MONOTONICITY OF DYNAMICAL DEGREES FOR HÉNON-LIKE AND POLYNOMIAL-LIKE MAPS

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Abstract. We prove that, for every invertible horizontal-like map (i.e., Hénon-like map) in any dimension, the sequence of the dynamical degrees is increasing until that of maximal value, which is the main dynamical degree, and decreasing after that. Similarly, for polynomial-like maps in any dimension, the sequence of dynamical degrees is increasing until the last one, which is the topological degree. This is the first time that such a property is proved outside of the algebraic setting. Our proof is based on the construction of a suitable deformation for positive closed currents, which relies on tools from pluripotential theory and the solution of the $d, \bar\partial$, and dd^c equations on convex domains.

Notation. $\mathbb{D}_r$ and $\mathbb{D}$ denote the disc of radius r and centre 0 and the unit disc in $\mathbb{C}$, respectively. For $k \geq 1$, $\mathbb{B} = \mathbb{B}_k$ denotes the unit ball in $\mathbb{C}^k$. M and N will denote open bounded convex subsets of $\mathbb{C}^p$ and $\mathbb{C}^{k-p}$, respectively, for some fixed $1 \leq p \leq k$. We will usually denote by M', M'' (resp. N', N'') open bounded convex subsets of $\mathbb{C}^p$ (resp. $\mathbb{C}^{k-p}$) which are slightly smaller than M (resp. N), with $M'' \Subset M' \Subset M$ and $N'' \Subset N' \Subset N$, and set $D' := M' \times N'$ and $D'' := M'' \times N''$. We also denote by $M^\star \subset \mathbb{C}^p$ and $N^\star \subset \mathbb{C}^{k-p}$ further auxiliary convex open sets satisfying $M'' \Subset M^\star \subseteq M$ and $N'' \Subset N^\star \subseteq N$. For $p = k$, N reduces to a point and we take $N'' = N' = N$.

The definition of a horizontal-like map f on $M \times N$ is given in Definition 3.1, see also Remark 3.2 for the special case of polynomial-like maps (corresponding to $p = k$). For such maps, the dynamical degrees of type I $\lambda_s^\pm$ and of type II $d_s^\pm$ are defined in Definitions 3.6 and 3.9, respectively[1]. For $0 \leq s \leq p$ (resp. $0 \leq s \leq k-p$) we denote by $\mathcal{H}_s(M^\star \times N^\star)$ (resp. $\mathcal{V}_s(M^\star \times N^\star)$) the space of horizontal (resp. vertical) positive closed currents of bi-dimension (s, s) on $M^\star \times N^\star$ and of finite mass. We will denote by $\mathcal{H}_s^{(1)}(M^\star \times N^\star)$ and $\mathcal{V}_s^{(1)}(M^\star \times N^\star)$ the subsets of $\mathcal{H}_s(M^\star \times N^\star)$ and $\mathcal{V}_s(M^\star \times N^\star)$ given by currents of mass 1, respectively. For $0 \leq s \leq p$ the semi-distance $\mathrm{dist}_{M^\star \times N^\star}$ on $\mathcal{H}_s(M^\star \times N^\star)$ is defined in (2.1). A similar semi-distance is defined on $\mathcal{V}_s(M^\star \times N^\star)$ for $0 \leq s \leq k-p$.

The pairing $\langle \cdot, \cdot \rangle$ is used for the integral of a function with respect to a measure or more generally the value of a current at a test form. By (s, s)-forms and (s, s)-currents, we mean forms and currents of bi-degree (s, s), respectively. The mass of a positive (resp. negative) (s, s)-current R on an open subset $U \subset \mathbb{C}^k$ is $\|R\|_U := \int_U R \wedge \omega^{k-s}$ (resp. $\|R\|_U := -\int_U R \wedge \omega^{k-s}$), where $\omega := dd^c \|z\|^2$ for $z \in \mathbb{C}^k$ is the standard Kähler form on $\mathbb{C}^k$. Recall that $d^c = \frac{i}{2\pi}(\bar\partial - \partial)$ and $dd^c = \frac{i}{\pi}\partial\bar\partial$. The definition generalizes to positive and negative currents on compact Kähler manifolds.

The notations $\lesssim$ and $\gtrsim$ stand for inequalities up to a multiplicative constant (usually depending on the domains under consideration). We will use the notation π to denote a projection. In general, given two domains A and B, we will denote by π_A and π_B the natural projections of the product $A \times B$ to the factors A and B, respectively.

1. Introduction

Let $f\colon X \to X$ be dominant rational self-map of a complex projective manifold, or more generally a dominant meromorphic self-map of a compact Kähler manifold of dimension k. Let

[1] The dynamical degrees $d_s^\pm$ were introduced in [9] for invertible horizontal-like maps. In the case of polynomial-like maps, the dynamical degrees were introduced in [10] and [15], and denoted by d_s and d_s^*, respectively. We choose here a notation to avoid confusion between them.

ω be a Kähler form on X. For any $0 \le s \le k$ one can define the sequence $\lambda^+_{s,n} := \|(f^n)_*(\omega^{k-s})\|_X$ of the masses of the positive closed $(k-s,k-s)$-currents $(f^n)_*(\omega^{k-s})$, and the *dynamical degree of order s of f* as

$$\lambda^+_s := \limsup_{n\to\infty} (\lambda^+_{s,n})^{1/n}.$$

For cohomological reasons, ω^{k-s} could be replaced by any smooth form or some positive closed current in the same cohomology class. In particular, the sequence $(\lambda^+_{s,n})_{n\in\mathbb{N}}$ detects the volume growth of s-dimensional subvarieties (whenever they exist) under the action of f^n, for $n \in \mathbb{N}$.

It turns out that the sequences $(\lambda^+_{s,n})_{n\in\mathbb{N}}$ are (almost) sub-multiplicative, hence the lim sup in the definition above is actually a limit [11, 12, 22, 29, 35], see also [5, 34]. For a precise behaviour of these sequences in a number of settings, see also [4, 6, 17, 26, 33] and references therein. It is also a consequence of the fundamental Khovanskii-Teissier inequalities that the sequence of the dynamical degrees λ^+_s is log-concave, i.e., the function $s \mapsto \log \lambda^+_s$ is concave [8, 20, 25, 32]. An immediate consequence of this property is that there exists $1 \le p \le k$ such that

$$\lambda^+_0 \le \lambda^+_1 \le \ldots \le \lambda^+_p \ge \ldots \ge \lambda^+_{k-1} \ge \lambda^+_k. \tag{1.1}$$

When X is projective, this means that the growth rate under the action of f of the volumes of s-dimensional analytic subsets, for $s \le p$, dominates that of $(s-1)$-dimensional analytic subsets, and the reversed property is true for $s > p$. Moreover, the so-called *algebraic entropy* $\log \lambda^+_p$ of f is larger than or equal to the (topological) entropy of the system [11, 12, 21, 36], see also [7, 37].

Let us stress that the proof of the log-concavity of the sequence of the degrees $\{\lambda^+_s\}_{0\le s\le k}$, and as a consequence that of (1.1), deeply relies on the algebraic setting and on cohomological arguments (Hodge-Riemann theorem). In particular, it breaks down when considering non-compact or local situations, even when λ^+_k is the degree of maximal value (i.e., when the system is somehow geometrically expanding). We address this problem in this paper, with new tools coming from pluripotential theory.

We consider in this paper invertible horizontal-like maps in any dimensions [9, 13] and polynomial-like maps in any dimensions [10, 15]. *Horizontal-like maps* are essentially holomorphic maps, defined on some bounded (convex, for simplicity) subset D of $\mathbb{C}^k$, that have an expanding behaviour in p directions and contracting behaviour in the remaining $k-p$ directions. Such expansion and contraction are of global nature, and these maps are in general not uniformly hyperbolic. Assuming $D = M \times N$ (with $M \Subset \mathbb{C}^p$ and $N \Subset \mathbb{C}^{k-p}$ bounded convex domains), the map f sends a vertical open subset of D to a horizontal one and, roughly speaking, the vertical (resp. horizontal) part of the boundary of the first to the vertical (resp. horizontal) part of the boundary of the second. As a particular case, when $p = k$ the set N reduces to a point and one recovers the notion of *polynomial-like maps*, proper holomorphic maps of the form $f: U \to V$, for some open bounded subsets $U \Subset V \Subset \mathbb{C}^k$, with V convex [10, 15].

Horizontal-like and polynomial-like maps can be seen as the building blocks of larger systems and, in particular, give a good setting to study local dynamical problems in larger dynamical systems. Small perturbations of such maps still belong to these classes (up to slightly shrinking the domain of definition), hence we get large classes of examples, and the families are infinite-dimensional. As examples, perturbations of lifts to $\mathbb{C}^{k+1}$ of holomorphic endomorphisms of $\mathbb{P}^k(\mathbb{C})$ give examples of polynomial-like maps. Perturbations of complex Hénon automorphisms of $\mathbb{C}^2$ [2, 19, 30] give horizontal-like maps with $k = 2$ and $p = 1$. Such maps were for instance considered in [16]. More generally, we call any invertible horizontal-like map a *Hénon-like map.*

Given a Hénon-like map or a polynomial-like map, one can introduce dynamical degrees as above. Denoting again by ω the standard Kähler form on $\mathbb{C}^k$, one can roughly define (see Section 3 for the formal definition)

$$\lambda^+_s := \limsup_{n\to\infty} (\lambda^+_{s,n})^{1/n}, \qquad \text{where} \qquad \lambda^+_{s,n} := \|(f^n)_*\omega^{k-s}\|_{M'\times N'}.$$

Here $M' \Subset M$ and $N' \Subset N$ are open convex sets slightly smaller than M and N respectively. In fact, we can show that λ_s^+ is independent of the choice of $D' := M' \times N'$, see Lemma 3.8. Because of the geometry of the problem, these definitions are only given for $0 \le s \le p$. On the other hand, for Hénon-like maps, since $p < k$, one can also define the remaining degrees as $\lambda_s^-(f) := \lambda_s^+(f^{-1})$ for $0 \le s \le k-p$ (since f^{-1} is a "vertical-like" map with $k-p$ expanding directions).

Observe that, a priori, the sequences $(\lambda_{s,n}^+)_{n\in\mathbb{N}}$ above need not be sub-multiplicative this time. More importantly, the lack of a Hodge theory means that, a priori, the resulting degree may change if one replaces ω^s with the integration on a given analytic set of dimension $k-s$, or more generally with a (positive closed) current of bi-degree (s,s). Hence, for $0 \le s \le p$ it is natural to also introduce the degree

$$d_s^+ := \limsup_{n\to\infty} (d_{s,n}^+)^{1/n}, \qquad \text{where} \qquad d_{s,n}^+ := \sup_S \|(f^n)_*(S)\|_{M'\times N}$$

and S runs over the set of all horizontal positive closed currents of bi-dimension (s,s) and of mass 1 on $D' := M' \times N'$. These definitions are also independent of the choice of D'. As for λ_s^-, for Hénon-like maps we can define the remaining dynamical degrees as $d_s^-(f) := d_s^+(f^{-1})$ for every $0 \le s \le k-p$.

The following theorems are our main results, which in particular answer [9, Question 6.3]. More complete and precise versions of Theorems 1.1 and 1.3 are given in Propositions 4.1 and 4.2 and Theorems 5.1 and 5.2.

Theorem 1.1. *Let $1 \le p < k$ be integers and f be a Hénon-like map from a vertical open subset of a bounded convex domain $D = M \times N \subset \mathbb{C}^p \times \mathbb{C}^{k-p}$ to a horizontal open subset of D. Then, the sequences $\{\lambda_s^+\}_{0\le s\le p}$, $\{\lambda_s^-\}_{0\le s\le k-p}$, $\{d_s^+\}_{0\le s\le p}$, and $\{d_s^-\}_{0\le s\le k-p}$ satisfy*

$$\lambda_0^+ \le \lambda_1^+ \le \ldots \le \lambda_p^+ = \lambda_{k-p}^- \ge \ldots \ge \lambda_1^- \ge \lambda_0^-$$

and

$$1 = d_0^+ \le d_1^+ \le \ldots \le d_p^+ = d_{k-p}^- \ge \ldots \ge d_1^- \ge d_0^-.$$

Moreover, we have $\lambda_p^+ = d_p^+ \in \mathbb{N}$.

Since $\log d_p^+$ is equal to the topological entropy $h_t(f)$ of f [9], we deduce the following immediate consequence of Theorem 1.1.

Corollary 1.2. *Let p, k, f be as in Theorem 1.1. Then, for all $0 \le s^+ \le p$ and $0 \le s^- \le k-p$ we have*

$$\log \lambda_{s^\pm}^\pm \le \log d_{s^\pm}^\pm \le \log d_p^+ = h_t(f).$$

As we will see, the proof of Theorem 1.1 can be applied also in the case of polynomial-like maps, even if these maps are in general not invertible.

Theorem 1.3. *Let $k \ge 1$ be an integer and take open sets $U \Subset V \Subset \mathbb{C}^k$ with V convex. Let $f\colon U \to V$ be a polynomial-like map of topological degree d_t. Then the sequences $\{\lambda_s^+\}_{0\le s\le k}$ and $\{d_s^+\}_{0\le s\le k}$ satisfy*

$$\lambda_0^+ \le \lambda_1^+ \le \ldots \le \lambda_k^+ = d_t \quad \text{and} \quad 1 = d_0^+ \le d_1^+ \le \ldots \le d_k^+ = d_t.$$

In particular, all the dynamical degrees of f are smaller than or equal to d_t.

As far as we are aware of, the only (non trivial) cases where the monotonicity of the dynamical degrees could be established until now are of algebraic nature, and such monotonicity is a consequence of the log-concavity property mentioned above. In a nutshell, in order to establish the monotonicity of the sequence $\{d_s^+\}_{0\le s\le p}$, given a (horizontal positive closed) current S of bi-dimension (s,s) with $s < p$, one needs to find another current, of bi-dimension $(s+1,s+1)$ whose mass growth under iteration bounds the mass growth of S under iteration. Although tricky, this is not a big problem when S is smooth (which essentially gives the monotonicity of the sequence $\{\lambda_s^+\}_{0\le s\le p}$). The main problem arises when S is not smooth, and already in the case where S is given by the integration on an analytic set. Even considering an analytic set containing the first one, it is not clear at all why the iterates of the first should behave nicely inside the iterates of the second.

Our solution to the problem can be roughly explained as follows. Given a (horizontal positive closed) current S in D, we first construct a "holomorphic" family of positive closed currents S_θ parametrized by $\theta \in \mathbb{D}$ and with $S_0 \geq S$ [2]. We then consider all these currents as the slices of a unique current $\mathcal{R}$, of bi-dimension $(s+1, s+1)$, on the space $D \times \mathbb{D}$, using the slices $D \times \{\theta\}$. The candidate to the role of current of bi-dimension $(s+1, s+1)$ in D would then be $R := (\pi_D)_*(\mathcal{R})$, where $\pi_D \colon D \times \mathbb{D} \to D$ is the natural projection. Two difficulties arise here. First, we need to make sure that R is well-defined and horizontal in D. In order to do this, we suitably modify $\mathcal{R}$ in the space $D \times \mathbb{D} = M \times N \times \mathbb{D}$, in order to make it become *horizontal in* $M \times (N \times \mathbb{D})$, i.e., to have support contained in $M \times K$, for some compact subset K of $N \times \mathbb{D}$. This makes both the projection $(\pi_D)_*(\mathcal{R})$ well-defined (since now the projection π_D is proper on the support of $\mathcal{R}$), and horizontal there (since the projection of the support of R on N is relatively compact). In order to do this, we exploit some results in the theory of the $d, \overline{\partial}$, and dd^c equations. Observe that, in order to get all these controls, it is crucial to work with the extra flexibility given by (positive closed) currents, and not only with analytic subsets.

Once R is well-defined, we still need to make sure that the growth of the mass of $(f^n)_*(R)$ dominates the growth of the mass of $(f^n)_*(S)$. In order to get this, we need to pay extra attention, and get further estimates, during the construction of $\mathcal{R}$ and R. More precisely, we make sure that the family of deformations S_θ, and the family of the *slices* $\mathcal{R}_\theta$ of $\mathcal{R}$ with $D \times \{\theta\}$ are sufficiently continuous in a suitable sense. Studying the growth of the mass of $(f^n)_*(R)$ in D amounts to study the mass growth of $(F^n)_*(\mathcal{R})$, where $F := (f, \mathrm{id})$ on $D \times \mathbb{D}$. We prove that the sequence of functions ϕ_n on $\mathbb{D}$, where $\phi_n(\theta)$ is the mass of $(f^n)_*(\mathcal{R}_\theta)$ (suitably normalized), is bounded with respect to a suitable norm (the DSH norm). A now-classical theorem by Skoda then implies that a large growth of this sequence at $\theta = 0$ must imply a large growth of this sequence for θ sufficiently close to 0. Going back to currents, this implies a bound (from below) on the growth of the mass of $(f^n)_*(\mathcal{R}_\theta)$, hence of $(F^n)_*(\mathcal{R})$, and hence of $(f^n)_*(R)$. The assertion then follows.

Organization of the paper. In Section 2 we give the main construction, that we will use in the sequel, to produce a current of bi-dimension $(s+1, s+1)$ in a product space from one of bi-dimension (s, s). The construction gives a good control on the support and norms. In Section 3 we define the dynamical degrees for horizontal-like maps, and we give their first properties. The two chains of inequalities in Theorems 1.1 and 1.3 are proved in Sections 4 and 5, respectively.

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2. Deformations of horizontal positive closed currents

We fix in this section integers $1 \leq p \leq k$ and convex open bounded subsets $M'' \Subset M' \Subset M \Subset \mathbb{C}^p$ and $N'' \Subset N' \Subset N \Subset \mathbb{C}^{k-p}$ and set $D := M \times N$, $D' := M' \times N'$, and $D'' := M'' \times N''$. Observe that, when $p = k$, we have $N = N' = N''$ and these sets reduce to a single point. We denote by π_M and π_N the natural projections of D on M and N, respectively, and use similar

[2] For technical reasons, we often need to reduce slightly the domain D, and the estimates that we obtain at any fixed n may depend on the chosen domain. On the other hand, we will show that the limit objects do not depend on such choice. The possible bad behaviour of the currents near the boundary of their domain of definition is a source of technical difficulty in all the paper. For the sake of simplicity, we do not specify this change of domain in this Introduction.

notations for the projections of D' to M' and N' and of D'' to M'' and N''. We say that a subset $E \subset M \times N$ is *horizontal* in $M \times N$ if $\pi_N(E) \Subset N$ and *vertical* if $\pi_M(E) \Subset M$. A current in $M \times N$ is horizontal (resp. vertical) if its support is horizontal (resp. vertical). These notions naturally generalize to subsets of and currents on other product spaces. Note that when $p = k$ any current in $M \times N$ is horizontal, since N is a single point.

Consider any convex open sets $M^\star$ and $N^\star$ with $M'' \Subset M^\star \subseteq M$ and $N'' \Subset N^\star \subseteq N$. For $0 \le s \le p$, we denote by $\mathcal{H}_s(D^\star)$ the set of all horizontal positive closed currents of bi-dimension (s,s) of finite mass on $D^\star := M^\star \times N^\star$. We consider the semi-distance on $\mathcal{H}_s(D^\star)$ given by

$$\text{dist}_{D^\star}(S, S') := \sup_{\Omega} |\langle S - S', \Omega \rangle|, \tag{2.1}$$

where Ω is a real smooth vertical (s,s)-form on $M'' \times N^\star$ whose $\mathcal{C}^1$-norm is at most 1. A similar semi-distance can be defined on the set $\mathcal{V}_s(D^\star)$ of all vertical positive closed currents of bi-dimension (s,s) of finite mass on $D^\star$, for $0 \le s \le k-p$, by testing against real smooth horizontal (s,s)-forms on $M^\star \times N''$ whose $\mathcal{C}^1$-norm is at most 1.

The main result of this section is the following technical theorem, which will be used in Section 5. It gives the deformation of a horizontal positive closed current S on D as described at the end of the Introduction.

Theorem 2.1. *Let S be a horizontal positive closed current of bi-dimension (s,s) and of mass 1 on $M \times N$ with support contained in $M \times N''$, for some $0 \le s \le p-1$. Then, there exist a positive closed current $\mathcal{R}$ of bi-dimension $(s+1, s+1)$ on $M' \times N \times \mathbb{D}$ and a constant $c > 0$ independent of S such that*

(i) *$\mathcal{R}$ is smooth outside $\pi_{\mathbb{D}}^{-1}(0)$;*
(ii) *the slice $\mathcal{R}_\theta := \langle \mathcal{R}, \pi_{\mathbb{D}}, \theta \rangle$ is well-defined as a current of $M' \times N$ of bi-dimension (s,s) and of mass at most 1 for every $\theta \in \mathbb{D}$, and $\|\mathcal{R}\| \le 1$;*
(iii) *$S \le c\mathcal{R}_0$ on $M' \times N$;*
(iv) *$\text{dist}_{D'}(\mathcal{R}_\theta, \mathcal{R}_{\theta'}) \le |\theta - \theta'|$ for all $\theta, \theta' \in \mathbb{D}$;*
(v) *$\mathcal{R}$ is horizontal in $M' \times (N \times \mathbb{D})$, with horizontal support in $M' \times (N' \times \mathbb{D}_{1/2})$.*

In particular, $(\pi_{M' \times N})_* \mathcal{R}$ is well-defined and $\|(\pi_{M' \times N})_* \mathcal{R}\| \le 1$.

Observe that the quantity $\text{dist}_{D'}(\mathcal{R}_\theta, \mathcal{R}_{\theta'})$ in the fourth item is well-defined by (2.1) since, for all $\theta \in \mathbb{D}$, the current $\mathcal{R}_\theta$ is horizontal on $M' \times N'$ by the fifth item.

We refer to [18] for the general theory of slicing of currents, and to [13, 16] in the particular case of horizontal positive closed currents. When well-defined, the slice $\langle \mathcal{R}, \pi_{\mathbb{D}}, \theta \rangle$ can be seen as the intersection current $\mathcal{R} \wedge [\pi_{\mathbb{D}}^{-1}(\theta)]$. In particular, it is a current of bi-dimension (s,s) on $M' \times N \times \mathbb{D}$, supported by $M' \times N \times \{\theta\}$, that we can identify with a current of bi-dimension (s,s) on $M' \times N$, see also [1]. In our case, since $\mathcal{R}$ is smooth outside of $\pi_{\mathbb{D}}^{-1}(0)$, for $\theta \neq 0$ the slice $\langle \mathcal{R}, \pi_{\mathbb{D}}, \theta \rangle$ is equal to the restriction of $\mathcal{R}$ to $\pi_{\mathbb{D}}^{-1}(\theta)$.

Although, a priori, the first property in the statement will not be needed in the sequel, we will use smooth deformations in order to obtain the other properties. In particular, the proof of Theorem 2.1 uses the following lemma, which relies on the solution of the d and $\bar\partial$ equations by means of integral formula, see for instance [3, 9, 23, 24, 28].

Lemma 2.2. *Let $m \ge 2$ and $0 \le l \le m-1$ be integers. Let U, U', V, V' be open convex domains of $\mathbb{C}^m$ with $V' \Subset V \subseteq U' \Subset U$. Let Ψ be a positive closed current of bi-degree $(m-l, m-l)$ on U, and assume the Ψ is smooth on V. Then there exists a negative L^1 form $\mathcal{U}_\Psi$ of bi-degree $(m-l-1, m-l-1)$ on U', which is smooth on V, and a positive constant c (depending on the domains but independent of Ψ) such that*

$$dd^c \mathcal{U}_\Psi = \Psi \text{ on } U', \qquad \|\mathcal{U}_\Psi\|_{U'} \le c\|\Psi\|_U, \qquad \text{and} \qquad \|\mathcal{U}_\Psi\|_{\mathcal{C}^2(V')} \le c\|\Psi\|_{\mathcal{C}^2(V)}. \tag{2.2}$$

We will also need the following basic lemma, which will be used several times in the paper.

Lemma 2.3. *Let $m \geq 2$, $1 \leq p \leq m-1$, and $0 \leq l \leq p$ be integers. Let $U \subset \mathbb{C}^p$ and $V \subset \mathbb{C}^{m-p}$ be bounded convex open domains. For every compact subset $K \Subset U \times V$ there exists a smooth horizontal positive closed $(m-l, m-l)$-form Ω_K on $U \times V$ such that Ω_K is strictly positive on K.*

Proof. Fix an open convex set $V' \Subset V$ such that $K \subset U \times V'$. Take $z \in K$. Since l satisfies $0 \leq l \leq p$, we can find an l-dimensional complex plane Π_z passing through z and contained in $U \times \{z\}$, so that it does not intersect $\overline{U} \times \partial V'$. By using a convolution, we can average small perturbations of $[\Pi_z]$ to obtain a closed positive horizontal $(m-l, m-l)$-form Ω_z on $U \times V'$, which is strictly positive at z. By continuity, Ω_z is actually strictly positive on a neighbourhood of z. By taking a finite sum of such Ω_z's, we can construct a positive closed horizontal form Ω on $U \times V'$ which is strictly positive in a neighbourhood of K. The assertion follows. □

Proof of Theorem 2.1. We fix $M^\star$ and $N^\star$ convex open sets satisfying $M' \Subset M^\star \Subset M$ and $N'' \Subset N^\star \Subset N'$, and set $D^\star := M^\star \times N^\star$. Fix $r > 0$ sufficiently small so that the $3r$-neighbourhood of M' (resp. $M^\star, N'', N^\star$) is contained in $M^\star$ (resp. $M, N^\star, N'$). Denote by $\mathbb{B}$ the unit ball of $\mathbb{C}^k$ and let $\mathbb{D}_2$ be the disc centred at 0 and of radius 2 in $\mathbb{C}$. For $a \in \mathbb{B}$ and every $\theta \in \mathbb{D}_2$ define the holomorphic automorphism $h_{a,\theta} : \mathbb{C}^k \to \mathbb{C}^k$ as

$$h_{a,\theta}(z) = z + r\theta a$$

and the holomorphic submersion $H_a : \mathbb{C}^k \times \mathbb{D}_2 \to \mathbb{C}^k$ as

$$H_a(z, \theta) := h_{a,\theta}^{-1}(z) = z - r\theta a.$$

Define $\mathcal{S}^a := (H_a^* S)_{|D^\star \times \mathbb{D}_2}$. Then $\mathcal{S}^a$ defines a current on $D^\star \times \mathbb{D}_2$ of bi-dimension $(s+1, s+1)$. Since N is convex and $h_{a,\theta}$ is close to the identity (by the choice of r), for every $\theta \in \mathbb{D}_2$ the set $(\operatorname{supp} \mathcal{S}^a) \cap \pi_{\mathbb{D}_2}^{-1}(\theta)$ is a horizontal set in $M^\star \times (N^\star \times \{\theta\})$, where $\pi_{\mathbb{D}_2} \colon D^\star \times \mathbb{D}_2 \to \mathbb{D}_2$ is the natural projection. Hence, the slice $(\mathcal{S}^a)_\theta := \langle \mathcal{S}^a, \pi_{\mathbb{D}_2}, \theta \rangle$ is a horizontal current of bi-dimension (s, s) in $M^\star \times N$, supported on $M^\star \times N^\star$ (where we identify $M^\star \times N \times \{\theta\}$ with $M^\star \times N$). Observe that, with this identification, we have $(\mathcal{S}^a)_\theta = (h_{a,\theta}^{-1})^* S$ on $D^\star$ and, in particular, $(\mathcal{S}^a)_0 = S$ on $D^\star$. Moreover, the dependence $\theta \mapsto (\mathcal{S}^a)_\theta$ is continuous for the weak topology of currents.

Let $\rho(a)$ be a fixed smooth positive form of maximal degree with compact support in $\mathbb{B}$ and of integral 1. It defines a probability measure on $\mathbb{B}$. Set

$$\mathcal{S} := \int_{\mathbb{B}} \mathcal{S}^a \rho(a).$$

Then $\mathcal{S}$ is a current on $M^\star \times N \times \mathbb{D}_2$, supported on $D^\star \times \mathbb{D}_2$. For $\theta \in \mathbb{D}_2$, we denote by $\mathcal{S}_\theta := \langle \mathcal{S}, \pi_{\mathbb{D}_2}, \theta \rangle$ the slice current of $\mathcal{S}$ by $\pi_{\mathbb{D}_2}^{-1}(\theta)$, and we identify it to a (positive closed) current on $D^\star$ of bi-dimension (s, s).

Claim. The current $\mathcal{S}$ satisfies the following properties:

(i) $\mathcal{S}_0 = S$ on $M^\star \times N$;

(ii) $\mathcal{S}_\theta$ is smooth for all $\theta \in \mathbb{D}_2 \setminus \{0\}$;

(iii) $\|\mathcal{S}\|_{\mathcal{C}^2} \leq \tilde{c}$ outside $M^\star \times N \times \mathbb{D}_{1/10}$ and $\|\mathcal{S}\| \leq \tilde{c}$ on $M^\star \times N \times \mathbb{D}_2$;

(iv) $\operatorname{dist}_{D'}(\mathcal{S}_\theta, \mathcal{S}_{\theta'}) \leq \tilde{c}|\theta - \theta'|$ for all $\theta, \theta' \in \mathbb{D}_2$;

where $\tilde{c}$ is a positive constant independent of S.

Observe that, as $\mathcal{S}_\theta$ is supported on $M^\star \times N^\star$ for all $\theta \in \mathbb{D}_2$, we can see its restriction to $M' \times N^\star$ as a (horizontal) current on $M' \times N'$. Hence, the quantity $\operatorname{dist}_{D'}(\mathcal{S}_\theta, \mathcal{S}_{\theta'})$ in the last item is well-defined by (2.1).

Proof of the Claim. It follows from the definition that $\mathcal{S}_0 = S$ on $M^\star \times N$. To prove the second item, we show that the coefficients of $\mathcal{S}_\theta$ are smooth. First, observe that

$$\mathcal{S}_\theta = \int_{\mathbb{B}} (S^a)_\theta \rho(a) \quad \text{for all } \theta \in \mathbb{D}_2.$$

For any two given multi-indices $I := (i_1, \dots, i_{k-s})$ and $J := (j_1, \dots, j_{k-s})$ with $i_l, j_l \in \{1, \dots, k\}$ for all $1 \le l \le k-s$, let $\gamma(z)$, $\gamma^a_\theta(z)$, and $\gamma_\theta(z)$ be the coefficients of $dz_I \wedge d\bar z_J := dz_{i_1} \wedge \cdots \wedge dz_{i_{k-s}} \wedge d\bar z_{j_1} \wedge \cdots \wedge d\bar z_{j_{k-s}}$ in S, $(\mathcal{S}^a)_\theta$, and $\mathcal{S}_\theta$, respectively. Consider the change of coordinates on $\mathbb{C}^k$ given by the translation

$$z \mapsto w = h^{-1}_{a,\theta}(z, \theta) = z - r\theta a. \tag{2.3}$$

Since $(\mathcal{S}^a)_\theta = (h^{-1}_{a,\theta})^* S$, we have $\gamma^a_\theta(z) = \gamma(w)$. Hence, we have

$$\gamma_\theta(z) = \int_{w\in\mathbb{C}^k} \gamma(w) \rho\left(\frac{z-w}{r\theta}\right).$$

Thus, γ_θ is smooth for $\theta \neq 0$. It follows that $\mathcal{S}_\theta$ is smooth for $\theta \in \mathbb{D}_2 \setminus \{0\}$. The second assertion follows.

We now prove the third item. In the variables $\zeta = (\zeta', \zeta_{k+1}) := (z, \theta)$ with $\zeta' := (\zeta_1, \dots, \zeta_k)$, let $\sigma^a(\zeta)$ and $\sigma(\zeta)$ be the coefficients of $d\zeta_I \wedge d\bar\zeta_J := d\zeta_{i_1} \wedge ... \wedge d\zeta_{i_{k-s}} \wedge d\bar\zeta_{j_1} \wedge ... \wedge d\bar\zeta_{i_{k-s}}$ in $\mathcal{S}^a$ and $\mathcal{S}$ respectively, where $I := (i_1, \dots, i_{k-s})$ and $J := (j_1, \dots, j_{k-s})$ with $i_l, j_l \in \{1, \dots, k+1\}$ for all $1 \le l \le k-s$. Consider this time the change of coordinates on $\mathbb{C}^{k+1}$ given by

$$(\zeta', \zeta_{k+1}) \mapsto (w, \zeta_{k+1}) = (\zeta' - r\zeta_{k+1} a, \zeta_{k+1}).$$

Then, we have

$$\sigma(\zeta) = \int_{w\in\mathbb{C}^k} \sigma^a(w, \zeta_{k+1}) \rho\left(\frac{\zeta' - w}{r\zeta_{k+1}}\right). \tag{2.4}$$

Since $\mathcal{S}^a = (H^*_a S)_{|D^\star\times\mathbb{D}_2}$, we can see that $\sigma^a(w, \zeta_{k+1})$ is independent of the coordinate ζ_{k+1}. So the right-hand side of (2.4) (and hence $\sigma(\zeta)$) is smooth outside $\zeta_{k+1} = 0$. Hence, $\mathcal{S}$ is smooth outside $\pi^{-1}_{\mathbb{D}_2}(0)$. As the estimates are uniform out of a neighbourhood of $\{\zeta_{k+1} = 0\} = \{\theta = 0\}$, this gives the proof of the first part of the item. For the second part, observe that

$$\mathcal{S} = (\pi_{D^\star\times\mathbb{D}_2})_* \left(\rho(a) \wedge H^*(S)|_{D^\star\times\mathbb{D}_2\times\mathbb{B}}\right)$$

where $H : \mathbb{C}^k \times \mathbb{D}_2 \times \mathbb{B} \to \mathbb{C}^k$ is defined as $H(z, \theta, a) = H_a(z, \theta)$, and $\pi_{D^\star\times\mathbb{D}_2} : D^\star \times \mathbb{D}_2 \times \mathbb{C}^k \to D^\star \times \mathbb{D}_2$ is the natural projection. Since H is a submersion, the masses of $\rho(a) \wedge H^*(S)$ on $D^\star \times \mathbb{D}_2 \times \mathbb{B}$ and of its push-forward to $D^\star \times \mathbb{D}_2$ are bounded above by a constant which only depends on ρ and the considered domains. This completes the proof of the third assertion in the claim.

Let us prove the last item. Let Ω be a smooth real (s, s)-form with vertical support in $M'' \times N'$ and $\mathcal{C}^1$-norm less than or equal to 1. Then, for all $\theta \in \mathbb{D}_2$, we have

$$\langle \mathcal{S}_\theta, \Omega\rangle = \int_{a\in\mathbb{B}} \langle (\mathcal{S}^a)_\theta, \Omega\rangle \rho(a) = \int_{a\in\mathbb{B}} \langle S, (h_{a,\theta})^*\Omega\rangle \rho(a) = \langle S, \Omega_\theta\rangle, \tag{2.5}$$

where we set

$$\Omega_\theta := \int_{a\in\mathbb{B}} (h_{a,\theta})^*(\Omega) \rho(a).$$

Since Ω is a vertical form with vertical support in $M'' \times N'$, Ω_θ is well-defined as a vertical form on $M^\star \times N^\star$. As S is supported on $M \times N''$, the pairing in last term in (2.5) is well-defined.

Since Ω is smooth and real, and $\|\Omega\|_{\mathcal{C}^0(M''\times N')} \le 1$, there exists a constant c_1 independent of Ω such that

$$-c_1\omega^s \|\Omega_{\theta_1} - \Omega_{\theta_2}\|_{\mathcal{C}^0(D^\star)} \le \Omega_{\theta_1} - \Omega_{\theta_2} \le c_1\omega^s \|\Omega_{\theta_1} - \Omega_{\theta_2}\|_{\mathcal{C}^0(D^\star)} \text{ for all } \theta_1, \theta_2 \in \mathbb{D}_2.$$

Moreover, since $\|\Omega\|_{\mathcal{C}^1(M''\times N')} \le 1$ there exists also a constant c_2 (depending on ρ but independent of Ω) such that

$$\|\Omega_{\theta_1} - \Omega_{\theta_2}\|_{\mathcal{C}^0(D^\star)} \le c_2|\theta_1 - \theta_2| \text{ for all } \theta_1, \theta_2 \in \mathbb{D}_2.$$

Since S has mass 1, we have

$$|\langle S_{\theta_1} - S_{\theta_2}, \Omega\rangle| = |\langle S, \Omega_{\theta_1} - \Omega_{\theta_2}\rangle| \le c_1 \|\Omega_{\theta_1} - \Omega_{\theta_2}\|_{\mathcal{C}^0(D^\star)} \le \tilde c|\theta_1 - \theta_2|,$$

where $\tilde{c} = c_1 c_2$ is independent of Ω. So, we have $\mathrm{dist}_{D'}(S_{\theta_1}, S_{\theta_2}) \leq \tilde{c}|\theta_1 - \theta_2|$. The proof of the claim is complete. □

The Claim implies that the current $\delta\mathcal{S}$ satisfies properties (i)-(iv) in the statement of Theorem 2.1 for any constant $\delta > 0$ small enough. We now need to modify this current in order to satisfy also the last property. Observe that $\pi_N(\mathrm{supp}\,\mathcal{S} \cap (M^\star \times N \times \overline{\mathbb{D}}_{1/4})) \Subset N^\star$ by the choice of r.

Since $\mathcal{S}$ is positive and closed, by using Lemma 2.2 for $\Psi = \mathcal{S}$ on $U' = D' \times \mathbb{D}$ and for $U := M^\star \times N \times \mathbb{D}_2$, $m = k+1$ and $l = s+1 \leq k$, we see that there exists a current $\mathcal{U}_\mathcal{S}$ on $D' \times \mathbb{D}$ such that $dd^c\mathcal{U}_\mathcal{S} = \mathcal{S}$ on $D' \times \mathbb{D}$. Moreover, $\mathcal{U}_\mathcal{S}$ is smooth on $D' \times (\mathbb{D} \setminus \{0\})$, with good $\mathcal{C}^2$ estimates on compact subsets of $D' \times (\mathbb{D} \setminus \{0\})$.

Denote $W := N' \times \mathbb{D}$. Let $0 \leq \chi \leq 1$ be a non-negative smooth cut-off function, horizontal in $M' \times W$, which is equal to 1 on a neighbourhood of $M' \times N^\star \times \overline{\mathbb{D}}_{1/4}$ and vanishes on $M' \times (N_1 \times \mathbb{D}_{1/3})^c$ for some open convex set N_1 such that $N^\star \Subset N_1 \Subset N'$. Define, on $M' \times W$,

$$\begin{aligned} \tilde{\mathcal{S}} := dd^c(\chi\mathcal{U}_\mathcal{S}) &= dd^c\chi \wedge \mathcal{U}_\mathcal{S} + d\mathcal{U}_\mathcal{S} \wedge d^c\chi + d\chi \wedge d^c\mathcal{U}_\mathcal{S} + \chi dd^c\mathcal{U}_\mathcal{S} \\ &= dd^c\chi \wedge \mathcal{U}_\mathcal{S} + d\mathcal{U}_\mathcal{S} \wedge d^c\chi + d\chi \wedge d^c\mathcal{U}_\mathcal{S} + \chi\mathcal{S}. \end{aligned}$$

Then $\tilde{\mathcal{S}}$ is horizontal and closed in $M' \times W$. It is smooth outside $\pi_\mathbb{D}^{-1}(0)$. Moreover, we have $\tilde{\mathcal{S}} = \mathcal{S}$ on $M' \times N^\star \times \mathbb{D}_{1/4}$ (since $dd^c\chi = d\chi = d^c\chi = 0$ and $\chi = 1$ there) and $\mathrm{supp}\,\tilde{\mathcal{S}} \subset M' \times \overline{N_1} \times \overline{\mathbb{D}}_{1/3}$ since χ vanishes on $M' \times (N_1 \times \mathbb{D}_{1/3})^c$. In particular, $\tilde{\mathcal{S}}$ satisfies (v). On the other hand, $\tilde{\mathcal{S}}$ is not necessarily positive on $M' \times N_1 \times (\overline{\mathbb{D}}_{1/3} \setminus \mathbb{D}_{1/4})$. However, it is bounded from below by some smooth negative form independent of S because the $\mathcal{C}^2$-norm of $\mathcal{U}_\mathcal{S}$ is bounded there by a constant independent of S. We now construct a smooth horizontal positive closed $(k-s, k-s)$-form Ω_+ on $M' \times N' \times \mathbb{D}_{1/2}$ such that $\tilde{\mathcal{S}} + \Omega_+$ is (horizontal, closed and) positive, and has good support and norm estimates.

Set $F := \overline{N_1} \times \overline{\mathbb{D}}_{1/3}$. By Lemma 2.3 applied with $m = k+1$, $l = s+1$ (which is possible since $s \leq p-1$ by assumption), $U = M$, $V = N' \times \mathbb{D}_{1/2}$, and $K = \overline{M'} \times F$, there exists a smooth horizontal positive closed $(k-s, k-s)$-form Ω_+ on $M \times N' \times \mathbb{D}_{1/2}$ which is strictly positive on $\overline{M'} \times F$. Since $\tilde{\mathcal{S}} = \mathcal{S}$ on $M' \times N^\star \times \mathbb{D}_{1/4}$, and outside this set the $\mathcal{C}^0$-norm of $\tilde{\mathcal{S}}$ is bounded by a constant independent of S, by taking large enough constants $b_1, b_2 > 0$ (independent of S) we can see that the current $\mathcal{R} := b_2^{-1}(\tilde{\mathcal{S}} + b_1\Omega_+)$ satisfies the requirements on the statement. This concludes the proof.

□

3. Dynamical degrees of horizontal-like maps

We again fix in this section integers $1 \leq p \leq k$ and a bounded convex domain $D = M \times N \subset \mathbb{C}^p \times \mathbb{C}^{k-p}$. We also set $\partial_v D := \partial M \times \overline{N}$ (resp. $\partial_h D := \overline{M} \times \partial N$) and we call it the *vertical* (resp. *horizontal*) boundary of D. We denote by π_1 and π_2 the first and the second projections of $D \times D$ on its factors, respectively.

Definition 3.1. A *horizontal-like map* f on D is a holomorphic map whose graph $\Gamma \subset D \times D$ satisfies the following properties:

- (i) Γ is a submanifold of $D \times D$ of pure dimension k (not necessarily connected);
- (ii) $(\pi_1)_{|\Gamma}$ is injective;
- (iii) $(\pi_2)_{|\Gamma}$ has finite fibers;
- (iv) $\overline{\Gamma}$ does not intersect $\partial_v D \times \overline{D}$ and $\overline{D} \times \partial_h D$.

Observe in particular that f does not need to be defined on the whole D, but only on the vertical open subset $\pi_1(\Gamma)$ of D. Likewise, the image of f is the horizontal subset $\pi_2(\Gamma)$ of D. We will write $D_{v,1} = \pi_1(\Gamma)$ and $D_{h,1} = \pi_2(\Gamma)$ in the following. More generally, for all $n \geq 1$ we consider the iterate $f^n = f \circ \cdots \circ f$ (n times). These are also horizontal-like maps. We denote by $D_{v,n}$ and $D_{h,n}$ the domain and the image of f^n. Observe that the sequences $(D_{v,n})_{n\geq 1}$ and $(D_{h,n})_{n\geq 1}$ are decreasing.

Remark 3.2. In the case where $p = k$, as N is a point, Definition 3.1 amounts to simply considering proper holomorphic maps $f\colon M_0 \to M$, where $M_0 := f^{-1}(M)$ is open, $M_0 \Subset M$, and $M \Subset \mathbb{C}^k$ is a convex open set (we set in this case $\partial N = \varnothing$ by convention, to keep the condition (iv) consistent). These are the so-called *polynomial-like maps*, see for instance [10]. A polynomial-like map $f\colon M_0 \to M$ defines a ramified covering over M. We usually call the degree d_t of the covering the *topological degree* of f.

The *filled (forward) Julia set* $\mathcal{K}_+$ of f is defined as

$$\mathcal{K}_+ := \{z \in D\colon f^n(z) \text{ is defined for all } n \geq 0\}.$$

Note that $\mathcal{K}_+ = \cap_{n\geq 1} D_{v,n}$. Define also $\mathcal{K}_- := \cap_{n\geq 1} D_{h,n}$. Both $\mathcal{K}_+$ and $\mathcal{K}_-$ are non-empty, and we have $f^{-1}(\mathcal{K}_+) = \mathcal{K}_+$ and $f(\mathcal{K}_-) = \mathcal{K}_-$. Observe that $\mathcal{K}_+$ (resp. $\mathcal{K}_-$) is a vertical (resp. horizontal) closed subsets of D. In particular, $\mathcal{K}_+ \cap \mathcal{K}_-$ is a compact subset of D.

We call any invertible horizontal-like map (i.e., any horizontal-like map such that $(\pi_2)_{|\Gamma}$ is also injective) a *Hénon-like map*. In the rest of this section, f will be either a Hénon-like map or a polynomial-like map in the sense of Remark 3.2. The sets M', M'', N', N'' will be as in the Notation at the beginning of the paper. We will assume in what follows that M' and M'' (resp. N' and N'') are chosen sufficiently close to M (resp. N) so that

$$D_{v,1} \subset M'' \times N \quad \text{and} \quad D_{h,1} \subset M \times N''. \tag{3.1}$$

In particular, we can replace D with D' or D'' and still get horizontal-like maps. We denote by $D'_{v,n}$, $D'_{h,n}$, $D''_{v,n}$, and $D''_{h,n}$ the domains and images of the restrictions of f^n to D' and D'', respectively. Note that, when f is a polynomial-like map, N, N', and N'' consist of a single point and we have $D_{v,n} = f^{-n}(M)$ and $D_{h,n} = M$ (and, similarly, $D'_{v,n} = f^{-n}(M')$, $D'_{h,n} = M'$, $D''_{v,n} = f^{-n}(M'')$, and $D''_{h,n} = M''$). In this case, condition (3.1) means that $f^{-1}(M) \subset M''$.

The following lemma gives some basic compactness estimates that follow from the inclusions (3.1) and that we will need in the sequel. Recall that the operators f_* and f^* are defined as

$$f_* := ((\pi_2)_{|\Gamma})_* \circ ((\pi_1)_{|\Gamma})^* \quad \text{and} \quad f^* = ((\pi_1)_{|\Gamma})_* \circ ((\pi_2)_{|\Gamma})^*.$$

The operator f_* is continuous on horizontal currents, and the operator f^* is continuous on vertical currents. Recall that we only consider Hénon-like maps and polynomial-like maps.

Lemma 3.3. *Let M, M', M'', N, N', N'', and f be as above. Then, there exists a constant A (depending on the domains and f) such that*

(i) $(f^n)^*(\omega^s) \leq A(f^{n-1})^*(\omega^s)$ *on $D'_{v,n}$ for all $0 \leq s \leq k$ and $n \geq 1$;*
(ii) $(f_*(S))_{|D'} \in \mathcal{H}_s(D')$ *and $\|f_*(S)\|_{M'\times N} \leq A\|S\|_{M''\times N}$ for all $0 \leq s \leq p$ and $S \in \mathcal{H}_s(D')$.*

Proof. Set $A := \max_{0\leq s\leq k} \|f^*(\omega^s)\|_{\mathcal{C}^1(D'_{v,1})}$. Then we have $f^*(\omega^s) \leq A\omega^s$ on $D'_{v,1}$ for all $0 \leq s \leq k$. The first assertion follows from the inclusion $D'_{v,n} \subset D'_{v,1}$.

Take now $S \in \mathcal{H}_s(D')$, for some $0 \leq s \leq p$. Using (3.1), we have

$$\begin{aligned}
\|f_*(S)\|_{M'\times N} &= \int_{M'\times N} f_*(S) \wedge \omega^s \leq \int_{M'\times N} f_*\left(S \wedge f^*(\omega^s)\right) \\
&= \int_{f^{-1}(M'\times N)} S \wedge f^*(\omega^s) \leq \int_{M''\times N} S \wedge f^*(\omega^s) \\
&= \int_{M''\times N'} S \wedge f^*(\omega^s) \leq A\|S\|_{M''\times N}.
\end{aligned}$$

In the last inequality, we used the fact that $S \in \mathcal{H}_s(D')$ and the first assertion with $n = 1$. Since f is horizontal-like, $(f_*(S))_{|D'}$ is a horizontal current on D'. By the above inequality its mass is finite and hence $(f_*(S))_{|D'} \in \mathcal{H}_s(D')$. This completes the proof. □

We now define quantities to measure the growth of the mass of the currents $(f^n)_*(S)$ (for S horizontal and of dimension up to p) and $(f^n)^*(R)$ (for R vertical and of dimension up to $k-p$). The case of dimension p and $k-p$, respectively, is given by Lemma 3.5 below. Recall that, by [13, Theorem 2.1] for any $S \in \mathcal{H}_p(D)$, the *slice measure* $\langle S, \pi_{M'}, z\rangle$ is well-defined for any $z \in M'$ and its mass, denoted by $\|S\|_h$, is independent of z and is equal to $\langle S, \pi_M^*(\Omega_M)\rangle$ for every smooth probability measure Ω_M with compact support in M'. Similarly, if $R \in \mathcal{V}_{k-p}(D)$ the slice measure $\langle R, \pi_{N'}, w\rangle$ is well-defined for any $w \in N'$ and its mass, denoted by $\|R\|_v$ is independent of w and equal to $\langle R, \pi_N^*(\Omega'_N)\rangle$ for every smooth probability measure Ω'_N with compact support in N'.

Lemma 3.4. *If $S \in \mathcal{H}_p(D)$ is supported in $M \times N'$, then $\|S\|_{D'} \lesssim \|S\|_h \lesssim \|S\|_{D'}$. Similarly, if $R \in \mathcal{V}_{k-p}(D)$ is supported in $M' \times N$ then $\|R\|_{D'} \lesssim \|R\|_v \lesssim \|R\|_{D'}$.*

Proof. It is enough to prove the assertion for $S \in \mathcal{H}_p(D)$. The inequality $\|S\|_h \lesssim \|S\|_{D'}$ is a consequence of the fact that $(\pi_{M'})_*(S_{|D'}) = \|S\|_h \cdot [M']$, which gives $\|S\|_{D'} \geq \|(\pi_{M'})_*(S_{|D'})\| \gtrsim \|S\|_h \cdot |M'|$, where $|M'|$ denotes the Lebesgue measure of M'. The proof of the other inequality is essentially given in [10, Lemma 3.3.3], see also [15, Lemma 2.5]. We recall here the idea of the proof.

Let ω_M and ω_N denote the standard Kähler forms on M and N, respectively. Then, $\omega = \omega_M + \omega_N$ is the standard Kähler form on D. For any $[p \times (k-p)]$-matrix A, consider the projection $\pi_{M,A}\colon (\mathbb{C}^p \times \mathbb{C}^{k-p}) \to \mathbb{C}^p$ defined as $\pi_{M,A}(z,w) = z + Aw$. If the entries of A are sufficiently small (depending on M, M', and N), $\pi_{M,A}$ is well-defined on D', with image in M. By the first part of the proof, the slice $\langle S, \pi_{M,A}, z\rangle$ is then well-defined for every $z \in M'$. Its mass is independent of A, and in particular equal to $\|S\|_h$. Hence, the integral $\int_{D'} S \wedge \pi_{M,A}^*(\omega_M^p)$ is well-defined and bounded by a constant times $\|S\|_h \cdot |M|$ for every A. To conclude, it is enough to observe that ω^p can be bounded on D' by a finite sum of forms $c_i \cdot \pi_{M,A_i}^*(\omega_M^p)$, where the entries of A_i can be taken small as above, and the c_i's are positive. As a consequence, we have

$$\|S\|_{D'} = \int_{D'} S \wedge \omega^p \leq \sum_i c_i \int_{D'} S \wedge \pi_{M,A_i}^*(\omega_M^p) \lesssim \|S\|_h.$$

The assertion follows. □

Lemma 3.5 ([13, Proposition 4.2])**.** *The operators $f_*\colon \mathcal{H}_p(D') \to \mathcal{H}_p(D')$ and $f^*\colon \mathcal{V}_{k-p}(D') \to \mathcal{V}_{k-p}(D')$ are well-defined and continuous. Moreover, there exists an integer $d \geq 1$ such that $\|f_*(S)\|_h = d\|S\|_h$ for all $S \in \mathcal{H}_p$ and $\|f^*(R)\|_v = d\|R\|_v$ for all $R \in \mathcal{V}_{k-p}(D')$.*

We call the integer d as in the last statement the *main dynamical degree* of f. Note that when f is a polynomial-like map (see Remark 3.2), since $\mathcal{H}_k(D')$ is the set of constant functions, the main dynamical degree is the topological degree d_t of f.

We now introduce some invariants in order to measure the growth of currents of arbitrary dimension and degree, as above. The first is an adaptation of the (smooth) dynamical degrees for polynomial-like maps introduced in [10].

Definition 3.6 (Dynamical degrees of type I)**.** Let M, M', N, N', and f be as above. For $0 \leq s \leq p$, we define the dynamical degree λ_s^+ as

$$\lambda_s^+ := \limsup_{n\to\infty} (\lambda_{s,n}^+)^{1/n}, \quad \text{where} \quad \lambda_{s,n}^+ := \|(f^n)_*(\omega^{k-s}_{|D'_{v,n}})\|.$$

For $0 \leq s \leq k-p$, we define the dynamical degree λ_s^- as

$$\lambda_s^- := \limsup_{n\to\infty} (\lambda_{s,n}^-)^{1/n}, \quad \text{where} \quad \lambda_{s,n}^- := \|(f^n)^*(\omega^{k-s}_{|D'_{h,n}})\|.$$

The mass in the definition of $\lambda_{s,n}^+$ (resp. $\lambda_{s,n}^-$) is taken on $M' \times N$ (resp. $M \times N'$), or equivalently on the image $D'_{h,n}$ (resp. the domain $D'_{v,n}$) of f^n.

Remark 3.7. Note that for polynomial-like maps we only consider λ_s^+ with $0 \leq s \leq k$. For Hénon-like maps, we can see that $\lambda_s^+(f) = \lambda_s^-(f^{-1})$ for all $0 \leq s \leq p$ and $\lambda_s^-(f) = \lambda_s^+(f^{-1})$ for all $0 \leq s \leq k-p$, where the degrees of f^{-1} can be defined reversing the role of f_* and f^* and using the fact that f^{-1} is a vertical-like map with $k-p$ expanding directions, see also [9, Section 3].

Lemma 3.8. *The definitions of λ_s^+ and λ_s^- are independent of the choice of M' and N' as above. Moreover, we have $\lambda_p^+ = \lambda_{k-p}^- = d$, the main dynamical degree.*

Proof. For the proof in the case of polynomial-like maps we refer to [15, Lemma 2.3]. So, we assume that f is a Hénon-like map. By Remark 3.7, it is enough to prove the statement for λ_s^+, for $0 \leq s \leq p$. We fix convex open sets $M'' \Subset M'$ and $N'' \Subset N'$ as above and $0 \leq s \leq p$, and we denote by $\tilde{\lambda}_{s,n}^+, \tilde{\lambda}_s^+$ the dynamical degrees of type I defined using $D''_{v,n}$ instead of $D'_{v,n}$ in Definition 3.6. Since

$$\|(f^n)_*(\omega^{k-s}_{|D''_{v,n}})\| \leq \|(f^n)_*(\omega^{k-s}_{|D'_{v,n}})\|$$

it is clear that, for all $n \geq 0$, we have $\tilde{\lambda}_{s,n}^+ \leq \lambda_{s,n}^+$ for $0 \leq s \leq p$, hence $\tilde{\lambda}_s^+ \leq \lambda_s^+$ for every s as before. So, we only need to prove the reverse inequality.

Recall that we are assuming that $D_{h,1} \subset M \times N''$ and $D_{v,1} \subset M'' \times N$, see (3.1). In particular, Lemma 3.3 holds and since $f^n : D'_{v,n} \to D'_{h,n}$ is bijective we have

$$f^{-1}(D'_{h,n}) = f^{n-1}(D'_{v,n}) \subset D''_{h,n-1} \cap D'_{v,1} \tag{3.2}$$

for all $n \geq 2$. Indeed, for all $n \geq 2$, since $D'_{h,n} \subset D'_{h,1}$ we have that $f^{-1}(D'_{h,n}) \subset D'_{v,1}$. Thanks to (3.1) we also have $D'_{v,n} = f^{-n}(D') = f^{-n}(M' \times N'') \subset f^{-n+1}(M'' \times N'') = D''_{v,n-1}$. So (3.2) follows.

Using (3.2) and the first assertion in Lemma 3.3 it follows that, for all $n \geq 2$ and $0 \leq s \leq k-p$, we have

$$\begin{aligned}
\|(f^n)_*(\omega^{k-s}_{|D'_{v,n}})\| &= \int_{D'_{h,n}} (f^n)_*(\omega^{k-s}_{|D'_{v,n}}) \wedge \omega^s \\
&= \int_{f^{-1}(D'_{h,n})} (f^{n-1})_*(\omega^{k-s}_{|D'_{v,n}}) \wedge f^*(\omega^s) \\
&\lesssim \int_{f^{-1}(D'_{h,n})} (f^{n-1})_*(\omega^{k-s}_{|D'_{v,n}}) \wedge \omega^s \\
&\leq \int_{D''_{h,n-1}} (f^{n-1})_*(\omega^{k-s}_{|D'_{v,n}}) \wedge \omega^s \\
&\leq \int_{D''_{h,n-1}} (f^{n-1})_*(\omega^{k-s}_{|D''_{v,n-1}}) \wedge \omega^s = \|(f^{n-1})_*(\omega^{k-s}_{|D''_{v,n-1}})\|,
\end{aligned}$$

where in the last inequality we used the inclusion $D'_{v,n} \subset D''_{v,n-1}$. Hence we have $\lambda_{s,n}^+ \lesssim \tilde{\lambda}_{s,n-1}^+$, which implies that $\lambda_s^+ \leq \tilde{\lambda}_s^+$ and the independence of λ_s^+ from the domains.

Finally, let us show that $\lambda_p^+ = d$. A similar proof also shows that $\lambda_{k-p}^- = d$, see also Remark 3.7. There exists $\alpha_-, \alpha_+ \in \mathcal{H}_p(D')$ such that $\|\alpha_\pm\|_h = 1$ and

$$\alpha_- \lesssim \omega^{k-p} \text{ on } D' \quad \text{ and } \quad \omega^{k-p} \lesssim \alpha_+ \text{ on } D''$$

(where the existence of α_+ follows from Lemma 2.3). We apply Lemma 3.5 with $S = \alpha_-$ and obtain that $\lambda_p^+ \geq d$. By the first part of the proof, the dynamical degree of type I is independent of the choice of D'. So we can replace D' with D'' in the definition of λ_p^+. Applying Lemma 3.5 with α_+ gives $\lambda_p^+ \leq d$. Hence, we have $\lambda_p^+ = d$. The proof is complete. □

We will also consider the following notion of dynamical degrees, which was introduced in [9], see [15] for the earlier definition in the setting of polynomial-like maps.

Definition 3.9 (Dynamical degrees of type II)**.** For $0 \leq s \leq p$, the dynamical degree d_s^+ is defined by

$$d_s^+ := \limsup_{n\to\infty} (d_{s,n}^+)^{1/n} \quad \text{where} \quad d_{s,n}^+ := \sup_S \|(f^n)_*(S)\|_{M'\times N},$$

and S runs over the set $\mathcal{H}_s^{(1)}(D')$ of all horizontal positive closed currents of bi-dimension (s,s) of mass 1 on $D' := M' \times N'$.

For $0 \leq s \leq k-p$, the dynamical degree d_s^- is defined by

$$d_s^- := \limsup_{n\to\infty} (d_{s,n}^-)^{1/n} \quad \text{where} \quad d_{s,n}^- := \sup_R \|(f^n)^*(R)\|_{M\times N'},$$

and R runs over the set $\mathcal{V}_s^{(1)}(D')$ of all vertical positive closed currents of bi-dimension (s,s) of mass 1 on $D' := M' \times N'$.

Remark 3.10. Similarly as for dynamical degrees of type I, for polynomial-like maps we have $p = k$, hence we only need to consider d_s^+ with $0 \leq s \leq k$. When f is a Hénon-like map, we can define the degrees of type II for the vertical-like map f^{-1} and we have $d_s^+(f) = d_s^-(f^{-1})$ for $0 \leq s \leq p$ and $d_s^-(f) = d_s^+(f^{-1})$ for $0 \leq s \leq k-p$.

The following lemma is proved in [15, Lemma 2.6] in the case of polynomial-like maps and in [9, Lemma 3.5] in the case of Hénon-like maps.

Lemma 3.11. *The dynamical degrees d_s^+ and d_s^- do not depend on the choice of M' and N'. Moreover, we have $d_0^+ = d_0^- = 1$ and $d_p^+ = d_{k-p}^- = d$.*

In particular, it follows from Lemma 3.5 that

$$d^n \lesssim d_{p,n}^+ \lesssim d^n \quad \text{and} \quad d^n \lesssim d_{k-p,n}^- \lesssim d^n \quad \text{as } n \to \infty.$$

Remark 3.12. By considering a current S given by the integration on a horizontal analytic subset of dimension $0 \leq s \leq p$ which intersects $\mathcal{K}_+$ we can see that $d_s^+ \geq 1$ because $(f^n)_*(S)$ satisfies the same property for all n, and hence these currents have mass bounded from below. Recall that for polynomial-like maps we only have to consider d_s^+ with $0 \leq s \leq k$. When f is a Hénon-like map, we can apply the above argument for f^{-1}. By Remark 3.10, we get that $d_s^- \geq 1$ for $0 \leq s \leq k-p$.

Remark 3.13. Take $0 \leq s \leq p$. By Lemma 2.3, one can bound $\omega^{k-s}_{|D''}$ from above with a smooth horizontal positive closed $(k-s, k-s)$-form Ω on D' (we used here that $s \leq p$). Therefore, it is easy to deduce that $\lambda_s^+ \leq d_s^+$ if we use D'' to compute λ_s^+ and D' to compute d_s^+, see Lemmas 3.8 and 3.11. Similarly, one can see that $\lambda_s^- \leq d_s^-$ for all $0 \leq s \leq k-p$.

Lemma 3.14. *The sequences $(d_{s,n}^+)_{n\in\mathbb{N}}$ (for $0 \leq s \leq p$) and $(d_{s,n}^-)_{n\in\mathbb{N}}$ (for $0 \leq s \leq k-p$) are sub-multiplicative, i.e., we have $d_{s,n+m}^\pm \leq d_{s,m}^\pm d_{s,n}^\pm$ for all $n, m \geq 0$. In particular, we have*

$$d_s^\pm = \lim_{n\to\infty} (d_{s,n}^\pm)^{1/n} = \inf_{n\geq 1} (d_{s,n}^\pm)^{1/n}.$$

Proof. We prove the assertion for d_s^+ for a given $0 \leq s \leq p$, the argument is the same for d_s^- for $0 \leq s \leq k-p$, see also Remark 3.10. As $d_{s,0}^+ = 1$ for all $0 \leq s \leq p$, we can assume that $n \geq 1$.

Take $S \in \mathcal{H}_s^{(1)}(D')$. Then, using that f is horizontal-like, for all $m \geq 0$ we have

$$\begin{aligned}
\|(f^{n+m})_*(S)\|_{M'\times N} &= \|(f^m)_*((f^n)_*(S))\|_{M'\times N} \\
&\leq \|(f^n)_*(S)\|_{M'\times N} \cdot \|(f^m)_*(T)\|_{M'\times N} \\
&\leq d_{s,n}^+ \cdot \|(f^m)_*(T)\|_{M'\times N},
\end{aligned}$$

where $T := (\|(f^n)_*(S)\|_{M'\times N})^{-1} \cdot ((f^n)_*(S))_{|M'\times N}$ is a current of mass 1 on $M' \times N$. More precisely, T is a horizontal positive closed current whose support is contained in the horizontal subset $D'_{h,n}$ of $M' \times N$. By (3.1), it is then also a horizontal positive closed current on $M' \times N'$.

In particular, T belongs to $\mathcal{H}_s^{(1)}(D')$. Therefore, by Definition 3.9, for all $m \geq 0$ we have $\|(f^m)_*(T)\|_{M'\times N} \leq d^+_{s,m}$. It follows that

$$\|(f^{n+m})_*(S)\|_{M'\times N} \leq d^+_{s,m} d^+_{s,n} \quad \text{for all } S \in \mathcal{H}_s^{(1)}(D'),$$

which implies that $d^+_{s,n+m} \leq d^+_{s,m} d^+_{s,n}$ for all $n \geq 1$ and $m \geq 0$. By the classical Fekete lemma, the second assertion is a consequence of the first one. □

4. Monotonicity of dynamical degrees of type I

In this section we prove the assertions in Theorems 1.1 and 1.3 involving the dynamical degrees of type I. We first deal with Hénon-like maps. Recall that, in this case, we fix integers $1 \leq p < k$, a bounded and convex domain $D = M \times N \subset \mathbb{C}^p \times \mathbb{C}^{k-p}$ and the convex open sets $M'' \Subset M' \Subset M$ and $N'' \Subset N' \Subset N$ are assumed to be sufficiently close to M and N so that (3.1) and Lemma 3.3 hold. The following result gives the first chain of inequalities in Theorem 1.1. The λ_s^+'s and λ_s^-'s in the statement are given by computing the masses on $M' \times N$ and $M \times N'$ in Definition 3.6, respectively.

Proposition 4.1. *Let $1 \leq p < k$ be integers, f be a Hénon-like map on $D = M \times N \subset \mathbb{C}^p \times \mathbb{C}^{k-p}$, and M', M'', N', N'', λ_s^+, λ_s^- be as above. Then, we have*

$$\lambda_s^+ \leq \lambda_{s+1}^+ \quad \textit{for} \quad 0 \leq s \leq p-1 \qquad \textit{and} \qquad \lambda_s^- \leq \lambda_{s+1}^- \quad \textit{for} \quad 0 \leq s \leq k-p-1.$$

Proof. It is enough to prove the inequalities for the degrees λ_s^+. The assertions on the degrees λ_s^- can be obtained by replacing f with f^{-1}, see Remark 3.10. We fix $0 \leq s < p$ in the following. Since $1 \leq p < k$, by Lemma 2.3 there exists a smooth horizontal closed $(k-p, k-p)$-form Ω on D' satisfying $\omega^{k-p}_{|D''} \lesssim \Omega \lesssim \omega^{k-p}_{D'}$. Let χ_1 be a cut-off function equal to 1 on $M'' \times N$ and which vanishes in a neighbourhood of $(M \setminus M') \times N$. Let χ_2 be another cut-off function equal to 1 on the support of χ_1 and vanishing in a neighbourhood of $(M \setminus M') \times N$. In particular, the supports of both χ_1 and χ_2 are vertical in $M' \times N$. Then, for all $n \geq 1$, using the fact that $s < p$ we have

$$\begin{aligned}
\|(f^{n+1})_*(\omega^{k-s}_{|D''_{v,n+1}})\|_{M''\times N'} &= \int_{D''_{h,n+1}} (f^{n+1})_*(\omega^{k-s}) \wedge \omega^s \\
&\lesssim \int_{D''_{h,n+1}} (f^{n+1})_*(\Omega \wedge \omega^{p-s}) \wedge \chi_1 \omega^s \\
&\leq \int_{D'_{v,n+1}} \Omega \wedge \omega^{p-s} \wedge (f^{n+1})^*(\chi_1 \omega^s) \\
&\lesssim \int_{M'\times N} dd^c \|z\|^2 \wedge \Omega \wedge \omega^{p-s-1} \wedge (f^{n+1})^*(\chi_1 \omega^s),
\end{aligned}$$

where z is the standard coordinate on $\mathbb{C}^k$ such that $\omega = dd^c\|z\|^2$ and in the last step we also used the fact that $D'_{v,n+1} \subset D'_{v,1} \subset M' \times N$, see (3.1). Since Ω is horizontal and closed and χ_1 is vertical in $M' \times N$, we can apply Stokes theorem and deduce that

$$\begin{aligned}
\|(f^{n+1})_*(\omega^{k-s}_{|D''_{v,n+1}})\|_{M''\times N'} &\lesssim \int_{M'\times N} \|z\|^2 \Omega \wedge \omega^{p-s-1} \wedge (f^{n+1})^*(dd^c \chi_1 \wedge \omega^s) \\
&\lesssim \int_{M'\times N} \Omega \wedge \omega^{p-s-1} \wedge (f^{n+1})^*(\chi_2 \omega^{s+1}) \\
&\lesssim \int_{M''\times N} \Omega \wedge \omega^{p-s-1} \wedge (f^{n})^*(\chi_2 \omega^{s+1}),
\end{aligned}$$

where in the last step we used again the fact that $D'_{v,n+1} \subset D'_{v,n} \subset M'' \times N$ (see (3.1)), Lemma 3.3, and the fact that $(f^{n+1})^*(\chi_2) \leq (f^n)^*(\chi_2)$ on $D'_{v,n+1}$. Using the fact that χ_2 is supported in $M' \times N$, and the above upper bound for Ω, we obtain

$$\begin{aligned}\|(f^{n+1})_*(\omega^{k-s}_{|D''_{v,n+1}})\|_{M''\times N'} &\lesssim \int_{D'} \Omega \wedge \omega^{p-s-1} \wedge (f^n)^*(\omega^{s+1}) \\ &= \int_{D'_{h,n}} (f^n)_*(\Omega \wedge \omega^{p-s-1}) \wedge \omega^{s+1} \\ &\lesssim \|(f^n)_*(\omega^{k-s-1}_{|D'_{v,n}})\|.\end{aligned}$$

This proves the inequality $\tilde{\lambda}^+_{s,n+1} \lesssim \lambda^+_{s+1,n}$, where $\tilde{\lambda}^+_{s,n}$ is the quantity defined using $D''_{v,n}$ instead of $D'_{v,n}$ in Definition 3.6, and the implicit constant is independent of n. The inequality $\lambda^+_s \leq \lambda^+_{s+1}$ follows by taking the powers $1/n$ and a lim sup for $n \to \infty$ in the previous one, and applying Lemma 3.8. The proof is complete. □

The following proposition gives the first part of Theorem 1.3.

Proposition 4.2. *Let $f : U \to V$ be a polynomial-like map where $U \Subset V$ is open and $V \subset \mathbb{C}^k$ is a bounded convex open set. Then $\lambda^+_s \leq \lambda^+_{s+1}$ for all $0 \leq s \leq k-1$.*

Proof. The proof is similar that of Proposition 4.1, hence we only sketch it.

Let V'' and V' be convex domains satisfying $U \Subset V'' \Subset V' \Subset V$. Note that $f^{-1}(V) = U \Subset V''$. In the sense of Remark 3.2, we have the identifications

$$M' \times N = V' = D'_{h,n}, \quad M'' \times N' = V'' = D''_{h,n}, \quad D'_{v,n} = f^{-n}(V'), \quad \text{and} \quad D''_{v,n+1} = f^{-n-1}(V'').$$

Recall that in this case $N = N' = N''$ is a single point.

As $p = k$, we have $k - p = 0$. By putting $\Omega \equiv 1$ and taking two cut-off functions χ_1 and χ_2 with compact support in V' and such that $\chi_1 = 1$ on V'' and $\chi_2 = 1$ on the support of χ_1, we can repeat the same computations as in the proof of Proposition 4.1 and obtain

$$\|(f^{n+1})_*(\omega^{k-s})\|_{V''} \lesssim \|(f^{n+1})_*(\omega^{k-s-1})\|_{V'}$$

for every $0 \leq s \leq k-1$, where the implicit constant is independent of n. Since λ^+_s is independent of the choice of V' (see Lemma 3.8) we have

$$\lambda^+_s = \limsup_{n\to\infty} \|(f^{n+1})_*(\omega^{k-s})\|^{1/n}_{V''} \leq \limsup_{n\to\infty} \|(f^{n+1})_*(\omega^{k-s-1})\|^{1/n}_{V'} = \lambda^+_{s+1}.$$

The proof is complete. □

5. Monotonicity of dynamical degrees of type II

The following theorems answer [9, Question 6.3] and complete the proof of Theorems 1.1 and 1.3. We work with the choice of M', M'', N', N'' as in the beginning of Section 4. In particular, we assume that (3.1) and Lemma 3.3 hold. As in the previous sections, the degrees $d^+_{s,n}$ and d^+_s are computed with respect to M' and N', as in Definition 3.9.

Theorem 5.1. *Let $1 \leq p < k$ be integers and f be a Hénon-like map on a bounded convex domain $D = M \times N \subset \mathbb{C}^p \times \mathbb{C}^{k-p}$. Then*

$$d^+_s \leq d^+_{s+1} \quad \text{for } 0 \leq s \leq p-1 \qquad \text{and} \qquad d^-_s \leq d^-_{s+1} \quad \text{for} \quad 0 \leq s \leq k-p-1.$$

In particular, all the dynamical degrees of f are smaller than or equal to $d^+_p = d^-_{k-p} = d$.

The following is the counterpart of the above statement for polynomial-like maps.

Theorem 5.2. *Let $k \geq 1$ be an integer and $f : U \to V$ be a polynomial-like map of topological degree d_t, where $U \Subset V \Subset \mathbb{C}^k$ are open sets and V is convex. Then $d^+_s \leq d^+_{s+1}$ for all $0 \leq s \leq k-1$. In particular, all the dynamical degrees of f are smaller than or equal to $d^+_k = d_t$.*

Since f is assumed to be invertible in Theorem 5.1, it is enough to prove only the assertion on the degrees d_s^+ in that statement. Because of the similarity of the proofs of the monotonicity of $\{d_s^+\}_{0\leq s\leq p}$ in Theorems 5.1 and 5.2, we will give their proofs in parallel. From now on, we work under the assumptions of Theorems 5.1 and 5.2 and we fix an s as in those statements. So, below f is either an invertible horizontal-like map or a polynomial-like map. In the case of a polynomial-like map (i.e., for $p = k$), we identify V with M and N to a single point, see Remark 3.2.

Fix an open convex set $M^\star$ with $M'' \Subset M^\star \Subset M'$. The quantities that we will introduce may depend on the domains $M', M^\star, M'', N', N''$, even if this is not explicitly stated. Recall that $\mathcal{H}_s(M^\star \times N')$ is the set of all horizontal positive closed currents of bi-dimension (s, s) of finite mass on $M^\star \times N'$. We will need the following basic lemma.

Lemma 5.3. *There is a constant $L > 0$ such that the action of $f_* : \mathcal{H}_s(M^\star\times N') \to \mathcal{H}_s(M^\star\times N')$ is L-Lipschitz with respect to the semi-distance* $\mathrm{dist}_{M^\star\times N'}$ *defined as in* (2.1).

Proof. Let Ω be a smooth form with vertical support in $M'' \times N'$ and whose $\mathcal{C}^1$-norm is at most 1. Since $f^*(\Omega)$ is also vertical, for $S, S' \in \mathcal{H}_s(M^\star \times N')$ we have

$$|\langle f_*(S) - f_*(S'), \Omega\rangle| = |\langle S - S', f^*(\Omega)\rangle| \leq \mathrm{dist}_{M^\star\times N'}(S, S')\|f^*(\Omega)\|_{\mathcal{C}^1(M''\times N')}.$$

Since $\|f^*(\Omega)\|_{\mathcal{C}^1(M''\times N')}$ is bounded by a constant independent from Ω, the result follows. □

Remark 5.4. Note that Lemma 5.3 is still true even when f is a non-invertible horizontal-like map. On the other hand, it is not clear how to get a version of Lemma 5.3 for the action of f^* in the case where f is not invertible. In particular, such a result cannot hold in general because, as soon as the critical set intersects the Julia set $\mathcal{J}^+ = \partial\mathcal{K}^+$, the $\mathcal{C}^1$, Lipschitz, and Hölder norms are not preserved by f_*.

Let $\tilde{d}^+_{s,n}$ be the quantity obtained by replacing M' with $M^\star$ in the definition of $d^+_{s,n}$ in Definition 3.9. Recall that the dynamical degree d^+_{s+1} does not depend on the choice of M' and N', see Lemma 3.11. The following more precise proposition implies Theorems 5.1 and 5.2.

Proposition 5.5. *Let f be either an invertible horizontal-like map or a polynomial like map as above. We have $d^+_{s,n+1} \lesssim n\tilde{d}^+_{s+1,n}$ as $n \to \infty$.*

Recall that we fix $0 \leq s \leq p-1$ in all this section. By the Definition 3.9 of $d^+_{s,n}$, for every $n \in \mathbb{N}$ there exists $S_{n-1} \in \mathcal{H}_s^{(1)}(D')$ such that

$$\|(f^n)_*(S_{n-1})\|_{M'\times N} \geq \frac{1}{2}d^+_{s,n}. \tag{5.1}$$

It follows from (5.1) (applied for $n+1$ instead of n) and Lemma 3.3(ii) that

$$\|(f^n)_*(S_n)\|_{M''\times N} \gtrsim \|(f^{n+1})_*(S_n)\|_{M'\times N} \gtrsim d^+_{s,n+1}. \tag{5.2}$$

For every $n \in \mathbb{N}$, we apply Theorem 2.1, with $M', M^\star$, and S_n instead of M, M', and S, respectively. This gives a corresponding current $\mathcal{R}_n$ on $M^\star \times N \times \mathbb{D}$ satisfying the properties (i)-(v) of $\mathcal{R}$ in that theorem. Fix a cut-off function $0 \leq \chi \leq 1$ with vertical support in $M^\star \times N$ and equal to 1 on $M'' \times N$. For each n, define the function $\phi_n : \mathbb{D} \to \mathbb{R}^+$ as

$$\phi_n(\theta) := \big\langle (d^+_{s,n+1})^{-1}(f^n)_*(\mathcal{R}_{n,\theta}), \chi\omega^s\big\rangle, \tag{5.3}$$

where $\mathcal{R}_{n,\theta} = \langle\mathcal{R}_n, \pi_\mathbb{D}, \theta\rangle$ is the slice current of $\mathcal{R}_n$ for $\theta \in \mathbb{D}$ and the integral in (5.3) is taken over $M^\star \times N$.

Proposition 5.5 will follow from the following three claims. We keep the assumptions and the notations as above.

Claim 1. *There exists a constant $\beta > 0$, independent of n, such that $\phi_n(0) \geq 2\beta$ for all n.*

Proof. By Theorem 2.1(iii) there exists $c > 0$ independent of n such that

$$\begin{aligned}\phi_n(0) &= \big\langle (d^+_{s,n+1})^{-1}(f^n)_*(\mathcal{R}_{n,0}), \chi\omega^s\big\rangle \geq c\,\big\langle (d^+_{s,n+1})^{-1}(f^n)_*(S_n), \chi\omega^s\big\rangle \\ &\geq c\int_{M''\times N} (d^+_{s,n+1})^{-1}(f^n)_*(S_n)\wedge\omega^s = c\,(d^+_{s,n+1})^{-1}\|(f^n)_*(S_n)\|_{M''\times N}.\end{aligned}$$

For the second inequality, we have used the fact that $\chi \equiv 1$ on $M'' \times N$. Finally, (5.2) implies that the last expression is larger than 2β, for some $\beta > 0$. The claim follows. □

We will see that $dd^c\phi_n$ is a signed measure on $\mathbb{D}$. When $dd^c\phi_n$ is a signed measure, we can define its mass $\|dd^c\phi_n\|$ as the sum of the masses of its positive and negative parts.

Claim 2. *The mass of $dd^c\phi_n$ satisfies*

$$\|dd^c\phi_n\| \lesssim (d^+_{s,n+1})^{-1}\tilde{d}^+_{s+1,n} \qquad \textit{as } n\to\infty.$$

Proof. Recall that we denote by $\pi_{M^\star\times N}$ the projection of $(M^\star\times N)\times\mathbb{D}$ to the factor $M^\star\times N$. Set

$$T_n := (d^+_{s,n+1})^{-1}\mathcal{R}_n \wedge \pi^*_{M^\star\times N}((f^n)^*(\chi\omega^s)),$$

with χ as above. By Theorem 2.1(v), the current $\mathcal{R}_n$ is horizontal on $M^\star\times(N'\times\mathbb{D})$. So, T_n is (well-defined and) compactly supported in $M^\star\times N'\times\mathbb{D}$. In particular, $\pi_{\mathbb{D}}$ is proper on the support of T_n. Hence, we have

$$(\pi_{\mathbb{D}})_*T_n = (\pi_{\mathbb{D}})_*\big((d^+_{s,n+1})^{-1}\mathcal{R}_n \wedge \pi^*_{M^\star\times N}((f^n)^*(\chi\omega^s))\big) = \phi_n.$$

It follows that

$$dd^c\phi_n = dd^c\big((\pi_{\mathbb{D}})_*T_n\big) = (\pi_{\mathbb{D}})_*(dd^cT_n) = (\pi_{\mathbb{D}})_*\big[(d^+_{s,n+1})^{-1}\mathcal{R}_n \wedge \pi^*_{M^\star\times N}((f^n)^*(dd^c\chi\wedge\omega^s))\big],$$

where we used that both $\mathcal{R}_n$ and ω are closed.

Since χ is smooth there exists a positive constant c such that $-c\omega \leq dd^c\chi \leq c\omega$ on $M^\star\times N$. Therefore, we have

$$\big|(d^+_{s,n+1})^{-1}\mathcal{R}_n \wedge \pi^*_{M^\star\times N}((f^n)^*(dd^c\chi\wedge\omega^s))\big| \lesssim (d^+_{s,n+1})^{-1}\mathcal{R}_n \wedge \pi^*_{M^\star\times N}\left((f^n)^*(\omega^{s+1})\right),$$

where we denoted by $|\cdot|$ the sum of the positive and negative parts of the measure in the left-hand side. It follows that

$$\begin{aligned}\|dd^c\phi_n\| &\lesssim \|(\pi_{\mathbb{D}})_*\big((d^+_{s,n+1})^{-1}\mathcal{R}_n \wedge \pi^*_{M^\star\times N}((f^n)^*(\omega^{s+1}))\big)\|_{\mathbb{D}} \\ &= (d^+_{s,n+1})^{-1}\|\mathcal{R}_n \wedge \pi^*_{M^\star\times N}((f^n)^*(\omega^{s+1}))\|_{M^\star\times N\times\mathbb{D}},\end{aligned}$$

where in the last step we used the fact that the pushforward operator $(\pi_{\mathbb{D}})_*$ preserves the mass of positive measures. To conclude, we need to bound the last term as in the statement.

Since $\mathcal{R}_n$ is horizontal on $M^\star\times(N'\times\mathbb{D})$, the projection $\pi_{M^\star\times N'}$ is proper on the support of $\mathcal{R}_n$. It follows that $(\pi_{M^\star\times N})_*(\mathcal{R}_n)$ is well-defined and is a horizontal positive closed current of bidimension $(s+1,s+1)$ on $M^\star\times N'$. Moreover, by Theorem 2.1(ii), the mass of $(\pi_{M^\star\times N})_*(\mathcal{R}_n)$ is less than or equal to 1. Using again the fact that $\pi_{M^\star\times N}$ preserves the mass of positive measures, it follows that

$$\begin{aligned}\big\|\mathcal{R}_n \wedge \pi^*_{M^\star\times N}((f^n)^*(\omega^{s+1}))\big\|_{M^\star\times N\times\mathbb{D}} &= \big\|(\pi_{M^\star\times N})_*\big[\mathcal{R}_n \wedge \pi^*_{M^\star\times N}((f^n)^*(\omega^{s+1}))\big]\big\|_{M^\star\times N} \\ &= \|(\pi_{M^\star\times N})_*(\mathcal{R}_n)\wedge(f^n)^*(\omega^{s+1})\|_{M^\star\times N} \\ &\leq \tilde{d}^+_{s+1,n},\end{aligned}$$

where in the last step we used the fact that $\|(\pi_{M^\star\times N})_*\mathcal{R}_n\| \leq 1$ and Definition 3.9 (recall that we replace M' with $M^\star$ to define $\tilde{d}^+_{s+1,n}$). We deduce from the inequalities above and the definition of ϕ_n that

$$\|dd^c\phi_n\| \lesssim (d^+_{s,n+1})^{-1}\tilde{d}^+_{s+1,n} \qquad \text{as } n\to\infty.$$

The proof of the claim is complete. □

Define now the function $\Phi_n : \mathcal{H}_s(M^\star \times N') \to \mathbb{R}$ by

$$\Phi_n(S) := \big\langle (d^+_{s,n+1})^{-1}(f^n)_*(S), \chi\omega^s \big\rangle,$$

where the integral is taken over $M^\star \times N$ and, for every n, is well-defined since $(f^n)_*(S)$ is horizontal and χ has vertical support in $M^\star \times N$.

Claim 3. *There exist a constant $L > 0$ independent of n such that*

$$|\Phi_n(S) - \Phi_n(S')| \lesssim L^n \operatorname{dist}_{M^\star \times N'}(S, S') \quad \text{as } n \to \infty.$$

Proof. Since χ is a smooth function with vertical support in $M^\star \times N$, by (3.1), $f^*(\chi\omega^s)$ is a vertical smooth form with support in $M'' \times N$. Hence, for all $n \geq 1$, by Lemma 5.3 we have

$$\begin{aligned} |\Phi_n(S) - \Phi_n(S')| &= (d^+_{s,n+1})^{-1} |\langle (f^{n-1})_*(S - S'), f^*(\chi\omega^s)\rangle| \\ &\leq (d^+_{s,n+1})^{-1} \|f^*(\chi\omega^s)\|_{\mathcal{C}^1(M''\times N')} L^{n-1} \operatorname{dist}_{M^\star\times N'}(S, S') \\ &\lesssim (d^+_{s,n+1})^{-1} L^n \operatorname{dist}_{M^\star\times N'}(S, S') \end{aligned}$$

for some positive constant L independent of n. Since $(d^+_{s,n+1})^{-1} \lesssim 1$ (see Remark 3.12), the assertion follows. □

We can now prove Proposition 5.5, which also concludes the proof of Theorems 5.1 and 1.1 (for f invertible horizontal-like map) and Theorems 5.2 and 1.3 (for f polynomial-like map).

Proof of Proposition 5.5. We keep the notations and definitions as above. We assume by contradiction that

$$\limsup_{n\to\infty} \frac{d^+_{s,n+1}}{n\tilde{d}^+_{s+1,n}} = +\infty.$$

Hence, there exist two sequences $n_j \to \infty$ and $c_j \to \infty$ such that

$$d^+_{s,n_j+1} = c_j n_j \tilde{d}^+_{s+1,n_j} \quad \text{for all } j. \tag{5.4}$$

By Theorem 2.1(v), we see that $\phi_{n_j}(\theta) = 0$ for all j and $1/2 < |\theta| < 1$. By Claim 2 and (5.4), we also have that $\|dd^c\phi_{n_j}\| \lesssim (d^+_{s,n_j+1})^{-1}\tilde{d}^+_{s+1,n_j} = (c_j n_j)^{-1}$ as $j \to \infty$. Hence, the family $\{c_j n_j \phi_{n_j}\}_{j\in\mathbb{N}}$ is a bounded family of DSH functions on $\mathbb{P}^1$, i.e., differences of quasi-psh functions on $\mathbb{P}^1$ (see [15, Appendix A.4]). Indeed, it is enough to apply Lemma 5.6 below with $X = \mathbb{P}^1$ and μ a smooth probability measure supported on $\mathbb{P}^1 \setminus \mathbb{D}$. Observe that, by the same lemma, the fact that the family is bounded does not depend on the choice of the measure μ used to define the norm.

According to Skoda-type estimates [31], there exist positive constants α, C, independent of j, such that

$$\int_{\mathbb{D}} e^{\alpha c_j n_j |\phi_{n_j}|} \leq C \quad \text{for all } j, \tag{5.5}$$

where the integral is with respect to the Lebesgue measure on $\mathbb{D}$.

Theorem 2.1(iv) and Claim 3 imply that

$$|\phi_{n_j}(0) - \phi_{n_j}(\theta)| \lesssim L^{n_j} \operatorname{dist}_{M^\star\times N'}(\mathcal{R}_{n_j,0}, \mathcal{R}_{n_j,\theta}) \lesssim L^{n_j}|\theta| \text{ for all } \theta \in \mathbb{D}.$$

Hence, for some r_j such that $L^{-n_j} \lesssim r_j \lesssim L^{-n_j}$, by Claim 1 we have

$$\phi_{n_j}(\theta) \geq \beta \quad \text{for} \quad |\theta| \leq r_j \quad \text{and all } j \text{ sufficiently large.}$$

This and the above estimate (5.5) imply that, for all j sufficiently large,

$$C \geq \int_{\mathbb{D}} e^{\alpha c_j n_j |\phi_{n_j}|} \geq \int_{\mathbb{D}_{r_j}} e^{\alpha c_j n_j |\phi_{n_j}|} \gtrsim e^{\alpha c_j n_j \beta} r_j^2 \simeq e^{\alpha c_j n_j \beta} L^{-2n_j} = e^{n_j(\alpha c_j \beta - 2\log L)}.$$

Since $c_j \to \infty$, this gives the desired contradiction, and completes the proof. □

Lemma 5.6 ([15, p.283]). *Let μ be a positive measure on a compact Kähler manifold X which integrates the* DSH *functions. Then*

$$\|u\|_\mu := \Big|\int u\mu\Big| + \min \|T^\pm\|$$

(where the minimum is taken over all positive closed $(1,1)$-currents $T^\pm$ such that $dd^c u = T^+ - T^-$) defines a norm on $\mathrm{DSH}(X)$. *Moreover, for all μ and μ' which integrate the DHS functions, the norms $\|\cdot\|_\mu$ and $\|\cdot\|_{\mu'}$ are equivalent.*

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