.8 are equivalent to the following one:

- **(S5)** there exists an acritical equilibrium web.

The following is a more general version of Theorem 1.3 that was announced in the Introduction. For any $\lambda \in M$, we denote by $C^+_{f_\lambda} := \cup_{m\geq 0} f^m_\lambda(C_{f_\lambda})$ the postcritical set of f.

Theorem 3.11. *Let M be a connected and simply connected complex manifold and let $(f_\lambda)_{\lambda \in M}$ be a holomorphic family of polynomial-like maps of large topological degree. Fix $\lambda_0 \in M$ and consider an ergodic f_{λ_0}-invariant probability measure ν_0 such that* $\text{Supp}\, \nu_0 \subseteq J_{\lambda_0}$*, the smallest Lyapunov exponent of ν_0 is strictly positive, and $\nu(C^+_{f_{\lambda_0}}) = 0$. Then, up to replacing M with a sufficiently small open neighbourhood M_{λ_0,ν_0} of λ_0, the conditions* **(S1)**-**(S4)** *in Theorem 3.8 and the condition* **(S5)** *above are equivalent to the following assertions:*

- **(S4')** *there exists a ν_0-lamination;*
- **(S5')** *there exists an ergodic acritical ν_0-web $\mathcal{M}$ such that, for all $\lambda \in M_{\lambda_0,\nu_0}$, the Lyapunov exponents of $(p_\lambda)_*\mathcal{M}$ are uniformly bounded from below by a strictly positive constant.*

The main reason why we may need to reduce M to M_{λ_0,ν_0} (possibly depending on ν_0) is due to the fact that the smallest Lyapunov exponent of the motion of ν_0 may, a priori, become negative at some $\lambda_1 \in M$. In our construction (and more precisely in the construction of the ν_0-lamination) we will need to restrict to the parameters where such an exponent stays positive.

It is proved in [8] that the existence of a graph $\gamma : M \to \mathbb{P}^k$ whose orbit does not intersect the postcritical set implies the stability of the family, and is then equivalent to it. The same result is proved in [10] for families of polynomial-like maps. In particular, it follows directly that the existence of an acritical web (associated to any measure, hence in particular **(S4')**) implies that the family is stable. The implication **(S5')** $\Rightarrow$ **(S4')** follows from similar arguments as in [8, 11, 7], that we will briefly recall for convenience and later reference, see Section 4.3 and the Appendix. The main point of Theorem 3.11 is the implication **(S1)** $\Rightarrow$ **(S5')**, and in particular the positivity of the Lyapunov exponents in **(S5')**.

By Theorem 2.7, Theorem 3.11 in particular applies when the measure-theoretic entropy of the measure ν_0 is strictly larger than $\log d^*_{k-1}$. In this case we can also find a uniformity for the neighbourhood M_{λ_0,ν_0}, depending only on the measure-theoretic entropy of ν_0. As we will see, this fact is also a consequence of the uniform bound on the Lyapunov exponents given by Theorem 2.7. The existence of $M_{\lambda_0,h}$ as in the statement below follows from the upper semicontinuity of the function $\lambda \mapsto d^*_{k-1}(f_\lambda)$.

Theorem 3.12. *Let M be a connected and simply connected complex manifold and let $(f_\lambda)_{\lambda\in M}$ be a stable family of polynomial-like maps of large topological degree. Fix $\lambda_0 \in M$ and $h \in \mathbb{R}$ such that $\log d^*_{k-1}(f_{\lambda_0}) < h < \log d_t$ and let $M_{\lambda_0,h}$ be a simply connected open neighbourhood of λ_0 such that $\log d^*_{k-1}(f_\lambda) < h$ for any $\lambda \in M_{\lambda_0,h}$. Then for any ergodic f_{λ_0}-invariant probability measure ν_0 supported in J_{λ_0} such that $h_{\nu_0}(f_{\lambda_0}) > h$, the properties* **(S4')** *and* **(S5')** *hold on $M_{\lambda_0,h}$.*

Moreover, there exists a dynamical lamination $\mathcal{L}$ satisfying

- **(Sh)** *$\nu(\{\gamma(\lambda) : \gamma \in \mathcal{L}\}) = 1$ for every $\lambda \in M_{\lambda_0,h}$ and every f_λ-invariant measure ν such that $h_\nu(f_\lambda) > h$.*

The following is a version of the above result for families of endomorphisms of $\mathbb{P}^k$, stating that, in this case, the neighbourhood $M_{\lambda_0,h}$ in Theorem 3.12 can be taken equal to M (recall that $d^*_{k-1}(f_\lambda) = d^{k-1}$ for all $\lambda \in M$ in this case). Observe that Theorem 1.3 corresponds to the case of Corollary 3.13 where M is an open subset of $\mathcal{H}_d(\mathbb{P}^k)$.

Corollary 3.13. *Let M be a connected and simply connected complex manifold and let $(f_\lambda)_{\lambda\in M}$ be a stable family of endomorphisms of $\mathbb{P}^k$ of algebraic degree $d \geq 2$. Fix $\lambda_0 \in M$ and let ν_0 be any ergodic f_{λ_0}-invariant probability measure with $h_{\nu_0}(f_{\lambda_0}) > (k-1)\log d$. Then, the properties* **(S4')** *and* **(S5')** *hold on M. In particular, there exists a dynamical lamination $\mathcal{L}$ satisfying*

- **(S*)** *$\nu(\{\gamma(\lambda) : \gamma \in \mathcal{L}\}) = 1$ for every $\lambda \in M$ and every f_λ-invariant measure ν such that $h_\nu(f_\lambda) > (k-1)\log d$.*

4. Proof of main results

In this section we give the proof of Theorems 3.11 and 3.12 and Corollary 3.13. In particular, this also proves Theorem 1.3. We fix a connected and simply connected complex manifold M and let $(f_\lambda)_{\lambda\in M}$ be a stable family of polynomial-like maps of large topological degree. In Section 4.1 we prove the existence of a special acritical ν-web, see Proposition 4.3. In Section 4.2 we define a Lyapunov function for a given ν-web and give a criterion for the pluriharmonicity of such function, that in particular applies to the web constructed in Proposition 4.3. As a consequence, we deduce the positivity of the Lyapunov exponents of the slices of that web in a neighbourhood of the starting parameter, proving **(S5')**. In Section 4.3, we prove the existence of a ν-lamination and conclude the proofs of the main results.

4.1. From stability to acritical ν-webs.

In this section we give the first part of the proof of the implication **(S1)** $\Rightarrow$**(S5')** in Theorem 3.11, namely we prove the existence of a suitable acritical web (with no requirement on the positivity of the associated Lyapunov exponents) under the stability assumption. Observe that we do not need to restrict the parameter space to get such property. For simplicity, we will set the following definition. Recall that we fix a stable family $(f_\lambda)_{\lambda\in M}$ in all this section.

Definition 4.1. Given $n \in \mathbb{N}$, a measure $\mathcal{M}$ on $\mathcal{J}$ is *critically n-aligned* if

$$\forall \gamma \in \mathrm{Supp}\mathcal{M}: \quad \Gamma_\gamma \cap f^n(C_f) \neq \emptyset \Rightarrow \Gamma_\gamma \subseteq f^n(C_f).$$

$\mathcal{M}$ is *critically aligned* if it is critically 0-aligned. It is *postcritically aligned* if it is critically n-aligned for all $n \geq 0$.

Lemma 4.2. *Take $\lambda_0 \in M$ and let ν be an f_{λ_0}-invariant measure such that $\nu(C^+_{f_{\lambda_0}}) = 0$. Let $\mathcal{M}$ be a postcritically aligned ν-web. Then $\mathcal{M}$ is acritical.*

Proof. By the $\mathcal{F}_*$-invariance of $\mathcal{M}$, it is enough to prove that $\mathcal{M}(\mathcal{J}^+_s) = 0$, where $\mathcal{J}^+_s := \left\{\gamma \in \mathcal{J} \colon \Gamma_\gamma \cap C^+_f \neq \emptyset\right\}$. Since $\mathcal{M}$ is postcritically aligned, by Definition 4.1 we have

$$\mathcal{M}\left(\left\{\gamma \in \mathcal{J} \colon \Gamma_\gamma \cap C^+_f \neq \emptyset\right\}\right) \leq \mathcal{M}\left(\left\{\gamma \in \mathcal{J} \colon \Gamma_\gamma \subseteq C^+_f\right\}\right)$$
$$= \mathcal{M}\left(\left\{\gamma \in \mathcal{J} \colon \gamma(\lambda_0) \in C^+_{f_{\lambda_0}}\right\}\right) = \nu\left(C^+_{f_{\lambda_0}}\right) = 0,$$

where the last equality follows from the assumption on ν. The assertion follows. □

The main result of this section is the following proposition.

Proposition 4.3. *Fix $\lambda_0 \in M$ and let ν be an ergodic f_{λ_0}-invariant probability measure supported in J_{λ_0} and such that $\nu(C^+_{f_{\lambda_0}}) = 0$. There exists a postcritically aligned ergodic ν-web $\mathcal{M}$.*

Observe that, in particular, $\mathcal{M}$ as in Proposition 4.3 is acritical by Lemma 4.2, and for every $\lambda \in M$ the measure $\nu_\lambda := (p_\lambda)_* \mathcal{M}$ is an ergodic f_λ-invariant probability measure.

We will need the following technical lemma.

Lemma 4.4. *Let X be a metric space and ν be a compactly supported probability measure on X. Let $X_j := \{x^j_1, x^j_2, \dots, x^j_{l(j)}\}$ be a sequence of finite sets such that*

(1) *the cardinalities $l(j)$ of X_j satisfy $l(j+1) \geq l(j)$;*
(2) *$L := \overline{\cup_j X_j}$ is a compact subset of X and* $\operatorname{Supp} \nu \subseteq L$*;*
(3) *$\overline{\cup_n X_{j_n}} = L$ for any subsequence $\{X_{j_n}\}$.*

Then there exists a sequence of sets $A_j := \{a^j_1, a^j_2, \dots, a^j_{l(j)}\}$ of non-negative real numbers with $\sum_{m=1}^{l(j)} a^j_m = 1$ such that

$$\nu_j := \sum_{m=1}^{l(j)} a^j_m \delta_{x^j_m} \to \nu,$$

where δ_x is the Dirac mass at $x \in X$.

Proof. For every j, let $\zeta_j := \{B^j_1, B^j_2, \dots, B^j_{l(j)}\}$ be a measurable partition of L such that $x^j_m \in B^j_m$ for all m. By the assumption (3) it follows the union of the X_j's is dense in L. Hence, we can assume that the maximum diameter of the elements of the partition ζ_j, goes to 0 as $j \to \infty$, i.e., that

$$\operatorname{diam} \zeta_j := \sup_{1 \leq m \leq l(j)} \operatorname{diam} B^j_m \to 0 \text{ as } j \to \infty. \tag{4.1}$$

For all j and $1 \leq m \leq l(j)$, set $a^j_m := \nu(B^j_m)$. By construction we have $|\nu_j| = 1$ for all j. Hence, by Banach-Alaoglu theorem, there exists a converging subsequence $\{\nu_{j_i}\}$ of $\{\nu_j\}$. Fix one such subsequence and denote $\widetilde{\nu} := \lim_{i \to \infty} \nu_{j_i}$. Since $|\nu_{j_i}| = 1$ it follows that $|\widetilde{\nu}| = 1$. Hence, it is enough to prove that $\nu \leq \widetilde{\nu}$. In order to do this, it is enough to show that $\nu(D) \leq \widetilde{\nu}(D)$ for all closed balls $D \subseteq X$ centered on $\operatorname{Supp} \nu$.

Let us fix a closed ball D as above and, for all $\varepsilon > 0$, let D_ε be the closed ε-neighbourhood of D, i.e.,

$$D_\varepsilon := \{x \in X : \operatorname{dist}(x, D) \leq \varepsilon\}.$$

By (4.1), there exists i_0 such that $\operatorname{diam} \zeta_{j_i} < \varepsilon/2$ for all $i > i_0$. It follows that, for $i \geq i_0$, all elements of ζ_{j_i} intersecting D are contained in D_ε. This implies that, for all $i \geq i_0$, we have $\nu(D) \leq \nu_{j_i}(D_\varepsilon)$. Hence, $\nu(D) \leq \widetilde{\nu}(D_\varepsilon)$. Finally, since $\lim_{\varepsilon \to 0} \widetilde{\nu}(D_\varepsilon) = \widetilde{\nu}(D)$, we obtain the desired inequality $\nu(D) \leq \widetilde{\nu}(D)$ by letting ε tend to 0. The proof is complete. □

Proof of Proposition 4.3. Since $(f_\lambda)_{\lambda\in M}$ is stable, by Theorem 3.8, for every $n \in \mathbb{N}^*$ there exists a collection $\{\gamma_{j,n} : 1 \le j \le N_d(n)\} \subset \mathcal{J}$ such that

$$\mathcal{R}_n(\lambda) = \{\gamma_{j,n}(\lambda) : 1 \le j \le N_d(n)\}$$

for all $\lambda \in M$, where $\mathcal{R}_n(\lambda)$ is a set of repelling n-periodic points in J_λ and $N_d(n) \sim d_t^n$. Let $A_n = \{a_1^n, \dots, a_{N_d(n)}^n\}$ be a sequence of sets of real numbers given by applying Lemma 4.4 with the measure ν and the sequence of sets $X_n = \mathcal{R}_n(\lambda_0)$. Observe that the assumptions in Lemma 4.4 are satisfied by the asymptotics of $N_d(n)$ and the equidistribution of periodic points with respect to μ_{λ_0} on $J_{\lambda_0} \supseteq \operatorname{Supp}\nu$, see [17, 22, 24]. Set

$$\mathcal{M}_n := \sum_{j=1}^{N_d(n)} a_j^n \delta_{\gamma_{j,n}}.$$

By definition, each $\mathcal{M}_n$ is a discrete probability measure supported on $\mathcal{J}$. In particular, $\cup_n \mathrm{Supp}\mathcal{M}_n$ is relatively compact. Hence, by Banach-Alaoglu theorem, there exists a converging subsequence $\mathcal{M}_{n_l} \to \widetilde{\mathcal{M}}$. Observe that also $\widetilde{\mathcal{M}}$ is a measure on $\mathcal{J}$.

We claim that $\widetilde{\mathcal{M}}$ is postcritically aligned. Indeed, by the assumption on the motion of the repelling cycles and Theorem 3.8, there are no Misiurewicz parameters in M. So, $\mathcal{M}_{n_l}$ is postcritically aligned for all l. It then follows from Hurwitz theorem that also $\widetilde{\mathcal{M}}$ is postcritically aligned.

By construction, we have

$$(p_{\lambda_0})_*\widetilde{\mathcal{M}} = \lim_{l\to\infty} (p_{\lambda_0})_*\mathcal{M}_{n_l} = \nu.$$

On the other hand, a priori, the measure $\widetilde{\mathcal{M}}$ may not be $\mathcal{F}$-invariant (hence, it may not be a web). We define $\mathcal{M}$ to be any limit of a subsequence of $n^{-1}\sum_{j=0}^{n-1} \mathcal{F}_*^j\widetilde{\mathcal{M}}$. Then, we have $\mathcal{F}_*\mathcal{M} = \mathcal{M}$. Since ν is f_{λ_0}-invariant, for every $j \in \mathbb{N}$ we have

$$(p_{\lambda_0})_*\mathcal{F}_*^j\widetilde{\mathcal{M}} = (f_{\lambda_0}^j)_*((p_{\lambda_0})_*\widetilde{\mathcal{M}}) = (f_{\lambda_0}^j)_*\nu = \nu.$$

In particular, for every $n \in \mathbb{N}$ we have $(p_{\lambda_0})_*\left(n^{-1}\sum_{j=0}^{n-1}\mathcal{F}_*^j\widetilde{\mathcal{M}}\right) = \nu$. It follows that $\mathcal{M}$ is a ν-web. As above, since $\mathcal{M}$ is a limit of postcritically aligned measures, it is postcritically aligned, too. Up to replacing $\mathcal{M}$ with one of its ergodic components (by means of Choquet theorem), we can also assume that $\mathcal{M}$ is a postcritically aligned ergodic web. The assertion follows. ☐

Remark 4.5. One can also construct webs as in Proposition 4.3 by using the equidistribution of preimages of generic points instead of repelling points, see [22]. Indeed, as mentioned above, by [8, 10] the stability of a family implies the existence of an element $\sigma \in \mathcal{J}\setminus\mathcal{J}_s$, i.e., a holomorphic map $\sigma\colon M \to V$ whose graph does not intersect the postcritical set. Hence, one can consider the measures

$$\mathcal{M}_{\gamma,n} := d_t^{-n} \sum_{\sigma\in\mathcal{F}^{-n}(\gamma)} a_\sigma^n \delta_\sigma$$

where $\{a_\sigma^n\colon n \in \mathbb{N}, \sigma \in \mathcal{F}^{-n}(\gamma)\}$ is chosen so that $(p_{\lambda_0})_*\mathcal{M}_{\gamma,n} \to \nu$. Observe that the support of $\mathcal{M}_n$ is discrete and disjoint from $\mathcal{J}_s$. In particular, every $\mathcal{M}_{\gamma,n}$ is postcritically aligned. Hence, as in the proof of Proposition 4.3, we can use the sequence $(\mathcal{M}_{\gamma,n})_{n\in\mathbb{N}}$ to construct a web $\mathcal{M}$ satisfying the properties in Proposition 4.3

The following is an immediate consequence of Proposition 4.3 and Corollary 2.4. Observe that, also in this case, $\mathcal{M}$ as in the statement is acritical by Lemma 4.2.

Corollary 4.6. *Fix $\lambda_0 \in M$ and let ν be an ergodic f_{λ_0}-invariant probability measure supported in J_{λ_0} such that $h_\nu(f_{\lambda_0}) > \log d_{k-1}^*(f_{\lambda_0})$. There exists a postcritically aligned ergodic ν-web $\mathcal{M}$.*

4.2. **Positivity of Lyapunov exponents.** Once the existence of a web as in Proposition 4.3 is established, in order to apply the ideas in [8] to prove **(S5')** we need to check that the function associating to every λ the smallest Lyapunov exponent of $(p_\lambda)_*\mathcal{M}$ is locally uniformly bounded from below by some strictly positive constant in a neighbourhood of the starting parameter λ_0. In order to do this, it is enough to show that this function is strictly positive and lower semicontinuous in a neighbourhood of λ_0 (possibly depending on ν_0).

Given a web $\mathcal{M}$, consider the current on $\mathcal{V}$ given by

$$W_{\mathcal{M}} := \int [\Gamma_\gamma] d\mathcal{M}(\gamma), \tag{4.2}$$

where $[\Gamma_\gamma]$ is the current of integration on the graph of the map γ. Observe that $W_{\mathcal{M}}$ is positive and closed.

Definition 4.7. Let $\mathcal{M}$ be a web. We define the function $L_{\mathcal{M}} \colon M \to \mathbb{R}$ as

$$L_{\mathcal{M}} = \pi_*(\log|\mathrm{Jac}\, f_\lambda(z)| W_{\mathcal{M}}),$$

where $W_{\mathcal{M}}$ is as in (4.2) and π denotes as usual the projection on the parameter space M.

Up to restricting to a small neighbourhood of a given parameter $\lambda_0 \in M$, we can assume that $W_{\mathcal{M}}$ is horizontal. Hence, the function $L_{\mathcal{M}}$ is well-defined. Moreover, since $\log|\mathrm{Jac}\, f_\lambda(z)|$ is a psh function on $\mathcal{V}$, by [23, Theorem 2.1] and [44, Proposition A.1], the function $L_{\mathcal{M}}$ is psh on M, or identically equal to $-\infty$. Observe that, when $(p_\lambda)_*\mathcal{M}$ is ergodic for all $\lambda \in M$, the function $L_{\mathcal{M}}(\lambda)$ is equal to the sum of the Lyapunov exponents of $(p_\lambda)_*\mathcal{M}$.

Lemma 4.8. *Take $\lambda_0 \in M$ and assume that ν is an ergodic f_{λ_0}-invariant measure supported in J_{λ_0} and such that the function $\log|\mathrm{Jac}\, f_{\lambda_0}|$ is ν-integrable. If $\mathcal{M}$ is a critically aligned ν-web, then $dd^c L_{\mathcal{M}}(\lambda) = 0$.*

Proof. Since the problem is local, we can fix $\lambda_0 \in M$ and work on a small ball B around $\lambda_0 \in M$ in the parameter space. The current as in (4.2) is then horizontal. We can also assume that B has dimension 1. Define $J(\lambda, z) := \mathrm{Jac}\, f_\lambda(z)$. Since the function $\log|J(\lambda_0, \cdot)|$ is ν-integrable, by [44, Theorem A.2], the currents $\log|J(\lambda, z)| W_{\mathcal{M}}$ and $dd^c(\log|J(\lambda, z)| W_{\mathcal{M}})$ are well-defined. Since $L_{\mathcal{M}} = \pi_*(\log|J(\lambda, z)| W_{\mathcal{M}})$, to get the assertion it is enough to show that $dd^c(\log|J(\lambda, z)| W_{\mathcal{M}}) = 0$ on $B \times V$. Observe that $dd^c(\log|J(\lambda, z)| W_{\mathcal{M}})$ is a positive measure on $B \times V$.

For this, we can follow the strategy in [8, Proposition 3.5]. For any small $\varepsilon > 0$, set

$$S_\varepsilon := \{\gamma \in \mathrm{Supp}\mathcal{M} : \Gamma_\gamma \cap \{|J(\lambda, z)| < \varepsilon\} \neq \emptyset\}$$

and define

$$W^\varepsilon_{\mathcal{M}} := \int_{S_\varepsilon} [\Gamma_\gamma] d\mathcal{M}(\gamma) \quad \text{and} \quad \widetilde{W}_{\mathcal{M}} := W_{\mathcal{M}} - W^\varepsilon_{\mathcal{M}}.$$

For any smooth test function ϕ compactly supported in $B \times V$, we have

$$\begin{aligned} \langle dd^c(\log|J(\lambda, z)| W_{\mathcal{M}}), \phi\rangle &= \langle \log|J(\lambda, z)| W_{\mathcal{M}}, dd^c\phi\rangle \\ &= \langle \log|J(\lambda, z)| W^\varepsilon_{\mathcal{M}}, dd^c\phi\rangle + \langle \log|J(\lambda, z)| \widetilde{W}_{\mathcal{M}}, dd^c\phi\rangle \\ &=: I_1(\varepsilon) + I_2(\varepsilon). \end{aligned} \tag{4.3}$$

Since the function $\lambda \mapsto \log|J(\lambda, \gamma(\lambda))|$ is harmonic for any $\gamma \in \mathrm{Supp}\mathcal{M} \setminus S_\varepsilon$, we have $\int_B \log|J(\lambda, \gamma(\lambda))| dd^c(\phi \circ \gamma) = 0$ for every such γ. This gives

$$I_2(\varepsilon) = \langle \log|J(\lambda, z)| \widetilde{W}_{\mathcal{M}}, dd^c\phi\rangle = \int_{\mathrm{Supp}\mathcal{M} \setminus S_\varepsilon} d\mathcal{M}(\gamma) \left(\int_B \log|J(\lambda, \gamma(\lambda))| dd^c(\phi \circ \gamma) \right) = 0,$$

i.e., the second term in the right hand side of (4.3) vanishes for all $\varepsilon > 0$.

In order to conclude, we need to prove that the first term $I_1(\varepsilon)$ tends to 0 as $\varepsilon \to 0$. The argument in [8] uses the fact that the measure of maximal entropy μ_{λ_0} is the Monge-Ampère of a $(1,1)$-current with Hölder-continuous local potentials, and in particular that it gives mass

$\lesssim \varepsilon^a$ (for some positive a) to balls of radius ε, but this is actually not necessary. The following claim will be enough to complete the proof.

Claim 1. For any $\varepsilon \ll 1$ there exists $c(\varepsilon) > 0$ with $c(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that

$$\nu\left(\left(C_{f_{\lambda_0}}\right)_\varepsilon\right) \le \frac{c(\varepsilon)}{|\log|\varepsilon||} \tag{4.4}$$

where $\left(C_{f_{\lambda_0}}\right)_\varepsilon$ is the ε-neighbourhood of $C_{f_{\lambda_0}}$.

Proof. Since $J(\lambda_0, z)$ is holomorphic there exists $A_1 > 0$ and ε small enough such that for every z such that $\operatorname{dist}(z, C_{f_{\lambda_0}}) < \varepsilon$ we have

$$|J(\lambda_0, z)| \le A_1 \operatorname{dist}(z, C_{f_{\lambda_0}}). \tag{4.5}$$

Since by assumption the function $z \mapsto \log|J(\lambda_0, z)|$ is ν-integrable, (4.5) implies that the function $z \mapsto \log \operatorname{dist}(z, C_{f_{\lambda_0}})$ is also ν-integrable. Hence, the measure $\widetilde{\nu} := |\log \operatorname{dist}(z, C_{f_{\lambda_0}})|\nu$ is finite and satisfies $\widetilde{\nu} \ll \nu$, i.e., we have $\widetilde{\nu}(E) = 0$ if $\nu(E) = 0$. In particular, we have $\widetilde{\nu}(C_{f_{\lambda_0}}) = 0$. Thus, there exists $c(\varepsilon) > 0$ with $c(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that $\widetilde{\nu}\left(\left(C_{f_{\lambda_0}}\right)_\varepsilon\right) \le c(\varepsilon)$. Hence, for all ε sufficiently small, we have

$$|\log|\varepsilon||\nu\left(\left(C_{f_{\lambda_0}}\right)_\varepsilon\right) \lesssim \widetilde{\nu}\left(\left(C_{f_{\lambda_0}}\right)_\varepsilon\right) \le c(\varepsilon).$$

This gives (4.4) and proves the claim. $\square$

Let now $(C_f)_\varepsilon$ be the ε-neighbourhood of C_f in $B \times V$. Since $J(\lambda, z)$ is holomorphic we can find a constant c_1 such that $(C_f)_\varepsilon \subset \{|J(\lambda, z)| < c_1\varepsilon\}$. By using Łojasiewicz inequality we see that there are constants $c_2, \beta_1 > 0$ such that $\{|J(\lambda, z)| < c_1\varepsilon\} \subset (C_f)_{c_2\varepsilon^{\beta_1}}$. Again by using Łojasiewicz inequality we have

$$\{z\colon (\lambda_0, z) \in (C_f)_\varepsilon\} \subset \left(C_{f_{\lambda_0}}\right)_{c_3\varepsilon^{\beta_2}} \text{ for some constants } c_3, \beta_2 > 0.$$

Fix now a ball $B' \Subset B$ centered at λ_0.

Claim 2. (see the Claim in [8, Lemma 3.6]) There exists $0 < \alpha \le 1$ such that $\sup_{B'}|\psi| \le |\psi(\lambda_0)|^\alpha$ for every holomorphic function $\psi : B \to \mathbb{D}^*$.

Take $\gamma \in \operatorname{Supp} \mathcal{M}$ such that $\Gamma_\gamma \cap C_f = \emptyset$ but $\Gamma_\gamma \cap (C_f)_\varepsilon \neq \emptyset$. Then by using Claim 2 for the holomorphic function $J(\lambda, \gamma(\lambda))$ we have $\Gamma_{\gamma|_{B'}} \subset (C_f)_{c_4\varepsilon^{\alpha\beta_3}}$ for some constants $c_4, \beta_4 > 0$. Since $\mathcal{M}$ is critically aligned, we have

$$\begin{aligned}\mathcal{M}\{\gamma \in \mathcal{J} : \Gamma_{\gamma|_{B'}} \cap (C_f)_\varepsilon \neq \emptyset\} &\le \mathcal{M}\{\gamma \in \mathcal{J} : \Gamma_{\gamma|_{B'}} \subseteq (C_f)_{c_4\varepsilon^{\alpha\beta_3}}\} \\ &\le \mathcal{M}\{\gamma \in \mathcal{J} : (\lambda_0, \gamma(\lambda_0)) \in (C_f)_{c_4\varepsilon^{\alpha\beta_3}}\} \\ &\le \nu\left\{\left(C_{f_{\lambda_0}}\right)_{c_3(c_4\varepsilon^{\alpha\beta_3})^{\beta_2}}\right\}.\end{aligned}$$

Setting $\varepsilon' := c_3(c_4\varepsilon^{\alpha\beta_3})^{\beta_2}$ for simplicity, we can now apply Claim 1 to get

$$\mathcal{M}\{\gamma \in \mathcal{J} : \Gamma_{\gamma|_{B'}} \cap (C_f)_\varepsilon \neq \emptyset\} \le \nu\left\{\left(C_{f_{\lambda_0}}\right)_{c_3(c_4\varepsilon^{\alpha\beta_3})^{\beta_2}}\right\} \le \frac{c(\varepsilon')}{|\log \varepsilon'|} \lesssim \frac{c(\varepsilon')}{\log|\varepsilon|} = \frac{c'(\varepsilon)}{\log|\varepsilon|}$$

for some positive function $c'(\varepsilon)$ with $c'(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Finally, by using the last inequality and the fact that

$$S_\varepsilon \subset S'_{c_5\varepsilon^{\beta_4}} := \{\gamma \in \mathcal{J} : \Gamma_{\gamma|_{B'}} \cap (C_f)_{c_5\varepsilon^{\beta_4}} \neq \emptyset\} \text{ for some positive constants } c_5, \beta_4$$

(which again follows from Łojasiewicz inequality), setting $\varepsilon'' = c_5 \varepsilon^{\beta_4}$ for simplicity we deduce that

$$I_1(\varepsilon) = \langle \log|J(\lambda, z)| W^{\varepsilon}_{\mathcal{M}}, dd^c \phi\rangle \lesssim |\log|\varepsilon|| \int_{S_\varepsilon} d\mathcal{M}(\gamma) = |\log|\varepsilon|| \cdot W^{\varepsilon}_{\mathcal{M}}(S_\varepsilon)$$

$$\lesssim |\log|\varepsilon|| \cdot W^{\varepsilon}_{\mathcal{M}}(S'_{\varepsilon''}) \leq |\log|\varepsilon|| \cdot \frac{c'(\varepsilon'')}{|\log|\varepsilon''||} \lesssim c'(\varepsilon'').$$

Since $c'(\varepsilon'') \to 0$ as $\varepsilon \to 0$, the assertion follows. □

Corollary 4.9. *Let $\mathcal{M}$ be a critically aligned web. Assume that $(p_\lambda)_*\mathcal{M}$ is ergodic for all $\lambda \in M$ and denote by $L^1_{\mathcal{M}}(\lambda) \geq \ldots \geq L^k_{\mathcal{M}}(\lambda)$ the k Lyapunov exponents of $(p_\lambda)_*\mathcal{M}$, counting multiplicities. Assume also that there exists $\lambda_0 \in M$ such that $L^k_{\mathcal{M}}(\lambda_0) > -\infty$. Then the function $\lambda \mapsto -L^k_{\mathcal{M}}(\lambda)$ of $\mathcal{M}$ is plurisubhamonic. In particular, it is upper semicontinuous.*

Proof. By Proposition 3.2 the upper partial sums of Lyapunov exponents are plurisubharmonic In particular, the function $\lambda \mapsto \sum_{j=1}^{k-1} L^j_{\mathcal{M}}(\lambda)$ is plurisubharmonic. Set $\nu_0 := (p_{\lambda_0})_*\mathcal{M}$. Since, by assumption, the smallest Lyapunov exponent of ν_0 is finite, the function $\log|\mathrm{Jac}\, f_{\lambda_0}|$ is ν_0-integrable. Thanks to Lemma 4.8 we have that $L_{\mathcal{M}}(\lambda)$ is pluriharmonic. Since $-L^k_{\mathcal{M}}(\lambda) = \sum_{j=1}^{k-1} L^j_{\mathcal{M}}(\lambda) - L_{\mathcal{M}}(\lambda)$, it follows that the function $-L^k_{\mathcal{M}}(\lambda)$ is plurisubharmonic. □

4.3. **Existence of the lamination and proofs of the main results.** The following proposition has the same proof as [8, Theorem 4.1], see also [10, Theorem 3.4.1] and [7, Section 7] for an overview of the arguments. We will give in the Appendix an intermediate statement, for later reference.

Proposition 4.10. *Fix $\lambda_0 \in M$. Let ν_0 be an ergodic f_{λ_0}-invariant measure with* $\mathrm{Supp}\nu_0 \subseteq J_{\lambda_0}$. *Assume that there exists an acritical ergodic ν_0-web $\mathcal{M}$ such that, for all $\lambda \in M$, the Lyapunov exponents $L^j_{\mathcal{M}}(\lambda)$ of the measures $(p_\lambda)_*\mathcal{M}$ are uniformly strictly positive. Then there exists a ν_0-lamination on M.*

Sketch of proof. Thanks to the locally uniform lower bound for the smallest Lyapunov exponents (and in particular to Proposition A.1 below), one can prove the following property, see the Fact in [7, Section 4.3]:

Fact. $\mathcal{M}(\mathcal{K}_\cap) = 0$ for every compact subset $\mathcal{K} \subseteq \mathcal{J}$, where

$$\mathcal{K}_\cap := \{\gamma \in \mathcal{J} \colon \exists j \in \mathbb{N}, \exists \gamma' \in \mathcal{K} \text{ such that } \Gamma_{\mathcal{F}^j(\gamma)} \cap \Gamma_{\gamma'} \neq \emptyset \text{ and } \Gamma_{\mathcal{F}^j(\gamma)} \neq \Gamma_{\gamma'}\}.$$

Once this Fact is establish, in order to construct a lamination one can follow the arguments in [8, Section 4.3] or [10, Section 3.4.3] to show that the set $\mathcal{L} := \mathcal{J} \setminus (\mathcal{J}_\cap \cup \mathcal{J}_s)$ gives the desired lamination associated to ν. □

We can now complete the proofs of our main results.

Proof of Theorem 3.11. Proposition 4.3 and Corollary 4.9 show that (up to possibly restricting M as in the statement of the theorem) **(S1)** implies **(S5')**. We used here the fact that, by assumption, the Lyapunov exponents of ν_0 are strictly positive, and that the function $\lambda \mapsto L^k_{\mathcal{M}}(\lambda)$ is lower semicontinuous by Corollary 4.9. Proposition 4.10 shows that **(S5')** implies **(S4')**. As remarked after the statement of the theorem, these conditions imply those of Theorem 3.8. The proof is complete. □

In order to prove Theorem 3.12 (and Corollary 3.13) we will need the following further lemma.

Lemma 4.11. *Take $\lambda_0 \in M$ and let ν be an f_{λ_0}-invariant measure. Let $\mathcal{M}$ be a ν-web and $\mathcal{L}$ a ν-lamination. Then, for every $\lambda \in M$, the measure-theoretic entropy of $(p_\lambda)_*\mathcal{M}$ is equal to $h_\nu(f_{\lambda_0})$.*

Proof. We denote $\nu_\lambda := (p_\lambda)_*\mathcal{M}$ for simplicity. In particular, we have $\nu = \nu_{\lambda_0}$. Let $A \subset J_{\lambda_0}$ be a measurable set with $\nu_{\lambda_0}(A) > 0$. Set

$$\mathcal{A}_A := \{\gamma \in \mathcal{L} : \gamma(\lambda_0) \in A\}.$$

For every $\lambda \in M$, set also $A_\lambda := p_\lambda(\mathcal{A}_A)$. Note that we have $\mathcal{M}(\mathcal{A}_A) = \nu_{\lambda_0}(A) = \nu_\lambda(A_\lambda)$ for all $\lambda \in M$.

Fix $M \ni \lambda_1 \neq \lambda_0$ and let $\xi := \{A^i\}$ be a measurable partition for ν_{λ_0}. Define the measurable partition $(\xi)_{\lambda_1} := \{A^i_{\lambda_1}\}$ for ν_{λ_1}. By construction, the entropy of ξ with respect to ν_{λ_0} [19, 38] is equal to the entropy of $(\xi)_{\lambda_1}$ with respect to ν_{λ_1}. By the definition of $\mathcal{L}$, for every $n \in \mathbb{N}$ we have

$$\Big(\bigvee_{j=0}^{n} f_{\lambda_0}^{-j}\xi\Big)_{\lambda_1} = \bigvee_{j=0}^{n} f_{\lambda_1}^{-j}(\xi)_{\lambda_1},$$

where we recall that, for every $\lambda \in M$ and a given partition η, the partition $\bigvee_{j=0}^{n} f_\lambda^{-j}\eta$ is defined as

$$\bigvee_{j=0}^{n} f_\lambda^{-j}\eta := \{f_\lambda^{-n}(B^n) \cap \ldots \cap f_\lambda^{-1}(B^1) \cap B^0 \colon B^0, \ldots, B^n \in \eta\}.$$

By the definition of measure-theoretic entropy [19, 38] we conclude that $h_{\nu_{\lambda_0}}(f_{\lambda_0}) \le h_{\nu_{\lambda_1}}(f_{\lambda_1})$. By reversing the roles of λ_0 and λ_1, we also see that $h_{\nu_{\lambda_1}}(f_{\lambda_1}) \le h_{\nu_{\lambda_0}}(f_{\lambda_0})$. So, we have $h_{\nu_\lambda}(f_\lambda) = h_{\nu_{\lambda_0}}(f_{\lambda_0})$ for all $\lambda \in M$. The assertion follows. □

Proof of Theorem 3.12. Theorems 3.11 and 2.7 show that **(S1)** implies **(S5')** on a neighbourhood of λ_0, a priori depending on ν_0. As in the proof of Theorem 3.11, Proposition 4.10 shows that **(S5')** implies **(S4')**, hence **(S4')** holds on the same neighbourhood.

Since $d^*_{k-1}(f_\lambda)$ depends upper semicontinuously on λ, it follows that the set

$$M'_{\lambda_0,h} := \{\lambda \colon d^*_{k-1}(f_\lambda) < h\}$$

is open (and non-empty by the assumption on λ_0). Define $M^0_{\lambda_0,h}$ to be the connected component of $M'_{\lambda_0,h}$ containing λ_0. It is enough to show that **(S4')** and **(S5')** hold on $M^0_{\lambda_0,h}$.

Assume that this is not the case. Let Ω be the maximal open subset of $M^0_{\lambda_0,h}$ where **(S4')** and **(S5')** hold, and let $\mathcal{M}$ be as in **(S5')**. By Lemma 4.11, the measure-theoretic entropy of $(p_\lambda)_*\mathcal{M}$ is constant on Ω. It follows from Theorem 2.7 that the function $u(\lambda) := -L^k_{\mathcal{M}}(\lambda)$ is uniformly bounded from above on Ω by a strictly negative constant. Recall that this function is plurisubharmonic on M by Corollary 4.9. We can assume that $u \ge 0$ on the boundary of Ω, since otherwise (thanks to the upper semicontinuity of this function), we would have $u < 0$ in a neighbourhood Ω_0 of a point of the boundary of Ω, and we could extend the lamination to an open set $\Omega' = \Omega \cup \Omega_0$ which is larger than Ω.

The function u is then bounded above by a strictly negative constant on Ω, and satisfies $u \ge 0$ on its boundary. Up to adding a negative constant, we can assume that there exists λ_1 in the boundary of Ω with $u(\lambda_1) = 0$. By upper semicontinuity, for every $\varepsilon > 0$, there exists a neighbourhood of λ_1 where $u \le \varepsilon$. Since λ_1 is a point of strictly positive density for Ω, this gives a contradiction with the mean inequality for u at λ_1. Hence, the only possibility is that Ω is equal to $M^0_{\lambda_0,h}$. This proves the claim.

To conclude, we just need to show that **(Sh)** holds. In order to do this, we apply the Fact in the proof of Proposition 4.10 with $\mathcal{K} = \mathcal{J}$. For all $\lambda \in M^0_{\lambda_0,h}$, ν as in the statement, and $\mathcal{M}$ an acritical ν-web, we have $\mathcal{M}(\mathcal{J}_\cap) = 0$. Since we also have $\mathcal{M}(\mathcal{J}_s) = 0$ for all such $\mathcal{M}$, we have $\mathcal{M}(\mathcal{J}_\cap \cup \mathcal{J}_s) = 0$ for all $\lambda, \nu, \mathcal{M}$ as above. It follows that $\mathcal{L} := \mathcal{J} \setminus (\mathcal{J}_\cap \cup \mathcal{J}_s)$ gives the desired lamination, see also the end of [8, Section 4] for details. The proof is complete. □

Recall that Corollary 3.13 in particular implies Theorem 1.3.

Proof of Corollary 3.13. By Theorem 3.12, we only need to show that we can take $M_{\lambda_0,h} = M$ for all $\lambda_0 \in M$ and $h > \log d^{k-1}$. This is clear since for every endomorphism of $\mathbb{P}^k$ we have $d^*_{k-1} = d^{k-1}$. The proof is complete. □

Appendix A. Exponential backward contraction along graphs

Recall that $\mathcal{J}$ is as in (3.1) and is a compact metric space, and that $\mathcal{F}: \mathcal{J} \to \mathcal{J}$ is defined as $(\mathcal{F}\gamma)(\lambda) = f_\lambda(\gamma(\lambda))$. A web $\mathcal{M}$ (associated to any f_λ-invariant measure ν at any parameter λ) is an $\mathcal{F}$-invariant probability measure on $\mathcal{J}$, and $\mathcal{M}$ is acritical if $\mathcal{M}(\mathcal{J}_s) = 0$, see Definition 3.10. In particular, $\mathcal{F}$ is surjective on $\mathcal{X} := \mathcal{J} \setminus \mathcal{J}_s$. The *natural extension* $(\hat{\mathcal{X}}, \hat{\mathcal{F}}, \hat{\mathcal{M}})$ of the system $(\mathcal{J}, \mathcal{F}, \mathcal{M})$ can be defined as follows (see for instance [19, Section 10.4]). An element $\hat{\gamma} \in \hat{\mathcal{X}}$ is a bi-infinite sequence $\hat{\gamma} := (\dots, \gamma_{-1}, \gamma_0, \gamma_1, \dots)$ of elements of $\mathcal{X}$ with the property that $\mathcal{F}(\gamma_j) = \gamma_{j+1}$. For $j \in \mathbb{Z}$, we denote by $\pi_j: \hat{\mathcal{X}} \to \mathcal{X}$ the projection $\hat{\gamma} \mapsto \gamma_j$. We also denote by $\hat{\mathcal{F}}$ the shift map on $\hat{\mathcal{X}}$, i.e., for a $\hat{\gamma}$ as above we set

$$\hat{\mathcal{F}}(\hat{\gamma}) := (\dots, \mathcal{F}(\gamma_{-1}), \mathcal{F}(\gamma), \mathcal{F}(\gamma_1), \dots) = (\dots, \gamma_0, \gamma_1, \gamma_2, \dots).$$

The map $\hat{\mathcal{F}}$ is invertible and satisfies $\pi_j \circ \hat{\mathcal{F}} = \mathcal{F} \circ \pi_j$ for all $j \in \mathbb{Z}$. There exists a probability measure $\hat{\mathcal{M}}$ on $\hat{\mathcal{X}}$ such that $(\pi_j)_* \hat{\mathcal{M}} = \mathcal{M}$ for all $j \in \mathbb{Z}$. This measure is ergodic if $\mathcal{M}$ is ergodic.

The graph of any element $\gamma \in \mathcal{X}$ does not intersect the (graph of the) critical orbit of the family $(f_\lambda)_{\lambda \in M}$. It follows that, for every $\gamma \in \mathcal{X}$, the inverse branches of the holomorphic map (λ, f_λ) are well-defined in a open neighbourhood of the graph of γ. Given $\hat{\gamma} \in \hat{\mathcal{X}}$, we will denote by $f^{-n}_{\hat{\gamma}}$ the inverse branch of order n, defined on an open set as above, sending the graph of γ to the graph of γ_{-n}. The following proposition, proved in [8, Propositions 4.2 and 4.3] in the case of the measure of maximal entropy, gives a uniform control on the size of the neighbourhoods where the inverse branches as above are defined, and an explicit control on their contraction. Given $\gamma \in \mathcal{X}$, $\eta > 0$, and a subset $\Omega \subset M$, we denote by $T_\Omega(\gamma, \eta)$ the η-neighbourhood of the graph of γ over Ω, i.e., we set

$$T_\Omega(\gamma, \eta) := \{(\lambda, z) \in \Omega \times \mathbb{C}^k : |z - \gamma(\lambda)| < \eta\}.$$

Proposition A.1. *Let M be a connected and simply connected complex manifold and $(f_\lambda)_{\lambda \in M}$ a holomorphic family of polynomial-like maps of large topological degree. Assume that there exists a constant $A_1 > 0$ and an ergodic acritical web $\mathcal{M}$ with the property that the Lyapunov exponents of $(p_\lambda)_* \mathcal{M}$ are strictly larger than A_1 for all $\lambda \in M$. Then, for every open subset $\Omega \Subset M$ and constant $0 < A < A_1$, there exists $p \geq 1$, a Borel subset $\hat{\mathcal{Y}} \subseteq \hat{\mathcal{X}}$ with $\hat{\mathcal{M}}(\hat{\mathcal{Y}}) = 1$, and two measurable functions $\hat{\eta}_{p,A}: \hat{\mathcal{Y}} \to]0, 1]$ and $\hat{l}_{p,A}: \hat{\mathcal{Y}} \to [1, +\infty[$ which satisfy the following properties.*

For every $\hat{\gamma} \in \mathcal{Y}$ and every $n \in p\mathbb{N}^*$ the iterated inverse branch $f^{-n}_{\hat{\gamma}}$ is defined on the tubular neighbourhood $T_\Omega(\gamma_0, \hat{\eta}_{p,A}(\hat{\gamma}))$ of the graph $\Gamma_{\gamma_0} \cap (\Omega \times \mathbb{C}^k)$ of γ_0, and we have

$$f^{-n}_{\hat{\gamma}}(T_\Omega(\gamma_0, \hat{\eta}_{p,A}(\hat{\gamma}))) \subset T_\Omega(\gamma_{-n}, e^{-nA}) \quad \text{and} \quad \widetilde{\text{Lip}}(f^{-n}_{\hat{\gamma}}) \leq \hat{l}_{p,A}(\hat{\gamma}) e^{-nA},$$

where $\widetilde{\text{Lip}}(f^{-n}_{\hat{\gamma}}) := \sup_{\lambda \in \Omega} \text{Lip}((f^{-n}_{\hat{\gamma}})_{|B(\gamma_0(\lambda), \hat{\eta}_{p,A})})$.

Observe that, without loss of generality, one can actually assume that $p = 1$ in the above statement.

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