EXISTENCE AND PROPERTIES OF EQUILIBRIUM STATES OF HOLOMORPHIC ENDOMORPHISMS OF $\mathbb{P}^k$

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Dedicated to the memory of Professor Nessim Sibony

Abstract. We develop a new method, based on pluripotential theory, to study the transfer (Perron-Frobenius) operator induced on $\mathbb{P}^k = \mathbb{P}^k(\mathbb{C})$ by a holomorphic endomorphism and a suitable continuous weight. This method not only allows us to prove the existence and uniqueness of the equilibrium state and conformal measure for very general weights (due to Denker-Przytycki-Urbański in dimension 1 and Urbański-Zdunik in higher dimensions, both in the case of Hölder continuous weights), but, most importantly, to establish the existence of a spectral gap for the transfer operator and its perturbations on various functional spaces, which is new even in dimension 1.

Our analytic method replaces all distortion estimates on inverse branches with a unique, global, estimate on dynamical currents, and allows us to reduce the dynamical questions to comparisons between currents and their potentials. The main issue to overcome is the rigidity of the complex objects, since the transfer operator is a *real* perturbation of the complex operator f_*. The system is moreover non-uniformly hyperbolic and one may have critical points on the Julia set. The construction of our norm requires the introduction and study of several intermediate new norms, and a careful combination of ideas from pluripotential and interpolation theory. As far as we know, this is the first time that pluripotential methods have been applied to solve a mixed real-complex problem.

Thanks to the spectral gap, we are able to obtain a complete statistical study for the equilibrium states and conformal measures. We establish the equidistribution of the backward orbits of points, with exponential speed of convergence, towards the conformal measure, and the equidistribution of repelling periodic points with respect to the equilibrium state. Moreover, we obtain a full list of statistical properties for the equilibrium states: K-mixing, mixing of all orders, exponential mixing, CLT, Berry-Esseen theorem, local CLT, ASIP, LIL, LDP, almost sure CLT. Many of these properties are new even in dimension one, some even in the case of zero weight function (i.e., for the measure of maximal entropy). Properties such as the local CLT seem to be unattainable without a spectral gap.

Contents

Notation. Throughout the paper, $\mathbb{P}^k$ denotes the complex projective space of dimension k endowed with the standard Fubini-Study form $\omega_{\rm FS}$. This is a Kähler $(1,1)$-form normalized so that $\omega_{\rm FS}^k$ is a probability measure. We will use the metric and distance $\mathrm{dist}(\cdot,\cdot)$ on $\mathbb{P}^k$ induced by $\omega_{\rm FS}$ and the standard ones on $\mathbb{C}^k$ when we work on open subsets of $\mathbb{C}^k$. We denote by

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$\mathbb{B}_{\mathbb{P}^k}(a,r)$ (resp. $\mathbb{B}^k_r, \mathbb{D}(a,r), \mathbb{D}_r$) the ball of center a and radius r in $\mathbb{P}^k$ (resp. the ball of center 0 and radius r in $\mathbb{C}^k$, the disc of center a and radius r in $\mathbb{C}$, and the disc of center 0 and radius r in $\mathbb{C}$). Leb denotes the standard Lebesgue measure on a Euclidean space or on a sphere. The currents ω_n and their dynamical potentials u_n are introduced in Section 3.1.

The pairing $\langle \cdot, \cdot \rangle$ is used for the integral of a function with respect to a measure or more generally the value of a current at a test form. If S and R are two $(1,1)$-currents, we will write $|R| \leq S$ when $\Re(\xi R) \leq S$ for every function $\xi \colon \mathbb{P}^k \to \mathbb{C}$ with $|\xi| \leq 1$, i.e., all currents $S - \Re(\xi R)$ with ξ as before are positive. Notice that this forces S to be real and positive. We also write other inequalities such as $|R| \leq |R_1| + |R_2|$ if $|R| \leq S_1 + S_2$ whenever $|R_1| \leq S_1$ and $|R_2| \leq S_2$. Recall that $d^c = \frac{i}{2\pi}(\overline{\partial} - \partial)$ and $dd^c = \frac{i}{\pi}\partial\overline{\partial}$. The notations $\lesssim$ and $\gtrsim$ stand for inequalities up to a multiplicative constant. The function identically equal to 1 is denoted by $\mathbb{1}$. We also use the function $\log^\star(\cdot) := 1 + |\log(\cdot)|$.

Consider a holomorphic endomorphism $f \colon \mathbb{P}^k \to \mathbb{P}^k$ of algebraic degree $d \geq 2$ satisfying the Assumption **(A)** in the Introduction. Denote respectively by T, $\mu = T^k$, $\mathrm{supp}(\mu)$ the Green $(1,1)$-current, the measure of maximal entropy (also called the Green measure or the equilibrium measure), and the small Julia set of f. If S is a positive closed $(1,1)$-current on $\mathbb{P}^k$, its dynamical potential is denoted by u_S and is defined in Section 2.2. If ν is an invariant probability measure, we denote by $\mathrm{Ent}_f(\nu)$ the metric entropy of ν with respect to f.

We also consider a weight ϕ which is a continuous function on $\mathbb{P}^k$. We often assume that ϕ is real. The transfer operator (Perron-Frobenius operator) $\mathcal{L} = \mathcal{L}_\phi$ is introduced in the Introduction together with the scaling ratio $\lambda = \lambda_\phi$, the conformal measure m_ϕ, the density function $\rho = \rho_\phi$, the equilibrium state $\mu_\phi = \rho m_\phi$, the pressure $P(\phi)$, see also Section 4. The measures m_ϕ and μ_ϕ are probability measures. The operator L is a suitable modification of $\mathcal{L}$ and is introduced in Section 4.5.

The oscillation $\Omega(\cdot)$, the modulus of continuity $m(\cdot,\cdot)$, the semi-norms $\|\cdot\|_{\log^p}$ and $\|\cdot\|_*$ of a function are defined in Section 2.1. Other norms and semi-norms $\|\cdot\|_p$, $\|\cdot\|_{p,\alpha}$, $\|\cdot\|_{\langle p,\alpha\rangle}$, $\|\cdot\|_{\langle p,\alpha\rangle,\gamma}$ for $(1,1)$-currents and functions are introduced in Section 3 and the norms $\|\cdot\|_{\diamond_1}$, $\|\cdot\|_{\diamond_2}$ in Section 5.3. The semi-norms we consider are almost norms: they vanish only on constant functions. It is easy to make them norms by adding a suitable functional such as $g \mapsto |\langle m_\phi, g\rangle|$. However, for simplicity, it is more convenient to work directly with these semi-norms. The versions of these semi-norms for currents are actually norms. The positive numbers q_0, q_1, q_2 are defined in Lemmas 3.10, 3.13, 3.16 and the families of weights $\mathcal{P}(q, M, \Omega), \mathcal{P}_0(q, M, \Omega), \mathcal{Q}_0$ are introduced in Sections 4.5 and 5.2.

1. Introduction and results

Let $f \colon \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic endomorphism of the complex projective space $\mathbb{P}^k = \mathbb{P}^k(\mathbb{C})$, with $k \geq 1$, of algebraic degree $d \geq 2$. Denote by μ the unique measure of maximal entropy for the dynamical system $(\mathbb{P}^k, f)$ [Lyu83, BD09, DS10a]. The support $\mathrm{supp}(\mu)$ of μ is called *the small Julia* set of f. The measure μ corresponds to the equilibrium state of the system in the case without weight, i.e., when the weight is zero. In this paper, we will consider the case where the weight, denoted by ϕ, is not necessarily equal to zero. This problem has been studied for Hölder continuous weights using a geometric approach, in dimension 1, see, e.g., Denker-Przytycki-Urbański [Prz90, DU91a, DU91b, DPU96] and Haydn [Hay99] just to name a few, and in higher dimensions, see Szostakiewicz-Urbański-Zdunik [UZ13, SUZ14]. We will develop here an analytic method which will allow us to obtain more general and more quantitative results. Many results are new even when $\phi = 0$ and even for $k = 1$.

Throughout this paper, we make use of the following technical assumption for f:

(A) the local degree of the iterate $f^n := f \circ \cdots \circ f$ (n times) satisfies

$$\lim_{n\to\infty} \frac{1}{n} \log \max_{a \in \mathbb{P}^k} \deg(f^n, a) = 0.$$

Here, $\deg(f^n, a)$ is the multiplicity of a as a solution of the equation $f^n(z) = f^n(a)$. Note that generic endomorphisms of $\mathbb{P}^k$ satisfy this condition, see [DS10b]. Our study still holds under a weaker condition that the exceptional set of f (i.e., the maximal proper analytic subset of $\mathbb{P}^k$ invariant by f^{-1}) is empty or more generally has no intersection with $\mathrm{supp}(\mu)$ (in particular, this condition is superfluous in dimension 1). However, this situation requires more technical conditions on the weight ϕ. We choose not to present this case here in order to simplify the notation and focus on the main new ideas introduced in this topic. Our first goal in this paper is to prove the following theorem (more precise statements will be given later).

Theorem 1.1. *Let f be an endomorphism of $\mathbb{P}^k$ of algebraic degree $d \geq 2$ and satisfying the Assumption* **(A)** *above. Let ϕ be a real-valued $\log^q$-continuous function on $\mathbb{P}^k$, for some $q > 2$, such that $\Omega(\phi) := \max\phi - \min\phi < \log d$. Then ϕ admits a unique equilibrium state μ_ϕ, whose support is equal to the small Julia set of f. This measure μ_ϕ is K-mixing and mixing of all orders, and repelling periodic points of period n (suitably weighted) are equidistributed with respect to μ_ϕ as n goes to infinity. Moreover, there is a unique conformal measure m_ϕ associated to ϕ. We have $\mu_\phi = \rho m_\phi$ for some strictly positive continuous function ρ on $\mathbb{P}^k$ and the preimages of points by f^n (suitably weighted) are equidistributed with respect to m_ϕ as n goes to infinity.*

We say that a function is $\log^q$-continuous if its oscillation on a ball of radius r is bounded by a constant times $(\log^\star r)^{-q}$, see Section 2.1 for details. See also Section 4.5 for the K-mixing and mixing of all orders.

An *equilibrium state* as in the statement above is defined as follows, see for instance [Rue72, Wal00, PU10]. Given a *weight*, i.e., a real-valued continuous function, ϕ as above, we define the *pressure* of ϕ as

$$P(\phi) := \sup\big\{ \mathrm{Ent}_f(\nu) + \langle \nu, \phi\rangle \big\},$$

where the supremum is taken over all Borel f-invariant probability measures ν and $\mathrm{Ent}_f(\nu)$ denotes the metric entropy of ν. An equilibrium state for ϕ is then an invariant probability measure μ_ϕ realizing a maximum in the above formula, that is,

$$P(\phi) = \mathrm{Ent}_f(\mu_\phi) + \langle \mu_\phi, \phi\rangle.$$

On the other hand, a *conformal measure* is defined as follows. Define the *Perron-Frobenius* (or *transfer*) operator $\mathcal{L}$ with weight ϕ as (we often drop the index ϕ for simplicity)

$$\mathcal{L}g(y) := \mathcal{L}_\phi g(y) := \sum_{x\in f^{-1}(y)} e^{\phi(x)} g(x), \tag{1.1}$$

where $g\colon \mathbb{P}^k \to \mathbb{R}$ is a continuous test function and the points x in the sum are counted with multiplicity. A conformal measure is an eigenvector for the dual operator $\mathcal{L}^*$ acting on positive measures.

Notice that, in the case where ϕ is Hölder continuous, a part of Theorem 1.1 was established by Urbański-Zdunik [UZ13] (also under a genericity assumption for f), see also [Prz90, DU91a, DU91b, DPU96] for previous results in dimension $k = 1$. When ϕ is constant, the operator $\mathcal{L}$ reduces to a constant times the push-forward operator f_* and we get $\mu_\phi = \mu$. For an account of the known results in this case, see for instance [DS10a].

A reformulation of Theorem 1.1 is the following: given ϕ as in the statement, there exist a number $\lambda > 0$ and a continuous function $\rho = \rho_\phi\colon \mathbb{P}^k \to \mathbb{R}$ such that, for every continuous function $g\colon \mathbb{P}^k \to \mathbb{R}$, the following uniform convergence holds:

$$\lambda^{-n}\mathcal{L}^n g(y) \to c_g \rho \tag{1.2}$$

for some constant c_g depending on g. By duality, this is equivalent to the convergence, uniform on probability measures ν,

$$\lambda^{-n}(\mathcal{L}^*)^n \nu \to m_\phi, \tag{1.3}$$

where m_ϕ is a conformal measure associated to the weight ϕ. The equilibrium state μ_ϕ is then given by $\mu_\phi = \rho m_\phi$, and we have $c_g = \langle m_\phi, g\rangle$.

To prove Theorem 1.1, in Section 4 we develop a new and completely different approach with respect to [UZ13] and to the previous studies in dimension 1. As we will see later, the flexibility of this method will allow for a more quantitative understanding of the convergences (1.2) and (1.3), and for the direct establishment of several statistical properties of the equilibrium states.

The main idea of our method is the following. Let us just consider for now the case where both of the functions g and ϕ are of class $\mathcal{C}^2$ (the general case requires suitable approximations of g and ϕ by $\mathcal{C}^2$ functions). Given such a function g, first we want to prove that the ratio between the maximum and the minimum of $\mathcal{L}^n g$ stays bounded with n. This allows us to define the good scaling ratio λ and to get that the sequence $\lambda^{-n}\mathcal{L}^n g$ is uniformly bounded. Next, we would like to prove that this sequence is actually equicontinuous. This, together with other technical arguments, would imply the existence and uniqueness of the limit function ρ.

In order to establish the above controls, we study the sequence of $(1,1)$-currents given by $dd^c\mathcal{L}^n g$. First we prove that suitably normalized versions of these currents are uniformly bounded by a common positive closed $(1,1)$-current R. This is the core of our method which replaces all controls on the distortion of inverse branches of f^n in the geometric method of [UZ13] by a unique, global, and flexible estimate. Namely, for every $n \in \mathbb{N}$ we can get an estimate of the form

$$\Big| dd^c \frac{\mathcal{L}^n g}{c_n}\Big| \lesssim \sum_{j=0}^{\infty} \Big(\frac{e^{\Omega(\phi)}}{d}\Big)^j \frac{(f_*)^j \omega_{\mathrm{FS}}}{d^{(k-1)j}} \quad \text{with} \quad c_n := \|g\|_{\mathcal{C}^2} \langle \omega_{\mathrm{FS}}^k, \mathcal{L}^n \mathbb{1}\rangle. \tag{1.4}$$

Here, ω_{FS} denotes the usual Fubini-Study form on $\mathbb{P}^k$ normalized so that ω_{FS}^k is a probability measure. Notice that the last infinite sum gives a key reason for the assumption $\Omega(\phi) < \log d$ made on the weight ϕ as the mass of the current $(f_*)^j\omega_{\mathrm{FS}}$ is equal to $d^{(k-1)j}$.

We will establish in Section 2 some general criteria, interesting in themselves, which allow one to bound the oscillation of $c_n^{-1}\mathcal{L}^n g$ in terms of the oscillation of the potentials of the current in the RHS of (1.4). This latter oscillation is actually controllable. Assumption **(A)** allows us to have a simple control which makes the estimates less technical but such a control exists without Assumption **(A)**.

Combining all these ingredients, the existence and uniqueness of the equilibrium state and conformal measure, as well as the equidistribution of preimages and the equality $P(\phi) = \log\lambda$, follow from standard arguments that we recall in Sections 4.5 and 4.6 for completeness. We also prove that the entropy of μ_ϕ is larger than $k\log d - \Omega(\phi) > (k-1)\log d$, and that all the Lyapunov exponents of μ_ϕ are strictly positive. This also leads to a lower bound for the Hausdorff dimension of μ_ϕ. In Section 4.7 we establish the equidistribution of repelling periodic points with respect to μ_ϕ, see Theorem 4.20, which completes the proof of Theorem 1.1. This result is due to Lyubich [Lyu83] (for $k = 1$) and Briend-Duval [BD99] (for any $k \geq 1$) when $\phi = 0$, and is new even for $k = 1$ otherwise.

Without extra arguments, given a continuous test function g the convergence as $n \to \infty$ in (1.2) is not uniform in g. Our next and main goal is to establish an exponential speed of convergence in (1.2). This requires to build a suitable (semi-)norm for (or equivalently, a suitable functional space on) which the operator $\lambda^{-1}\mathcal{L}$ turns out to be a contraction.

Establishing the following statement is then our main goal in the current paper. As far as we know, this is the first time that the existence of a spectral gap for the perturbed Perron-Frobenius operator is proved in this context even in dimension 1, except for hyperbolic endomorphisms or for weights with ad-hoc conditions (see for instance [Rue92, MS00]). This is one of the most desirable properties in dynamics.

Theorem 1.2. *Let f, q, ϕ, ρ, m_ϕ be as in Theorem 1.1 and $\mathcal{L}, \lambda$ the above Perron-Frobenius operator and scaling factor associated to ϕ. Let $A > 0$ and $0 < \Omega < \log d$ be two constants. Then, for every constant $0 < \gamma \leq 1$, there exist two explicit equivalent norms for functions on*

$\mathbb{P}^k$: $\|\cdot\|_{\diamond_1}$, *depending on* f, γ, q *and independent of* ϕ, *and* $\|\cdot\|_{\diamond_2}$, *depending on* f, ϕ, γ, q, *such that*

$$\|\cdot\|_\infty + \|\cdot\|_{\log^q} \lesssim \|\cdot\|_{\diamond_1} \simeq \|\cdot\|_{\diamond_2} \lesssim \|\cdot\|_{\mathcal{C}^\gamma} .$$

Moreover, there are positive constants $c = c(f, \gamma, q, A, \Omega)$ *and* $\beta = \beta(f, \gamma, q, A, \Omega)$ *with* $\beta < 1$, *both independent of* ϕ *and* n, *such that when* $\|\phi\|_{\diamond_1} \le A$ *and* $\Omega(\phi) \le \Omega$ *we have*

$$\|\lambda^{-n}\mathcal{L}^n\|_{\diamond_1} \le c, \quad \|\rho\|_{\diamond_1} \le c, \quad \|1/\rho\|_{\diamond_1} \le c, \quad \text{and} \quad \left\|\lambda^{-1}\mathcal{L}g\right\|_{\diamond_2} \le \beta \|g\|_{\diamond_2}$$

for every function $g\colon \mathbb{P}^k \to \mathbb{R}$ *with* $\langle m_\phi, g\rangle = 0$. *Furthermore, given any constant* $0 < \delta < d^{\gamma/(2\gamma+2)}$, *when* A *is small enough, the norm* $\|\cdot\|_{\diamond_2}$ *can be chosen so that we can take* $\beta = 1/\delta$.

The construction of the norms $\|\cdot\|_{\diamond_1}$ and $\|\cdot\|_{\diamond_2}$ is quite involved. We use here ideas from the theory of interpolation between Banach spaces [Tri95] combined with techniques from pluripotential theory and complex dynamics. Roughly speaking, an idea from interpolation theory allows us to reduce the problem to the case where $\gamma = 1$. The definition of the above norms in this case requires a control of the derivatives of g (in the distributional sense), and this is where we use techniques from pluripotential theory. This also explains why these norms are bounded by the $\mathcal{C}^1$ norm. Note that we should be able to bound the derivatives of $\mathcal{L}g$ in a similar way. A quick expansion of the derivatives of $\mathcal{L}g$ using (1.1) gives an idea of the difficulties that one faces. The existence of these norms is still surprising to us.

Let us highlight two among these difficulties. First, the objects from complex analysis and geometry are too rigid for perturbations with a non-constant weight: none of the operators f_*, d, and dd^c commutes with the operator $\mathcal{L}$. In particular, the dd^c-method developed by the second author and Sibony (see for instance [DS10a]) cannot be applied in this context, even for small perturbations of the weight $\phi = 0$. Moreover, we have critical points on the support of the measure, which cause a loss in the regularity of functions under the operators f_* and $\mathcal{L}$ (see Section 3). Notice that we do not assume that our potential degenerates at the critical points.

Our solution to these problems is to define a new invariant functional space and norm in this mixed real-complex setting, that we call the *dynamical Sobolev space* and *semi-norm*, taking into account both the regularity of the function (to cope with the rigidity of the complex objects) and the action of f (to take into account the critical dynamics), see Definitions 3.12 and 3.15. The construction of this norm requires the definition of several intermediate semi-norms and the precise study of the action of the operator f_* with respect to them, and is carried out in Section 3. Some of the intermediate estimates already give new or more precise convergence properties for the operator f_* and the equilibrium measure μ, see for instance Theorem 3.6.

A spectral gap for the Perron-Frobenius operator and its perturbations is one of the most desirable properties in dynamics. It allows us to obtain several statistical properties of the equilibrium state. In the present setting, we have the following result, see Appendix A for the definitions.

Theorem 1.3. *Let* $f, \phi, \mu_\phi, m_\phi, \|\cdot\|_{\diamond_1}$ *be as in Theorems 1.1 and 1.2 ,* λ *the scaling ratio associated to* ϕ, *and assume that* $\|\phi\|_{\diamond_1} < \infty$. *Then the equilibrium state* μ_ϕ *is exponentially mixing for observables with bounded* $\|\cdot\|_{\diamond_1}$ *norm and the preimages of points by* f^n *(suitably weighted) equidistribute exponentially fast towards* m_ϕ *as* n *goes to infinity. The measure* μ_ϕ *satisfies the LDP for all observables with finite* $\|\cdot\|_{\diamond_1}$ *norm, the ASIP, CLT, almost sure CLT, LIL for all observables with finite* $\|\cdot\|_{\diamond_1}$ *norm which are not coboundaries, and the local CLT for all observables with finite* $\|\cdot\|_{\diamond_1}$ *norm which are not* $(\|\cdot\|_{\diamond_1}, \phi)$*-multiplicative cocycles. Moreover, the pressure* $P(\phi)$ *is equal to* $\log \lambda$ *and is analytic in the following sense: for* $\|\psi\|_{\diamond_1} < \infty$ *and* t *sufficiently small, the function* $t \mapsto P(\phi + t\psi)$ *is analytic.*

In particular, all the properties in Theorem 1.3 hold when the weight ϕ and the observable are Hölder continuous and satisfy the necessary coboundary/cocycles requirements. In this assumption, some of the above properties were previously obtained with ad hoc arguments, see [PUZ89, DU91a, DU91b, DPU96, Hay99, DNS07, SUZ15] when $k = 1$, [UZ13, SUZ14] for mixing, CLT, LIL when $k \geq 1$, and [Dup10] for the ASIP when $k \geq 1$ and $\phi = 0$. Our results are more general, sharper, and with better error control. Note that the LDP and the local CLT are new even for $\phi = 0$ (for all $k \geq 1$ and for all $k > 1$, respectively). Our method of proof of Theorem 1.3 exploits the spectral gap established in Theorem 1.2 and is based on the theory of perturbed operators. This approach was first developed by Nagaev [Nag57] in the context of Markov chains, see also [RE83, Bro96, Gou15], and provides a unified treatment for all the statistical study. Observe in particular that the fine control given for instance by the local CLT is simply impossible to prove using weaker arguments, such as martingales, see, e.g., [Gou15, p. 163].

For the reader's convenience, the above statistic properties (exponential mixing, LDP, ASIP, CLT, almost sure CLT, LIL, local CLT) and the notions of coboundary and multiplicative cocycle will be recalled in Section 6 and Appendix A at the end of the paper.

Outline of the organization of the paper. In Section 2, we introduce some useful notions and establish comparison principles for currents and potentials that will be the technical key to prove Theorems 1.1 and 1.2. In Section 3 we introduce the main (semi-)norms that we will need, and study the action of the operator f_* with respect to these (semi-)norms. Section 4 is dedicated to the proof of Theorem 4.1. For this purpose, we develop our method to get the uniform boundedness and equicontinuity for the sequence $\mathcal{L}^n g$, properly normalized, that lead to the good definition of the scaling ratio λ. Once this is done, the deduction of most of Theorem 1.1 is classical, and we just recall the proof of this in Sections 4.5 and 4.6 for the reader's convenience. Section 5 is dedicated to the proof of Theorem 1.2, which is built on the method developed in Section 4, made quantitative with respect to the semi-norms that we introduced in Section 3. Finally, in Section 6 we develop the statistical study of the equilibrium states. This section contains the proof of Theorem 1.3 and more precise statements. In Appendix A we recall statistical properties and criteria in abstract settings that we use to prove results in Section 6.

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2. Preliminaries and some comparison principles

2.1. **Some definitions.** We collect here some notions that we will use throughout the paper.

Definition 2.1. Given a subset U of $\mathbb{P}^k$ or $\mathbb{C}^k$ and a real-valued function $g\colon U \to \mathbb{R}$, define the *oscillation* $\Omega_U(g)$ of g as

$$\Omega_U(g) := \sup g - \inf g$$

and its continuity modulus $m_U(g, r)$ at distance r as

$$m_U(g, r) := \sup_{x,y\in U\colon \operatorname{dist}(x,y)\leq r} |g(x) - g(y)|.$$

We may drop the index U when there is no possible confusion.

The following semi-norms will be the main building blocks for all the semi-norms that we will construct and study in the sequel.

Definition 2.2. The semi-norm $\|\cdot\|_{\log^p}$ is defined for every $p > 0$ and $g\colon \mathbb{P}^k \to \mathbb{R}$ as

$$\|g\|_{\log^p} := \sup_{a,b\in\mathbb{P}^k} |g(a) - g(b)| \cdot (\log^\star \mathrm{dist}(a,b))^p = \sup_{r>0, a\in\mathbb{P}^k} \Omega_{\mathbb{B}_{\mathbb{P}^k}(a,r)}(g) \cdot (1 + |\log r|)^p,$$

where $\mathbb{B}_{\mathbb{P}^k}(a,r)$ denotes the ball of center a and radius r in $\mathbb{P}^k$.

Definition 2.3. The norm $\|\cdot\|_*$ of a $(1,1)$-current R is given by

$$\|R\|_* := \inf \|S\|$$

where the infimum is taken over all positive closed $(1,1)$-currents S such that $|R| \leq S$, see the Notation at the beginning of the paper. When such a current S does not exist, we put $\|R\|_* := +\infty$. The semi-norm $\|\cdot\|_*$ of an integrable function $g\colon \mathbb{P}^k \to \mathbb{R}$ is given by

$$\|g\|_* := \|dd^c g\|_* .$$

Note that when R is a real closed $(1,1)$-current the above norm is equivalent to the usual one defined as

$$\|R\|_* := \inf(\|S^+\| + \|S^-\|)$$

where the infimum is taken over all positive closed $(1,1)$-currents $S^\pm$ on $\mathbb{P}^k$ such that $R = S^+ - S^-$. In particular, for $R = dd^c g$, the currents S^+ and S^- are cohomologous and thus have the same mass, i.e., $\|S^+\| = \|S^-\|$, see [DS10a, App. A.4] for details.

In this paper we only consider continuous functions g. So the above semi-norms (and the others that we will introduce later) are almost norms: they only vanish when g is constant. In particular, they are norms on the space of functions g satisfying $\langle \nu, g\rangle = 0$ for some fixed probability measure ν. We will use later $\nu = m_\phi$ or $\nu = \mu_\phi$ to obtain a spectral gap for the Perron-Frobenius operator and to study the statistical properties of μ_ϕ.

2.2. **Dynamical potentials.** Let T denote the Green $(1,1)$-current of f. It is positive closed and of unit mass. Let S be any positive closed $(1,1)$-current of mass m on $\mathbb{P}^k$. There is a unique function $u_S\colon \mathbb{P}^k \to \mathbb{R} \cup \{-\infty\}$ which is p.s.h. modulo mT and such that

$$S = mT + dd^c u_S \qquad \text{and} \qquad \langle \mu, u_S\rangle = 0.$$

Locally, u_S is the difference between a potential of S and a potential of mT. We call it *the dynamical potential* of S. Observe that the dynamical potential of T is zero, i.e., $u_T = 0$.

Recall that T has Hölder continuous potentials. So, u_S is locally the difference between a p.s.h. function and a Hölder continuous one. The dynamical potential of S behaves well under the push-forward and pull-back operators associated to f. Indeed, because of the invariance properties of T, we have

$$f^*S = md \cdot T + dd^c(u_S \circ f) \qquad \text{and} \qquad f_*S = md^{k-1} \cdot T + dd^c(f_* u_S),$$

which, together with the invariance properties of μ, imply

$$u_{f^*S} = u_S \circ f \qquad \text{and} \qquad u_{f_*S} = f_* u_S.$$

We refer the reader to [DS10a] for details. In this paper, we only need currents S such that u_S is continuous.

2.3. **Comparisons between currents and their potentials.** A technical key point in the proof of our main theorems will be based on the following general idea: if u and v are two functions on some domain in $\mathbb{C}^k$ such that $|dd^c u| \leq dd^c v$ or $i\partial u \wedge \overline{\partial} u \leq dd^c v$, then u inherits some of the regularity properties of v. This section and the next are devoted to make this idea precise and quantitative for our purposes. We start with the simplest occurrence of this fact in the first case in terms of the sup-norm.

Lemma 2.4. *There exists a positive constant A such that, for every positive closed $(1,1)$-current S_0 on $\mathbb{P}^k$ of mass 1 and for every positive closed $(1,1)$-current S on $\mathbb{P}^k$ with $S \leq S_0$, we have $\Omega(u_S) \leq A + \Omega(u_{S_0})$, where u_{S_0} and u_S denote the dynamical potentials of S_0 and S, respectively.*

Proof. We assume that $\Omega(u_{S_0})$ is finite, since otherwise the assertion trivially holds. Observe that the mass m of S is at most equal to 1 because $S \leq S_0$. Recall that u_S and u_{S_0} satisfy

$$S = mT + dd^c u_S, \quad S_0 = T + dd^c u_{S_0}, \quad \langle \mu, u_S \rangle = 0, \quad \text{and} \quad \langle \mu, u_{S_0} \rangle = 0.$$

The last identity implies that $\sup u_{S_0}$ is non-negative.

We first prove that u_S is bounded above by a constant. As mentioned above, the correspondence between positive closed $(1,1)$-currents and their dynamical potentials is a bijection. Moreover, we know that quasi-p.s.h. functions (i.e., functions that are locally difference between a p.s.h. and a smooth function) are integrable with respect to μ [DS10a, Th. 1.35]. Since the set of positive closed $(1,1)$-currents of mass less than or equal to 1 is compact, u_S belongs to a compact family of p.s.h. functions modulo mT. We deduce that there is a constant $A > 0$ independent of S such that $u_S \leq A/2$ on $\mathbb{P}^k$, see [DS10a, App. A.2] for more details. It follows that $\sup u_S \leq \sup u_{S_0} + A/2$ because $\sup u_{S_0}$ is non-negative.

Consider the current $S' := S_0 - S$ which is positive closed and smaller than S_0. By the uniqueness of the dynamical potential, we have $u_{S'} = u_{S_0} - u_S$, which implies $u_S = u_{S_0} - u_{S'}$. Since $S' \leq S$, as above, we also have $\sup u_{S'} \leq A/2$. It follows that

$$\inf u_S \geq \inf u_{S_0} - \sup u_{S'} \geq \inf u_{S_0} - A/2.$$

This estimate and the above inequality $\sup u_S \leq \sup u_{S_0} + A/2$ imply the lemma. □

Corollary 2.5. *There exists a positive constant A such that for every positive closed $(1,1)$-current S_0 on $\mathbb{P}^k$ and for every continuous function $g\colon \mathbb{P}^k \to \mathbb{R}$ with $|dd^c g| \leq S_0$ we have $\Omega(g) \leq A\,\|S_0\| + 3\Omega(u_{S_0})$.*

Proof. By linearity we can assume that S_0 is of mass $1/2$. Define $R := dd^c g$ and write it as a difference of positive closed currents, $R = (R + S_0) - S_0$. Since $R + S_0$ and S_0 belong to the same cohomology class, they have the same mass $1/2$. We denote as usual by u_{R+S_0} and u_{S_0} the dynamical potentials of $R + S_0$ and S_0 respectively.

A direct computation gives $dd^c(g - u_{R+S_0} + u_{S_0}) = 0$ which implies that $g - u_{R+S_0} + u_{S_0}$ is a constant function. Thus,

$$\Omega(g) = \Omega(u_{R+S_0} - u_{S_0}) \leq \Omega(u_{R+S_0}) + \Omega(u_{S_0}).$$

The assertion follows from Lemma 2.4 applied to $R + S_0, 2S_0$ instead of S, S_0. We use here the fact that $R + S_0 = dd^c g + S_0 \leq 2S_0$ and that $2S_0$ is of mass 1. We also use a constant A which is equal to twice the one in Lemma 2.4. □

The following result gives a quantitative control on the oscillation of u in terms of the oscillation of v. Notice in particular that it implies that, if v is Hölder or $\log^p$-continuous for some $p > 0$, then u enjoys the same property with possibly a loss in the Hölder exponent, but not in the $\log^p$-exponent.

Proposition 2.6. *Let u and v be two p.s.h. functions on $\mathbb{B}^k_3$ such that $dd^c u \leq dd^c v$ and v is continuous. Then u is continuous and for every $0 < s \leq 1$ there is a positive constant A (independent of u and v) such that, for every $0 < r \leq 1/2$, we have*

$$m_{\mathbb{B}^k_1}(u, r) \leq m_{\mathbb{B}^k_2}(v, r^s) + A m_{\mathbb{B}^k_2}(u, r^s) r^{1-s} \leq m_{\mathbb{B}^k_2}(v, r^s) + A\Omega_{\mathbb{B}^k_2}(u) r^{1-s}.$$

Proof. The continuity of u is a well-known property. Indeed, since $dd^c v - dd^c u$ is a positive closed $(1,1)$-current, there is a p.s.h. function u' such that $dd^c u' = dd^c v - dd^c u$. So, both $u + u'$ and v are potentials of $dd^c v$. We deduce that they differ by a pluriharmonic function. Hence $u + u'$ is continuous. We then easily deduce that both u and u' are continuous because both are p.s.h. (and hence u.s.c.).

We prove now the estimate in the lemma. Let $x, y \in \mathbb{B}^k_1$ be such that $\|x - y\| \leq r$. We need to bound $u(y) - u(x)$. Without loss of generality, we can reduce the problem to the case $k = 1$ by restricting ourselves to the complex line through x and y. Moreover, by translating and adding constants to u and v, we can assume that $x=0$, $|y|\le r$, $u(x)=v(x)=0$, and $u(y)\ge 0$. It is then enough to prove that

$$u(y)\le m_{\mathbb{D}_1}(v,r^s)+A\Omega_{\mathbb{D}_{r^s}}(u)r^{1-s}$$

for some positive constant A and for u,v defined on $\mathbb{D}_2$. Note that $\Omega_{\mathbb{D}_{r^s}}(u)\le 2m_{\mathbb{D}_1}(u,r^s)$.

Claim. *We have, for some positive constant A,*

$$u(y)\le \frac{1}{\mathrm{Leb}(\partial\mathbb{D}_{r^s})}\int_{|z|=r^s} u(z)d\,\mathrm{Leb}(z)+A\Omega_{\mathbb{D}_{r^s}}(u)r^{1-s}.$$

Assuming the claim, we first complete the proof of the lemma. Let $\widetilde{u}$ (resp. $\widetilde{v}$) be the radial subharmonic function on $\mathbb{D}_2$ such that $\widetilde{u}(z)$ (resp. $\widetilde{v}(z)$) is equal to the mean value of u (resp. v) on the circle of center 0 and radius $|z|$. Using the Claim, in order to obtain the lemma, it is enough to show that $\widetilde{u}\le\widetilde{v}$.

Recall that $v-u$ is a subharmonic function vanishing at 0. Therefore, $\widetilde{v}-\widetilde{u}$ is a radial subharmonic function vanishing at 0. Radial subharmonic functions are increasing in $|z|$. Thus, $\widetilde{v}-\widetilde{u}$ is a non-negative function and the lemma follows. □

Proof of the Claim. Define $u'(z):=u(zr^s)$ and $y':=y/r^s$. We need to show that, for $|y'|\le r^{1-s}$,

$$u'(y')\le\frac{1}{\mathrm{Leb}(\partial\mathbb{D}_1)}\int_{\partial\mathbb{D}_1}u'(z)d\,\mathrm{Leb}(z)+A\Omega_{\mathbb{D}_1}(u')r^{1-s}.$$

We can assume, without loss of generality, that $y'=\alpha\in\mathbb{R}^+$ and $\alpha\le r^{1-s}$. Consider the automorphism Ψ of the unit disc given by $\Psi(z)=\frac{z+\alpha}{1+\alpha z}$. The map Ψ satisfies $\Psi(0)=y'$ and moreover Ψ extends smoothly to $\partial\mathbb{D}_1$ and tends to the identity in the $\mathcal{C}^1$ norm as $\alpha\to 0$. It follows that $\left\|\Psi^{\pm 1}-\mathrm{id}\right\|_{\mathcal{C}^1}\le A'\alpha\le A'r^{1-s}$ for some positive constant A'.

Define $u'':=u'\circ\Psi$ and denote by ν the normalized standard Lebesgue measure on the unit circle. We deduce from the last inequalities that $\Psi_*\nu-\nu$ is given by a smooth 1-form on $\partial\mathbb{D}_1$ and $\|\Psi_*\nu-\nu\|_\infty=O(r^{1-s})$. Applying the submean inequality to the subharmonic function u'' we get

$$u'(y')=u''(0)\le\langle\nu,u''\rangle=\langle\nu,u'\circ\Psi\rangle=\langle\Psi_*\nu,u'\rangle=\langle\nu,u'\rangle+\langle\Psi_*\nu-\nu,u'\rangle.$$

Since $\Psi_*\nu$ and ν are probability measures, the integral $\langle\Psi_*\nu-\nu,u'\rangle$ does not change if we add to u' a constant c. With the choice $c=-\inf_{\mathbb{D}_1}u'$ (observe that u' is continuous on $\overline{\mathbb{D}}_1$) we get

$$u'(y')\le\int_{\partial\mathbb{D}_1}u'\,d\nu+\sup_{\mathbb{D}_1}|u'+c|\,O(r^{1-s})\le\int_{\partial\mathbb{D}_1}u'\,d\nu+A\Omega_{\mathbb{D}_1}(u')r^{1-s}$$

for some positive constant A. This implies the desired inequality. □

Corollary 2.7. *Let v be a continuous p.s.h. function on $\mathbb{B}^k_3$. Let u be a continuous real-valued function on $\mathbb{B}^k_3$ such that $|dd^cu|\le dd^cv$. Then for every $0<s\le 1$ we have for $0<r\le 1/2$*

$$m_{\mathbb{B}^k_1}(u,r)\le 3m_{\mathbb{B}^k_2}(v,r^s)+A\left(\Omega_{\mathbb{B}^k_2}(u)+\Omega_{\mathbb{B}^k_2}(v)\right)r^{1-s},$$

where A is a positive constant independent of u and v.

Proof. Since $|dd^cu|\le dd^cv$, we have $dd^c(u+v)=dd^cu+dd^cv\ge 0$. So the function $u+v$ is p.s.h.; observe also that $dd^c(u+v)=dd^cu+dd^cv\le 2dd^cv$. Therefore, we can apply Proposition 2.6 to $u+v,2v$ instead of u,v. This gives

$$\begin{aligned}m_{\mathbb{B}^k_1}(u,r)&\le m_{\mathbb{B}^k_1}(u+v,r)+m_{\mathbb{B}^k_1}(v,r)\le m_{\mathbb{B}^k_2}(2v,r^s)+A\Omega_{\mathbb{B}^k_2}(u+v)r^{1-s}+m_{\mathbb{B}^k_2}(v,r)\\&\le 3m_{\mathbb{B}^k_2}(v,r^s)+A\left(\Omega_{\mathbb{B}^k_2}(u)+\Omega_{\mathbb{B}^k_2}(v)\right)r^{1-s},\end{aligned}$$

which is the desired estimate. □

Corollary 2.8. *Let S_0 be a positive closed $(1,1)$-current on $\mathbb{P}^k$ with continuous local potentials. Let $\mathcal{F}(S_0)$ denote the set of all continuous real-valued functions g on $\mathbb{P}^k$ such that $|dd^cg|\le S_0$. Then $\mathcal{F}(S_0)$ is equicontinuous.*

Proof. Let g be as in the statement. We cover $\mathbb{P}^k$ with a finite family of open sets of the form $\Phi_j(\mathbb{B}^k_{1/2})$ where Φ_j is an injective holomorphic map from $\mathbb{B}^k_4$ to $\mathbb{P}^k$. Write $S_0 = dd^c v_j$ for some continuous p.s.h. function v_j on $\Phi_j(\mathbb{B}^k_4)$ and define $V_j := \Phi_j(\mathbb{B}^k_3)$.

We apply Corollary 2.7 to g, v_j restricted to V_j instead of u, v and to $s = 1/2$. Taking into account the distortion of the maps Φ_j, we see that for all r smaller than some constant $r_0 > 0$

$$m_{\mathbb{P}^k}(g, r) \leq 3 \max_j m_{V_j}(v_j, c\sqrt{r}) + A\Big(\Omega_{\mathbb{P}^k}(g) + \max_j \Omega_{V_j}(v_j)\Big)\sqrt{r},$$

where $c \geq 1$ is a constant. Since $\Omega_{\mathbb{P}^k}(g)$ is bounded by Corollary 2.5, the RHS of the last inequality is bounded by a constant ϵ_r which is independent of g and tends to 0 when r tends to 0. It is now clear that the family $\mathcal{F}(S_0)$ is equicontinuous. □

2.4. **Further comparisons.** In our study we will be naturally lead to also consider currents of the form $i\partial u \wedge \bar\partial u$. These currents are always positive. In this section we study the regularity of u under the assumption that $i\partial u \wedge \bar\partial u \leq dd^c v$ for some v of given regularity. Recall that, given a smooth bounded open set $\Omega \subset \mathbb{C}^k$, the Sobolev space $W^{1,2}(\Omega)$ is defined as the space of functions $u\colon \Omega \to \mathbb{R}$ such that $\|u\|_{W^{1,2}(\Omega)} := \|u\|_{L^2(\Omega)} + \|\partial u\|_{L^2(\Omega)} < \infty$, where the reference measure is the standard Lebesgue measure on Ω. The Poincaré-Wirtinger's inequality implies that $\|u\|_{W^{1,2}(\Omega)} \lesssim \|\partial u\|_{L^2(\Omega)} + \big|\int_\Omega u \, d\,\mathrm{Leb}\big|$. We will need the following lemmas.

Lemma 2.9. *There is a universal positive constant c such that*

$$\int_K |u| d\,\mathrm{Leb} \leq c\,\mathrm{Leb}(K)(\log^\star \mathrm{Leb}(K))^{1/2}.$$

for every compact set $K \subset \mathbb{D}_1$ *with* $\mathrm{Leb}\, K > 0$ *and function* $u\colon \mathbb{D}_1 \to \mathbb{R}$ *such that* $\|u\|_{W^{1,2}(\mathbb{D}_1)} \leq 1$.

Proof. By Trudinger-Moser's inequality [Mos71], there are positive constants c_0 and α such that

$$\int_{\mathbb{D}_1} e^{2\alpha|u|^2} d\,\mathrm{Leb} \leq c_0.$$

Let m denote the restriction of the measure Leb to K multiplied by $1/\mathrm{Leb}(K)$. This is a probability measure. It follows from Cauchy-Schwarz's inequality that

$$\int e^{\alpha|u|^2} dm \leq \Big(\int e^{2\alpha|u|^2} dm\Big)^{1/2} \lesssim \mathrm{Leb}(K)^{-1/2}.$$

Observe that the function $t \mapsto e^{\alpha t^2}$ is convex on $\mathbb{R}^+$ and its inverse is the function $t \mapsto \alpha^{-1/2}(\log t)^{1/2}$. By Jensen's inequality, we obtain

$$\int |u| dm \leq \alpha^{-1/2}\Big[\log \int e^{\alpha|u|^2} dm\Big]^{1/2} \lesssim (\log^\star \mathrm{Leb}(K))^{1/2}.$$

The lemma follows. □

Lemma 2.10. *Let* $u\colon \mathbb{D}_2 \to \mathbb{R}$ *be a continuous function and* $\chi\colon \mathbb{D}_2 \to \mathbb{R}$ *a smooth function with compact support in* $\mathbb{D}_2$ *and equal to* 1 *on* $\overline{\mathbb{D}}_1$. *Set* $\chi_z := \partial\chi/\partial z$. *Then we have, for all* $0 < r < s < 1$,

$$u(0) - u(r) = \frac{1}{2\pi}\Big\langle i\partial u, \chi(s^{-1}z)\frac{r}{\overline{z}(\overline{z} - r)} d\overline{z}\Big\rangle + \frac{1}{2\pi}\Big\langle u, \chi_z(s^{-1}z)\frac{r}{s\overline{z}(\overline{z} - r)} idz \wedge d\overline{z}\Big\rangle. \tag{2.1}$$

Proof. Denote by δ_ξ the Dirac mass at $\xi \in \mathbb{C}$. Observe that

$$\frac{i}{2\pi}\partial\frac{d\overline{z}}{\overline{z} - \overline{\xi}} = dd^c \log|z - \xi| = \delta_\xi,$$

where the equalities are in the sense of currents on $\mathbb{C}$. Hence, for $|\xi| < s$,

$$\frac{i}{2\pi}\partial\Big[\frac{\chi(s^{-1}z)d\overline{z}}{\overline{z} - \overline{\xi}}\Big] = \frac{\chi_z(s^{-1}z)idz \wedge d\overline{z}}{2\pi s(\overline{z} - \overline{\xi})} + \chi(s^{-1}z)\delta_\xi = \frac{\chi_z(s^{-1}z)idz \wedge d\overline{z}}{2\pi s(\overline{z} - \overline{\xi})} + \delta_\xi.$$

Applying this identity for $\xi = 0$ and $\xi = r$, and since $u(0) - u(r) = \langle u, \delta_0 - \delta_r \rangle$, we obtain

$$\begin{aligned} u(0) - u(r) &= \frac{1}{2\pi} \Big\langle u, i\partial \Big[\chi(s^{-1}z) \Big(\frac{1}{\overline{z}} - \frac{1}{\overline{z} - r} \Big) d\overline{z} \Big] \Big\rangle - \frac{1}{2\pi s} \Big\langle u, \chi_z(s^{-1}z) \Big(\frac{1}{\overline{z}} - \frac{1}{\overline{z} - r} \Big) idz \wedge d\overline{z} \Big\rangle \\ &= \frac{1}{2\pi} \Big\langle i\partial u, \chi(s^{-1}z) \frac{r}{\overline{z}(\overline{z} - r)} d\overline{z} \Big\rangle + \frac{1}{2\pi} \Big\langle u, \chi_z(s^{-1}z) \frac{r}{s\overline{z}(\overline{z} - r)} idz \wedge d\overline{z} \Big\rangle. \end{aligned}$$

The assertion is proved. □

The following is a main result in this section. It will be a crucial technical tool in the construction of the norms with respect to which the transfer operator has a spectral gap.

Proposition 2.11. *Let $u\colon \mathbb{B}_5^k \to \mathbb{R}$ be continuous and such that $\|\partial u\|_{L^2(\mathbb{B}_5^k)} < \infty$. Assume that $i\partial u \wedge \overline{\partial} u \leq dd^c v$ where $v\colon \mathbb{B}_5^k \to \mathbb{R}$ is continuous, p.s.h., and such that*

$$\int_0^1 m_{\mathbb{B}_4^k}(v, t)(\log\log^\star t)^4 t^{-1} dt < +\infty. \tag{2.2}$$

Then there is a positive constant c such that, for all $0 < r \leq 1/2$, we have

$$\begin{aligned} m_{\mathbb{B}_1^k}(u, r) \leq c \Big(\int_0^{r^{1/2}} m_{\mathbb{B}_4^k}(v, t)(\log\log^\star t)^2 t^{-1} dt \Big)^{1/2} \\ + c m_{\mathbb{B}_4^k}(v, r)^{1/3} \Omega_{\mathbb{B}_4^k}(v)^{1/6} (\log^\star r)^{1/2} + c\Omega_{\mathbb{B}_4^k}(v)^{1/2} r^{1/2} (\log^\star r)^{1/2}. \end{aligned} \tag{2.3}$$

Proof. Let $x, y \in \mathbb{B}_1^k$ be such that $\mathrm{dist}(x, y) \leq r \leq 1/2$. We need to bound $|u(x) - u(y)|$ by the RHS of (2.3). We can assume without loss of generality that $\mathrm{dist}(x, y) = r$. By a change of coordinates and restricting to the complex line through x and y we can reduce the problem to the case of dimension 1. More precisely, we can assume that $x = 0$ and $y = r$ in $\mathbb{C}$ and that u, v are defined on $\mathbb{D}_4$. By subtracting constants, we can assume that $v(0) = 0$ and $\int_{\mathbb{D}_3} u(z) idz \wedge d\overline{z} = 0$. By multiplying u and v by suitable constants γ and γ^2, we can assume that $m_{\mathbb{D}_3}(v, 1) = 1/8$, which implies that $|v| \leq 1$ on $\mathbb{D}_3$. In order to establish (2.3) it is enough to show that $|u(0) - u(r)|$ is bounded by a constant times

$$\Big(\int_0^{r^{1/2}} m_{\mathbb{D}_3}(v, t)(\log\log^\star t)^2 t^{-1} dt \Big)^{1/2} + m_{\mathbb{D}_3}(v, r)^{1/3} (\log^\star r)^{1/2} + r^{1/2} (\log^\star r)^{1/2}. \tag{2.4}$$

Since v is bounded, by Chern-Levine-Nirenberg's inequality [CLN69] the mass of $dd^c v$ on $\mathbb{D}_2$ is bounded by a constant. Thus, by the hypotheses on u and v, the L^2-norm of ∂u on $\mathbb{D}_2$ is bounded by a constant and therefore, by Poincaré-Wirtinger's inequality, $\|u\|_{W^{1,2}(\mathbb{D}_2)}$ is also bounded by a constant.

Fix a smooth function $0 \leq \chi(z) \leq 1$ with compact support in $\mathbb{D}_2$ and such that $\chi = 1$ on $\overline{\mathbb{D}}_1$. Define $\chi_z := \partial\chi/\partial z$ and set

$$s := r \min\big\{ r^{-1/2}, m_{\mathbb{D}_3}(v, r)^{-1/3} \big\}.$$

We have

$$\sqrt{2} r \leq s \leq r^{1/2} < 1 \tag{2.5}$$

because $m_{\mathbb{D}_3}(v, r) \leq m_{\mathbb{D}_3}(v, 1) = 1/8$ and $0 < r \leq 1/2$. The functions u and χ satisfy the assumptions of Lemma 2.10. Thus, (2.1) holds for the above s and r. The second term in the RHS of (2.1) is an integral over $\mathbb{D}_{2s} \setminus \mathbb{D}_s$ because χ_z has support in $\mathbb{D}_2 \setminus \mathbb{D}_1$. Moreover, for $z \in \mathbb{D}_{2s} \setminus \mathbb{D}_s$, we have $\frac{r}{s\overline{z}(z - r)} = O(rs^{-3})$ because of (2.5). Using that $\|u\|_{W^{1,2}(\mathbb{D}_2)} \lesssim 1$ and that $\mathrm{Leb}(\mathbb{D}_{2s}) \lesssim s^2$, Lemma 2.9 implies that the considered term has modulus bounded by a constant times

$$rs^{-1}(\log^\star s)^{1/2} \lesssim \max\big\{ r^{1/2}, m_{\mathbb{D}_3}(v, r)^{1/3} \big\} (\log^\star r)^{1/2}.$$

The last expression is bounded by the sum in (2.4).

In order to conclude, it remains to bound the first term in the RHS of (2.1). Choose a smooth decreasing function $h(t)$ defined for $t > 0$ and such that $h(t) := (-\log t)(\log(-\log t))^2$

for t small enough and $h(t)=1$ for t large enough. Define $\eta(z):=h(|z|)+h(|z-1|)$. We will also use the function $\tilde{v}(z):=v(z)-r^{-1}v(r)\Re(z)$. This function satisfies $dd^c\tilde{v}=dd^cv$ and $\tilde{v}(0)=\tilde{v}(r)=0$. By Cauchy-Schwarz's inequality we have for the first term in the RHS of (2.1)

$$\Big|\Big\langle i\partial u, \chi(s^{-1}z)\frac{r}{\overline{z}(\overline{z}-r)}d\overline{z}\Big\rangle\Big|^2 \leq \big\langle i\partial u\wedge\overline{\partial}u, \chi(s^{-1}z)\eta(r^{-1}z)\big\rangle\int\frac{\chi(s^{-1}z)}{\eta(r^{-1}z)}\frac{r^2}{|z^2(z-r)^2|}idz\wedge d\overline{z}.$$

Using the change of variable $z\mapsto rz$, the fact that $0\leq\chi\leq 1$, and the definition of η we see that the last integral is bounded by

$$\int_{\mathbb{C}}\frac{idz\wedge d\overline{z}}{\big[h(|z|)+h(|z-1|)\big]|z^2(z-1)^2|}.$$

Using polar coordinates for z and for $z-1$ and the definition of h it is not difficult to see that the last integral is finite. Therefore, since $i\partial u\wedge\overline{\partial}u\leq dd^cv=dd^c\tilde{v}$, we get

$$\Big|\Big\langle i\partial u, \chi(s^{-1}z)\frac{r}{\overline{z}(\overline{z}-r)}d\overline{z}\Big\rangle\Big|^2 \lesssim \big\langle dd^c\tilde{v}, \chi(s^{-1}z)\eta(r^{-1}z)\big\rangle.$$

Define $\hat{v}(z):=\tilde{v}(sz)$. The RHS in the last expression is then equal to

$$\big\langle dd^c\hat{v}, \chi(z)\eta(r^{-1}sz)\big\rangle = \big\langle dd^c\hat{v}, \chi(z)h(r^{-1}s|z|)\big\rangle + \big\langle dd^c\hat{v}, \chi(z)h(|r^{-1}sz-1|)\big\rangle.$$

In order to conclude the proof of the proposition, it is enough to show that each term in the last sum is bounded by a constant times

$$\int_0^s m_{\mathbb{D}_3}(v,t)(\log^\star|\log t|)^2t^{-1}dt + m_{\mathbb{D}_3}(v,r)^{2/3}\log^\star r. \tag{2.6}$$

We will only consider the first term. The second term can be treated in a similar way using the coordinate $z':=z-rs^{-1}$. Since h is decreasing, the first term we consider is bounded by

$$\big\langle dd^c\hat{v}, \chi(z)h(|z|)\big\rangle.$$

Claim. *We have*

$$\big\langle dd^c\hat{v}, \chi(z)h(|z|)\big\rangle = \int_{\mathbb{D}_2\setminus\{0\}}\hat{v}(z)dd^c[\chi(z)h(|z|)]. \tag{2.7}$$

We assume the claim for now and conclude the proof of the proposition. Notice that the assumption (2.2) will be used in the proof of this claim.

Using the definitions of h and χ, we can bound the RHS of (2.7) by a constant times

$$\begin{aligned}\int_{\mathbb{D}_2\setminus\{0\}}|\hat{v}(z)z^{-2}|(\log\log^\star|z|)^2idz\wedge d\overline{z} &= \int_{\mathbb{D}_{2s}\setminus\{0\}}|\tilde{v}(z)z^{-2}|(\log\log^\star|z/s|)^2idz\wedge d\overline{z}\\ &\lesssim \int_{\mathbb{D}_{2s}\setminus\{0\}}|\tilde{v}(z)z^{-2}|(\log\log^\star|z|)^2idz\wedge d\overline{z},\end{aligned}$$

where we used the change of variable $z\mapsto sz$ and the fact that $\log^\star|z/s|\lesssim\log^\star|z|$ for $0<|z|<2s<2$. Moreover, by the definition of $\tilde{v}$ and using that $v(0)=\tilde{v}(0)=0$, we have for $|z|<2s$

$$|\tilde{v}(z)|\leq m_{\mathbb{D}_3}(v,|z|)+r^{-1}|v(r)\Re(z)|\leq m_{\mathbb{D}_3}(v,|z|)+m_{\mathbb{D}_3}(v,r)r^{-1}|z|.$$

Therefore, using polar coordinates, we see that the last integral is bounded by a constant times

$$\int_0^{2s}m_{\mathbb{D}_3}(v,t)(\log\log^\star t)^2t^{-1}dt+m_{\mathbb{D}_3}(v,r)r^{-1}s(\log\log^\star(2s))^2.$$

The first term in this sum is bounded by a constant times the integral in (2.6) because $m_{\mathbb{D}_3}(v,t')\leq 4m_{\mathbb{D}_3}(v,t)$ for $s/2\leq t\leq s\leq t'\leq 2s$. The second one is bounded by a constant times the second term in (2.6) by the definition of s and (2.5). The proposition follows. □

Proof of the claim. Observe that $h(|z|)$ tends to infinity when z tends to 0. Let $\vartheta : \mathbb{R} \to \mathbb{R}$ be a smooth increasing concave function such that $\vartheta(t) = t$ for $t \leq 0$ and $\vartheta(t) = 1$ for $t \geq 2$. Define $\vartheta_n(t) := \vartheta(t-n) + n$. This is a sequence of smooth functions increasing to the identity. Define $l(z) := \chi(z)h(|z|)$. Using an integration by parts, we see that the LHS of (2.7) is equal to

$$\begin{aligned}\lim_{n\to\infty} \langle dd^c\hat{v}, \vartheta_n(l(z))\rangle &= \lim_{n\to\infty} \int_{\mathbb{D}_3} \hat{v}(z) dd^c \vartheta_n(l(z)) \\ &= \lim_{n\to\infty} \int_{\mathbb{D}_3} \hat{v}(z)\vartheta_n'(l(z)) dd^c l(z) + \lim_{n\to\infty} \int_{\mathbb{D}_3} \hat{v}(z)\vartheta_n''(l(z)) dl(z) \wedge d^c l(z).\end{aligned}$$

The first term in the last sum converges to the RHS of the identity in the claim using Lebesgue's dominated convergence theorem and (2.2). We need to show that the second term tends to 0. Since $\chi(z) = 1$ for z near 0, for n large enough, the considered term has an absolute value bounded by a constant times

$$\lim_{n\to\infty} \int_{\{h(|z|)>n\}} |\hat{v}(z)| i\partial h(|z|) \wedge \overline{\partial} h(|z|) \lesssim \lim_{n\to\infty} \int_{\{h(|z|)>n\}} |\hat{v}(z) z^{-2}| (\log\log^\star |z|))^4 i dz \wedge d\overline{z}.$$

Using the arguments as at the end of the proof of Proposition 2.11 and the assumption (2.2) on v, we see that the last integrand is an integrable function on $\mathbb{D}_1$. Since the set $\{h(|z|) > n\}$ decreases to $\{0\}$ when n tends to infinity, the last limit is zero according to Lebesgue's dominated convergence theorem. This ends the proof of the claim. □

Corollary 2.12. *Let S_0 be a positive closed $(1,1)$-current on $\mathbb{P}^k$ whose dynamical potential u_S satisfies $\|u_S\|_{\log^p} \leq 1$ for some $p > 3/2$. Let $\mathcal{F}(S_0)$ denote the set of all continuous functions $g \colon \mathbb{P}^k \to \mathbb{R}$ such that $i\partial g \wedge \bar{\partial} g \leq S_0$. Then for any positive number $q < \frac{p}{3} - \frac{1}{2}$ we have $\|g\|_{\log^q} \leq c$ for some positive constant $c = c(p,q)$. In particular, the family $\mathcal{F}(S_0)$ is equicontinuous.*

Proof. Notice that (2.2) is satisfied for all v such that $\|v\|_{\log^p} < \infty$ for some $p > 1$. It follows that if u and v are as in Proposition 2.11 and v is $\log^p$-continuous for some $p > 1$ then u is $\log^q$-continuous on $\mathbb{B}_1^k$ for all q as in the statement, with $\|u\|_{\mathbb{B}_1^k, \log^q} \leq c \, \|v\|^{1/2}_{\mathbb{B}_5^k, \log^p}$ for some positive constant c independent of u, v. The result is thus deduced from Proposition 2.11 by means of a finite cover of $\mathbb{P}^k$, in the same way as in the proof of Corollary 2.8. □

3. Some semi-norms and equidistribution properties

In this section we consider the action of the operator $(f^n)_*$ on functions and currents. We also introduce the semi-norms which are crucial in our study. Some results and ideas here are of independent interest. Recall that we always assume that f satisfies the Assumption **(A)** in the Introduction.

3.1. Bounds on the potentials of $(f^n)_*\omega_{\rm FS}$.

We start by giving estimates on the potentials of the currents $(f^n)_*\omega_{\rm FS}$. As explained in the Introduction, these estimates will allow us to globally control the distortion of f^n. Define

$$\omega_n := d^{-(k-1)n} (f^n)_* \omega_{\rm FS}.$$

Recall that f_* multiplies the mass of a positive closed $(1,1)$-current by d^{k-1}. Therefore, all currents ω_n have unit mass. We denote by u_n the dynamical potential of ω_n. In particular, u_0 is the dynamical potential of $\omega_{\rm FS}$. It is known that u_0 is Hölder continuous, see [Kos97, DS10a].

Observe that $d^{-1} f^* \omega_{\rm FS}$ is a smooth positive closed $(1,1)$-form of mass 1. Therefore, there is a unique smooth function v such that

$$dd^c v = d^{-1} f^* \omega_{\rm FS} - \omega_{\rm FS} \qquad \text{and} \qquad \langle \mu, v \rangle = 0.$$

Lemma 3.1. *We have*

$$u_n = d^{-(k-1)n} (f^n)_* u_0 \qquad \textit{and} \qquad u_0 = -\sum_{n=0}^{\infty} d^{-n} v \circ f^n.$$

Proof. We prove the first identity. Denote by u'_n the RHS of this identity, which is a continuous function. By the definition of u_n and the invariance of T, we have

$$dd^c(u_n - u'_n) = (\omega_n - T) - d^{-(k-1)n}(f^n)_*(\omega_{\rm FS} - T) = (\omega_n - T) - (\omega_n - T) = 0.$$

Therefore, $u_n - u'_n$ is pluriharmonic and hence constant on $\mathbb{P}^k$. Moreover, the invariance of μ implies that

$$\langle \mu, u'_n\rangle = d^{-(k-1)n}\langle (f^n)^*\mu, u_0\rangle = d^n\langle \mu, u_0\rangle = 0.$$

By the definition of u_n, we also have $\langle \mu, u_n\rangle = 0$. We deduce that $u_n = u'_n$, which implies the first identity in the lemma.

It is clear that the sum in the RHS of the second identity in the lemma converges uniformly. Therefore, this RHS is a continuous function that we denote by u'_0. The invariance of μ also implies that $\langle \mu, u'_0\rangle = 0$. A direct computation gives

$$dd^c u'_0 = \lim_{N\to\infty}\Big(-\sum_{n=0}^{N-1} d^{-n} dd^c(v\circ f^n)\Big) = \lim_{N\to\infty} \omega_{\rm FS} - d^{-N}(f^N)^*\omega_{\rm FS} = \omega_{\rm FS} - T,$$

where the last identity is a consequence of the definition of T. Since $dd^c u_0$ is also equal to $\omega_{\rm FS} - T$, we obtain that $u_0 - u'_0$ is constant on $\mathbb{P}^k$. Finally, using that

$$\langle \mu, u_0\rangle = \langle \mu, u'_0\rangle = 0,$$

we conclude that $u_0 = u'_0$. This ends the proof of the lemma. □

In the sequel, we will need explicit bounds on the oscillation $\Omega(u_n)$ of u_n. These are provided in the next result.

Lemma 3.2. *For every constant $A > 1$, there exists a positive constant c independent of n such that $\|u_n\|_\infty \le cA^n$ and $\Omega(u_n) \le cA^n$ for all $n \ge 0$.*

Proof. Observe that the second assertion is deduced from the first one by replacing c with $2c$. We prove now the first assertion. By Lemma 3.1 we have, for any given $z \in \mathbb{P}^k$,

$$\begin{aligned} u_n(z) &= d^{-(k-1)n}\left((f^n)_* u_0\right)(z) = \left\langle \delta_z, d^{-(k-1)n}(f^n)_* u_0\right\rangle \\ &= d^n\left\langle d^{-kn}(f^n)^*\delta_z, u_0\right\rangle = d^n\Big\langle d^{-kn}(f^n)^*\delta_z, -\sum_{m=0}^{\infty} d^{-m} v\circ f^m\Big\rangle \\ &= -d^n\Big\langle d^{-kn}(f^n)^*\delta_z, \sum_{m=0}^{n} d^{-m} v\circ f^m\Big\rangle - \Big\langle d^{-kn}(f^n)^*\delta_z, \sum_{m=n+1}^{\infty} d^{-m+n} v\circ f^m\Big\rangle. \end{aligned}$$

The absolute value of the second term in the last line is bounded by $\|v\|_\infty$ because $d^{-kn}(f^n)^*\delta_z$ is a probability measure. Observe that $(f^n)_*(v\circ f^m) = d^{km}(f^{n-m})_* v$ for all $n \ge m$. Hence, the absolute value of the first term is equal to

$$\Big|\sum_{m=0}^{n} d^{n-m}\Big\langle \delta_z, d^{-k(n-m)}(f^{n-m})_* v\Big\rangle\Big| \le \sum_{j=0}^{n} d^j \|d^{-kj}(f^j)_* v\|_\infty. \tag{3.1}$$

Under the Assumption **(A)**, it is known that $\|d^{-kj}(f^j)_* v\|_\infty \lesssim \delta^{-j}$ for every $0 < \delta < d$. Indeed, the Assumption **(A)** implies the property **(A1)** below, see [DS10b, Cor. 1.2].

(A1) Let $g\colon \mathbb{P}^k \to \mathbb{R}$ be $\mathcal{C}^2$ and such that $\langle \mu, g\rangle = 0$. For every constant $1 < \delta < d$, there is a positive constant c independent of g and n such that

$$\|d^{-kn}(f^n)_* g\|_\infty \le c\|g\|_{\mathcal{C}^2}\delta^{-n}.$$

By choosing $\delta > d/A$, we can bound the RHS of (3.1) by a constant times A^n. This ends the proof of the lemma. □

As an application of the previous estimates, we have the following lemma that can be used to study the regularity of functions $g\colon \mathbb{P}^k \to \mathbb{R}$.

Lemma 3.3. *Let $g\colon \mathbb{P}^k \to \mathbb{R}$ be a continuous function and $0 < \beta < 1$ a constant such that*

$$|dd^c g| \leq \sum_{n=0}^{\infty} \beta^n \omega_n. \tag{3.2}$$

Then, for every $q > 0$, there is a positive constant $c = c(q, \beta)$ independent of g such that

$$\|g\|_{\log^q} \leq c.$$

Proof. We bound the continuity modulus $m(g, r)$ of g by means of Corollary 2.7. We only need to consider $0 < r \leq 1/2$. For this purpose, since T has Hölder continuous local potentials, it suffices to bound the continuity modulus of the dynamical potential of the RHS of (3.2), see also Lemma 3.7 below. This dynamical potential is equal to

$$u := \sum_{n=0}^{\infty} \beta^n u_n.$$

Fix a constant $1 < A < 1/\beta$. By Lemma 3.2, we have $\|u_n\|_\infty \lesssim A^n$. Hence, for every N, we have

$$m(u, r) \lesssim \sum_{n \leq N} \beta^n m(u_n, r) + \sum_{n > N} (A\beta)^n \lesssim \sum_{n \leq N} \beta^n m(u_n, r) + (A\beta)^N.$$

Applying [DS10b, Cor. 4.4] inductively to some iterate of f, we see that the Assumption **(A)** implies:

(A2) for every constant $\kappa > 1$, there are an integer $n_\kappa \geq 0$ and a constant $c_\kappa > 0$ independent of n such that for all $x, y \in \mathbb{P}^k$ and $n \geq n_\kappa$ we can write $f^{-n}(x) = \{x_1, \ldots, x_{d^{kn}}\}$ and $f^{-n}(y) = \{y_1, \ldots, y_{d^{kn}}\}$ (counting multiplicity) with the property that

$$\operatorname{dist}(x_j, y_j) \leq c_\kappa \operatorname{dist}(x, y)^{1/\kappa^n} \quad \text{for } \ j = 1, \ldots, d^{kn}.$$

By definition, the function u_0 is γ-Hölder continuous for some Hölder exponent γ because T has Hölder continuous local potentials. The above property **(A2)** implies that $(f^n)_* u_0$ is $\gamma\kappa^{-n}$-Hölder continuous for all $n \geq n_\kappa$. More precisely, we have

$$m(d^{-kn}(f^n)_* u_0, r) \leq c' r^{\gamma\kappa^{-n}} \qquad \text{and hence} \qquad m(u_n, r) \leq c' d^n r^{\gamma\kappa^{-n}}$$

for some positive constant c' independent of $n \geq n_\kappa$ and r. Observe also that for $0 \leq n \leq n_\kappa$ all the u_n are α_κ-Hölder continuous for some $\alpha_\kappa > 0$. Indeed, as the multiplicity of f^n at a point is at most d^{kn}, we have (see again [DS10b, Cor. 4.4]):

(A2') there is a constant $c_0 > 0$ such that for every $n \geq 0$, for all $x, y \in \mathbb{P}^k$, we can write $f^{-n}(x) = \{x_1, \ldots, x_{d^{kn}}\}$ and $f^{-n}(y) = \{y_1, \ldots, y_{d^{kn}}\}$ (counting multiplicity) with the property that

$$\operatorname{dist}(x_j, y_j) \leq c_0 \operatorname{dist}(x, y)^{1/d^{kn}} \quad \text{for } \ j = 1, \ldots, d^{kn}.$$

Therefore, we have

$$m(u, r) \lesssim r^{\alpha_\kappa} + \sum_{n_\kappa \leq n \leq N} (\beta d)^n \, r^{\gamma\kappa^{-n}} + (A\beta)^N. \tag{3.3}$$

Choose κ close enough to 1 so that $2q \log \kappa < |\log(A\beta)|$ and take

$$N = \frac{1}{2 \log \kappa} \log |\log r|$$

(recall that we only need to consider $r \leq 1/2$). Then, the last term in (3.3) satisfies

$$(A\beta)^N = e^{N \log(A\beta)} < e^{-2Nq \log \kappa} = |\log r|^{-q}.$$

It remains to prove that the sum in (3.3) satisfies a similar estimate. We have

$$\sum_{n \leq N} (\beta d)^n \, r^{\gamma\kappa^{-n}} \leq \sum_{n \leq N} \beta^n d^N r^{\gamma\kappa^{-N}} \lesssim d^N r^{\gamma\kappa^{-N}} = e^{\frac{\log d}{2 \log \kappa} \log|\log r|} e^{\gamma (\log r) e^{-\frac{1}{2} \log|\log r|}} = \frac{|\log r|^{\frac{\log d}{2\log\kappa}}}{e^{\gamma\sqrt{|\log r|}}}.$$

The last expression is smaller than a constant times $|\log r|^{-q}$ because $e^t \gg t^M$ when $t \to \infty$ for every $M \geq 0$. This, together with the above estimates, gives $m(u,r) \lesssim |\log r|^{-q}$ and ends the proof of the lemma. $\square$

Remark 3.4. The conclusion of Lemma 3.3 stays true if the assumption (3.2) is replaced by

$$i\partial g \wedge \bar{\partial} g \leq \sum_{n=0}^{\infty} \beta^n \omega_n.$$

It suffices to use Corollary 2.12 instead of Corollary 2.7.

3.2. **Bounds with respect to the semi-norm $\|\cdot\|_{\log^p}$.** In this section we study the action of the operator f_* on functions with bounded semi-norm $\|\cdot\|_{\log^p}$. We first prove that, with respect to this semi-norm, the operator f_* is Lipschitz.

Lemma 3.5. *For every constant $A > 1$, there exists a positive constant $c = c(A)$ such that for every $n \geq 0$, $p > 0$, and continuous function $g\colon \mathbb{P}^k \to \mathbb{R}$, we have*

$$\left\|d^{-kn}(f^n)_* g\right\|_{\log^p} \leq c^p A^{pn} \|g\|_{\log^p}.$$

Proof. We use the notations in the proof of Lemma 3.3 and the properties **(A2)** and **(A2')** stated there. We have

$$\left\|d^{-kn}(f^n)_* g\right\|_{\log^p} = \sup_{x,y \in \mathbb{P}^k} d^{-kn} \left|(f^n)_* g(x) - (f^n)_* g(y)\right| \cdot (\log^\star \mathrm{dist}(x,y))^p.$$

We need to bound the RHS by $c^p A^{pn} \|g\|_{\log^p}$. Fix $\kappa < A$. We have, for $n \geq n_\kappa$,

$$\begin{aligned} d^{-kn} \left|(f^n)_* g(x) - (f^n)_* g(y)\right| (\log^\star \mathrm{dist}(x,y))^p &\leq \max_j |g(x_j) - g(y_j)| \, (\log^\star \mathrm{dist}(x,y))^p \\ &\leq \max_j \frac{\|g\|_{\log^p}}{(\log^\star \mathrm{dist}(x_j,y_j))^p} (\log^\star \mathrm{dist}(x,y))^p \\ &= \max_j \Big(\frac{\log^\star \mathrm{dist}(x,y)}{\kappa^n \log^\star \mathrm{dist}(x_j,y_j)}\Big)^p \|g\|_{\log^p} \kappa^{pn}. \end{aligned}$$

We need to check that the expression in the last parentheses is bounded by a constant. Fix a large constant $M > 0$. Since $\log^\star \mathrm{dist}(x_j,y_j)$ is bounded from below by 1, when $\log^\star \mathrm{dist}(x,y)$ is bounded by $2M\kappa^n$ the considered expression is bounded by some constant c as desired. Assume now that $\log^\star \mathrm{dist}(x,y) \geq 2M\kappa^n$. Since M is large, we deduce that $\log \mathrm{dist}(x,y) \leq -2M\kappa^n + 1 \leq -M\kappa^n$. Hence, by **(A2)**, since M is large, we have

$$\log \mathrm{dist}(x_j,y_j) \leq \log c_\kappa + \kappa^{-n} \log \mathrm{dist}(x,y) \leq \frac{1}{2} \kappa^{-n} \log \mathrm{dist}(x,y).$$

It is now clear that $\kappa^n \log^\star \mathrm{dist}(x_j,y_j) \geq \frac{1}{2} \log^\star \mathrm{dist}(x,y)$ which implies that the considered expression is bounded, as desired. This implies the lemma for $n \geq n_\kappa$.

When $n \leq n_\kappa$, it is enough to use **(A2')** instead of **(A2)**. Since n_κ is fixed it is clear that $\frac{\log^\star \mathrm{dist}(x,y)}{\log^\star \mathrm{dist}(x_j,y_j)} \lesssim d^{kn_\kappa}$, which is bounded. The proof is complete. $\square$

We will need the following result which is an improvement of [DS10b, Th. 1.1] in the case where f satisfies the Assumption **(A)**. This assumption is likely necessary here to get the estimate in the norm $\|\cdot\|_\infty$.

Theorem 3.6. *Let f be an endomorphism of $\mathbb{P}^k$ of algebraic degree $d \geq 2$ and satisfying the Assumption* **(A)**. *Consider a real number $p > 0$. Let $g\colon \mathbb{P}^k \to \mathbb{R}$ be such that $\|dd^c g\|_* \leq 1$, $\langle \mu, g\rangle = 0$ and $\|g\|_{\log^p} \leq 1$. Then, for every constant $\eta > d^{-p/(p+1)}$, there is a positive constant c independent of g such that for every $n \geq 0$*

$$\left\|d^{-kn}(f^n)_* g\right\|_\infty \leq c\eta^n.$$

Proof. Set $g_n := d^{-kn}(f^n)_* g$. Recall (see, e.g., [Sko72, DNS10]) that there exists a positive constant c_0 independent of g and n such that

$$\int_{\mathbb{P}^k} e^{d^n|g_n|} \leq c_0, \tag{3.4}$$

where the integral above is taken with respect to the Lebesgue measure associated to the volume form ω_{FS}^k on $\mathbb{P}^k$.

Fix a constant $A > 1$ such that $\eta > (A/d)^{p/(p+1)}$. Suppose by contradiction that for infinitely many n there exists a point $a_n \in \mathbb{P}^k$ such that $|g_n(a_n)| \geq 3\eta^n$ for some g as above. Choose $r := e^{-c_A A^n \eta^{-n/p}}$ with c_A the constant given by Lemma 3.5 (we write c_A instead of c in order to avoid confusion). By that lemma, when $\text{dist}(z, a_n) < r$, we have

$$|g_n(z)| \geq |g_n(a_n)| - |g_n(z) - g_n(a_n)| \geq 3\eta^n - c_A^p A^{pn}(1 + |\log r|)^{-p} \geq \eta^n.$$

This implies that

$$c_0 \geq \int_{\mathbb{P}^k} e^{d^n|g_n|} \geq \int_{\text{dist}(z,a_n)<r} e^{d^n|g_n(z)|} \gtrsim r^{2k} e^{d^n \eta^n} \gtrsim e^{-2kc_A A^n \eta^{-n/p} + d^n \eta^n}.$$

By the choice of A, the last expression diverges when n tends to infinity. This is a contradiction. The theorem follows. □

3.3. **The semi-norm $\|\cdot\|_p$.** In this section we combine the semi-norms $\|\cdot\|_{\log^p}$ and $\|\cdot\|_*$ to build a new semi-norm $\|\cdot\|_p$ and study its first properties. For our convenience we will use dynamical potentials of currents, but this is avoidable.

For every positive closed $(1,1)$-current S on $\mathbb{P}^k$ we first define

$$\|S\|'_p := \|S\| + \|u_S\|_{\log^p},$$

where u_S is the dynamical potential of S. When R is any $(1,1)$-current we define

$$\|R\|_p = \min \|S\|'_p,$$

where the minimum is taken over all positive closed $(1,1)$-currents S such that $|R| \leq S$, and we set $\|R\|_p := \infty$ when no such S exists. Finally, for all $g\colon \mathbb{P}^k \to \mathbb{R}$, define

$$\|g\|_p := \|dd^c g\|_p.$$

The following lemma shows in particular that the norm $\|\cdot\|_p$ is equivalent to the norm $\|\cdot\|'_p$ when both are defined. We will thus just consider the norm $\|\cdot\|_p$ in the sequel.

Lemma 3.7. *Let S be a positive closed $(1,1)$-current on $\mathbb{P}^k$ and let $g\colon \mathbb{P}^k \to \mathbb{R}$ be a continuous function. Then*

$$\|u_S\|_p \leq 2\|S\|'_p, \qquad \|S\|_p \leq \|S\|'_p \leq c\,\|S\|_p, \qquad \text{and} \qquad \|g\|_{\log^p} \leq c\,\|g\|_p$$

for some positive constant $c = c(p)$ independent of S and g.

Proof. Define $m := \|S\|$. We have $\|S\|'_p \geq m$. Since $u_T = 0$, we have $\|T\|_p = \|T\|'_p = 1$ and

$$\|u_S\|_p = \|dd^c u_S\|_p = \|S - mT\|_p \leq \|S\|_p + m\|T\|_p \leq \|S\|'_p + m \leq 2\|S\|'_p.$$

This proves the first assertion in the lemma.

We prove now the second assertion. The first inequality is true by definition. For the second one, it is enough to prove that if $\tilde{S}$ is also positive closed, and such that $S \leq \tilde{S}$, then $\|S\|'_p \leq c\|\tilde{S}\|'_p$ for some constant c independent of $S, \tilde{S}$. It is clear that $\|S\| \leq \|\tilde{S}\|$. So we can assume that $\|\tilde{S}\| = 1$ and we only need to check that $\|u_S\|_{\log^p}$ is bounded by $c(1 + \|u_{\tilde{S}}\|_{\log^p})$ for some constant c. For this purpose, we will apply Corollary 2.7 for the charts $\Phi_j(\mathbb{B}_4^k)$ of a finite atlas of $\mathbb{P}^k$ as in Corollary 2.8.

By Corollary 2.5, we have that $\Omega(u_S)$ is bounded by a constant. Let h_j denote a potential of T on $\Phi_j(\mathbb{B}_4^k)$. This is a Hölder continuous function. By definition of dynamical potential,

$v_j := u_{\tilde S} + h_j$ is a potential of $\tilde S$ on $\Phi_j(\mathbb{B}^k_4)$. Using the notation in the proof of Corollary 2.8, as in that corollary, we obtain for $r \le r_0$

$$m_{\mathbb{P}^k}(u_S, r) \lesssim \max_j m_{V_j}(v_j, c\sqrt{r}) + \sqrt{r} \lesssim (1 + \|u_{\tilde S}\|_{\log^p})|\log r|^{-p}.$$

This proves the second assertion.

Finally, let us consider the last inequality in the statement. By linearity, we can assume that there exists a positive closed $(1,1)$-current $\tilde S$ of mass $\|\tilde S\| = 1$ such that $|dd^c g| \le \tilde S$ and prove that $\|g\|_{\log^p} \le c(1 + \|u_{\tilde S}\|_{\log^p})$ for some positive constant c independent of g and $\tilde S$. Observe that $dd^c g + \tilde S$ is a positive closed current and $dd^c g + \tilde S \le 2\tilde S$. So we can apply the arguments in the previous paragraph to $g + u_{\tilde S}, 2u_{\tilde S}$ instead of $u_S, u_{\tilde S}$. We obtain $\|g + u_{\tilde S}\|_{\log^p} \lesssim 1 + \|u_{\tilde S}\|_{\log^p}$. This implies the last assertion, and completes the proof of the lemma. □

The following lemma can be applied to a non-convex $\mathcal{C}^2$ function χ as it can be written as the difference of two convex functions. We can also apply it to χ Lipschitz and convex because such a function can be approximated by smooth convex functions with bounded first derivatives. The statement for the smooth convex case that we need is however simpler.

Lemma 3.8. *There is a positive constant $c = c(p)$ such that for all continuous functions $g, h\colon \mathbb{P}^k \to \mathbb{R}$ with finite $\|\cdot\|_p$ semi-norms and any $\mathcal{C}^2$ convex function $\chi : \mathbb{R} \to \mathbb{R}$ we have*

$$\big\|\partial g \wedge \bar\partial h\big\|_p \le c(\Omega(g)\|h\|_p + \|g\|_p\Omega(h)) \quad \text{and} \quad \|\chi(g)\|_p \le c\|\chi'(g)\|_\infty \|g\|_p.$$

Proof. We can write $|dd^c g| \le S$ with $\|S\| \lesssim \|g\|_p$ and $\|u_S\|_{\log^p} \lesssim \|g\|_p$. We first prove the second inequality. Set $A := \|\chi'(g)\|_\infty$. We have

$$dd^c\chi(g) = \chi'(g)dd^c g + \frac{1}{\pi}\chi''(g) i\partial g \wedge \overline\partial g = \Big[\chi'(g)dd^c g + AS + \frac{1}{\pi}\chi''(g) i\partial g \wedge \overline\partial g\Big] - [AS].$$

Write the last expression as $R^+ - R^-$ where R^+ (resp. R^-) is the expression in the first (resp. second) brackets. Using the definition of A, the inequality $|dd^c g| \le S$, the convexity of χ, and the fact that $i\partial g \wedge \overline\partial g$ is always positive, we deduce that both R^+ and R^- are positive currents. Clearly, the current R^- is closed and its mass is $\lesssim A\|g\|_p$. The current R^+ is cohomologous to R^- because $dd^c\chi(g)$ is an exact current. It follows that R^+ is also a positive closed current of mass $\lesssim A\|g\|_p$.

We have $\|u_{R^-}\|_{\log^p} = A\|u_S\|_{\log^p} \lesssim A\|g\|_p$. This and the above estimate for the mass imply that $\|R^-\|_p \lesssim A\|g\|_p$. On the other hand, the above identities imply that $\chi(g) + u_{R^-}$ differs from u_{R^+} by a constant. We deduce that

$$\|u_{R^+}\|_{\log^p} \le \|u_{R^-}\|_{\log^p} + \|\chi(g)\|_{\log^p} \lesssim \|u_{R^-}\|_{\log^p} + A\|g\|_{\log^p} \lesssim A\|g\|_p.$$

Therefore, we also have $\|R^+\|_p \lesssim A\|g\|_p$. It is now clear that $\|dd^c\chi(g)\|_p \lesssim A\|g\|_p$. Hence, we get the second assertion in the lemma.

For the first assertion, we first consider the case where $g = h$. We can replace g by $g - \min g$ in order to assume that $\min g = 0$ and hence $\|g\|_\infty = \Omega(g)$. The above computation gives

$$0 \le i\partial g \wedge \bar\partial g = \pi\big(dd^c g^2 - 2g dd^c g\big) \lesssim dd^c g^2 + 2\|g\|_\infty S.$$

We thus have

$$\|i\partial g \wedge \bar\partial g\|_p \lesssim \|dd^c g^2\|_p + \|g\|_\infty \|S\|_p \lesssim \|dd^c g^2\|_p + \|g\|_\infty \|g\|_p.$$

We obtain the desired estimate by applying the second assertion in the lemma to $\|dd^c g^2\|_p$, using the function $\chi(t) := t^2$.

Finally, let us consider the first inequality for g and h. As above, we can assume that $\min h = 0$ and hence $\|h\|_\infty = \Omega(h)$. It follows from Cauchy-Schwarz's inequality that

$$\big|\partial g \wedge \bar\partial h \pm \partial h \wedge \bar\partial g\big| \lesssim \frac{\|h\|_\infty}{\|g\|_\infty} i\partial g \wedge \bar\partial g + \frac{\|g\|_\infty}{\|h\|_\infty} i\partial h \wedge \bar\partial h.$$

The assertion thus follows from the particular case considered above. This completes the proof of the lemma. □

3.4. **The dynamical norm $\|\cdot\|_{p,\alpha}$.** In this section we define the main norms $\|\cdot\|_{p,\alpha}$ for $(1,1)$-currents that we will use to quantify the convergence (1.2). Based on the results in the previous sections, we will see later that these norms satisfy the inequalities

$$\|\cdot\|_q \lesssim \|\cdot\|_{p,\alpha} \lesssim \|\cdot\|_p$$

for some explicit q depending on p, α, and d. In particular, the new norms are at the same time weaker than the previous norm $\|\cdot\|_p$, but still inherit the main properties of a similar norm $\|\cdot\|_q$ which are obtained in the previous section.

Definition 3.9. Given a positive closed $(1,1)$-current S on $\mathbb{P}^k$ and a real number α such that $d^{-1} \leq \alpha < 1$, we define the current S_α by

$$S_\alpha := \sum_{n=0}^{\infty} \alpha^n \frac{(f^n)_*(S)}{d^{(k-1)n}}\cdot$$

For any $(1,1)$-current R on $\mathbb{P}^k$ and real number $p > 0$, we define

$$\|R\|_{p,\alpha} := \inf\Big\{c \in \mathbb{R} \colon \exists S \text{ positive closed} \colon \|S\|_p \leq 1, |R| \leq cS_\alpha\Big\} \tag{3.5}$$

and we set $\|R\|_{p,\alpha} := \infty$ if such a number c does not exist.

Note that when $\|R\|_{p,\alpha}$ is finite, by compactness, the infimum in (3.5) is actually a minimum. We have the following lemma where the assumption $d^{-1} \leq \alpha < d^{-1/(p+1)}$ is equivalent to $0 < q_0 \leq p$.

Lemma 3.10. *Let α and p be positive and such that $d^{-1} \leq \alpha < d^{-1/(p+1)}$. Then, for every $0 < q < q_0 := \frac{|\log \alpha|}{\log d}(p+1) - 1$, there are positive constants $c_1 = c_1(p, \alpha)$ and $c_2 = c_2(p, \alpha, q)$ such that, for every $(1,1)$-current R,*

$$\|R\|_{p,\alpha} \leq c_1 \|R\|_p \qquad \text{and} \qquad \|R\|_q \leq c_2 \|R\|_{p,\alpha}.$$

Proof. The first inequality holds by the definition of $\|\cdot\|_{p,\alpha}$ and Lemma 3.7. We prove the second inequality. Consider a current R such that $\|R\|_{p,\alpha} = 1$. We have to show that $\|R\|_q$ is bounded by a constant.

From the definition of $\|\cdot\|_{p,\alpha}$, we can find a positive closed current S such that $\|S\|_p = 1$ and $|R| \leq S_\alpha$. By the definition of the norm $\|\cdot\|_q$ and Lemma 3.7 applied to S_α, it is enough to show that $\|S_\alpha\|'_q$ is bounded. Denote by u_α the dynamical potential of S_α. Since the mass of S_α is bounded, we only need to show that $\|u_\alpha\|_{\log^q}$ is bounded. By definition of S_α, we have

$$u_\alpha = \sum_{n=0}^{\infty} \alpha^n \frac{(f^n)_* u_S}{d^{(k-1)n}}\cdot$$

It follows that, for every positive number N,

$$m(u_\alpha, r) \leq \sum_{n \leq N} \alpha^n d^{-(k-1)n} \|(f^n)_* u_S\|_{\log^p} (\log^\star r)^{-p} + 2 \sum_{n > N} \alpha^n d^{-(k-1)n} \|(f^n)_* u_S\|_\infty.$$

Fix constants $A > 1$ close enough to 1, $\eta > d^{-p/(p+1)}$ close enough to $d^{-p/(p+1)}$ and $\alpha' > \alpha A^p$ close enough to α. In particular, by the assumption on α and the choice of η, we have that $\alpha d \eta$ is close to $\alpha d^{1/(p+1)}$ and smaller than 1. By Lemma 3.5 and Theorem 3.6 we know that $\|(f^n)_* u_S\|_{\log^p} \lesssim d^{kn} A^{pn}$ and $\|(f^n)_* u_S\|_\infty \lesssim d^{kn} \eta^n$. This, the above estimate on $m(u_\alpha, r)$, and the fact that $\alpha' d > \alpha d \geq 1$ imply that

$$m(u_\alpha, r) \lesssim \sum_{n \leq N} \alpha^n d^n A^{pn} (\log^\star r)^{-p} + \sum_{n > N} \alpha^n d^n \eta^n \lesssim (\alpha' d)^N (\log^\star r)^{-p} + (\alpha d \eta)^N.$$

Finally, choose $N = \frac{p+1}{\log d} \log \log^\star r$. Observe that if we replace α' by α and $\alpha d\eta$ by $\alpha d^{1/(p+1)}$, the last sum is equal to $2(\log^\star r)^{-q_0}$. So, this sum is bounded by a constant times $(\log^\star r)^{-q}$ for $q < q_0$ because α' is chosen close to α and $\alpha d\eta$ is close to $\alpha d^{1/(p+1)}$. This concludes the proof of the lemma. □

The following shifting property of the norm $\|\cdot\|_{p,\alpha}$ is very useful when we work with the action of f, and is the key property that we need of this norm.

Lemma 3.11. *For every $n \geq 0$ and every $(1,1)$-current R on $\mathbb{P}^k$, we have*

$$\left\| d^{-kn} \left(f^n\right)_* R \right\|_{p,\alpha} \leq \frac{1}{d^n \alpha^n} \left\| R \right\|_{p,\alpha}.$$

Proof. We can assume that $\|R\|_{p,\alpha} = 1$, so that there is a positive closed current S with $\|S\|_p = 1$ and $|R| \leq S_\alpha$, see the Notation at the beginning of the paper. Consider any function $\xi \colon \mathbb{P}^k \to \mathbb{C}$ such that $|\xi| \leq 1$ and define $\xi_n := \xi \circ f^n$. Since $|\xi_n| \leq 1$, we have

$$\Re\Big(\xi d^{-kn} \left(f^n\right)_* R\Big) = \frac{1}{d^n} \Re\Big(\frac{\left(f^n\right)_* \left(\xi_n R\right)}{d^{(k-1)n}}\Big) \leq \frac{1}{d^n \alpha^n} \sum_{j=0}^{\infty} \alpha^{n+j} \frac{(f^n)_*}{d^{(k-1)n}} \frac{(f^j)_*}{d^{(k-1)j}} S \leq \frac{1}{d^n \alpha^n} S_\alpha.$$

The lemma follows. □

3.5. **The dynamical Sobolev semi-norm $\|\cdot\|_{\langle p,\alpha\rangle}$.** We can now define the first semi-norm for functions $g \colon \mathbb{P}^k \to \mathbb{R}$ with respect to which we will be able to prove the existence of a spectral gap for the transfer operator. We can also define this norm for 1-forms.

Definition 3.12. Let α be a real number such that $d^{-1} \leq \alpha < 1$. For any function $g \colon \mathbb{P}^k \to \mathbb{R}$ we set

$$\|g\|_{\langle p,\alpha\rangle} := \left\| i\partial g \wedge \bar{\partial} g \right\|_{p,\alpha}^{1/2}.$$

The following two lemmas give the main properties of the semi-norm $\|\cdot\|_{\langle p,\alpha\rangle}$ that we will need in Section 5, together with Lemma 3.11. Recall that q_0 is defined in Lemma 3.10. Note that the hypothesis $p > 3/2$ ensures that $d^{-1} < d^{-5/(2p+2)}$ and the hypothesis on α ensures that $q_0 > 3/2$, and hence that q_1 is positive.

Lemma 3.13. *Let α and p be positive numbers such that $p > 3/2$ and $d^{-1} \leq \alpha < d^{-5/(2p+2)}$. Then, for every $0 < q < q_1 := \frac{q_0}{3} - \frac{1}{2}$, there are positive constants $c_1 = c_1(p,\alpha,q)$ and $c_2 = c_2(p,\alpha)$ such that for every $g \colon \mathbb{P}^k \to \mathbb{R}$ we have*

$$\|g\|_{\log^q} \leq c_1 \|g\|_{\langle p,\alpha\rangle}, \quad \|g\|_{\langle p,\alpha\rangle} \leq c_2 \|g\|_p, \quad \text{and} \quad \|g\|_{\langle p,\alpha\rangle} \leq c_2 \|g\|_{\mathcal{C}^1}.$$

Proof. We can assume that $\|g\|_{\langle p,\alpha\rangle} \leq 1$. By the definition of the norm $\|\cdot\|_{\langle p,\alpha\rangle}$ and Lemma 3.10, $\left\| i\partial g \wedge \bar{\partial} g \right\|_{q'}$ is bounded by a constant for any $q' < q_0$. Therefore, we have $i\partial g \wedge \bar{\partial} g \leq R$ for some positive closed current R such that $\|R\|$ and $\|u_R\|_{\log^{q'}}$ are bounded by a constant. The first inequality follows from Corollary 2.12. The second assertion follows from Lemmas 3.10 and 3.8. The last assertion follows from Definition 3.12. □

Lemma 3.14. *Let α and p be positive numbers such that $d^{-1} \leq \alpha < 1$. Then for all functions $g, h \colon \mathbb{P}^k \to \mathbb{R}$ we have*

$$\|gh\|_{\langle p,\alpha\rangle} \leq \sqrt{2}\big(\|g\|_{\langle p,\alpha\rangle} \|h\|_\infty + \|g\|_\infty \|h\|_{\langle p,\alpha\rangle} \big).$$

Proof. Using an expansion of $i\partial(gh)\wedge\bar\partial(gh)$ and Cauchy-Schwarz's inequality we have

$$\begin{aligned}\left\|i\partial(gh)\wedge\bar\partial(gh)\right\|_{p,\alpha} &\leq \|h\|_\infty^2\left\|i\partial g\wedge\bar\partial g\right\|_{p,\alpha}+\|g\|_\infty^2\left\|i\partial h\wedge\bar\partial h\right\|_{p,\alpha}\\ &\quad+\|g\|_\infty\|h\|_\infty\left\|i\partial g\wedge\bar\partial h+i\partial h\wedge\bar\partial g\right\|_{p,\alpha}\\ &\leq \|h\|_\infty^2\left\|i\partial g\wedge\bar\partial g\right\|_{p,\alpha}+\|g\|_\infty^2\left\|i\partial h\wedge\bar\partial h\right\|_{p,\alpha}\\ &\quad+\|g\|_\infty\|h\|_\infty\Big(\frac{\|h\|_\infty}{\|g\|_\infty}\left\|i\partial g\wedge\bar\partial g\right\|_{p,\alpha}+\frac{\|g\|_\infty}{\|h\|_\infty}\left\|i\partial h\wedge\bar\partial h\right\|_{p,\alpha}\Big)\\ &\leq 2\|h\|_\infty^2\left\|i\partial g\wedge\bar\partial g\right\|_{p,\alpha}+2\|g\|_\infty^2\left\|i\partial h\wedge\bar\partial h\right\|_{p,\alpha}.\end{aligned}$$

The assertion follows from Definition 3.12. □

3.6. **The semi-norm $\|\cdot\|_{\langle p,\alpha\rangle,\gamma}$.** The following semi-norm defines the final space of functions that we will use in our study of the transfer operator. We use here some ideas from the theory of interpolation between Banach spaces, see also [Tri95].

Definition 3.15. For all real numbers $d^{-1}\leq\alpha<1$, $\gamma>0$ and $p>0$, we define for a continuous function $g\colon\mathbb{P}^k\to\mathbb{R}$

$$\begin{aligned}\|g\|_{\langle p,\alpha\rangle,\gamma}:=\inf\Big\{c\geq 0\colon\forall\, 0<\epsilon\leq 1\ \exists\, g_\epsilon^{(1)},g_\epsilon^{(2)}\colon\\ g=g_\epsilon^{(1)}+g_\epsilon^{(2)},\ \|g_\epsilon^{(1)}\|_{\langle p,\alpha\rangle}\leq c(1/\epsilon)^{1/\gamma},\ \|g_\epsilon^{(2)}\|_\infty\leq c\epsilon\Big\}.\end{aligned}\tag{3.6}$$

When such a number c does not exist, we set $\|g\|_{\langle p,\alpha\rangle,\gamma}:=\infty$.

The following two lemmas are the counterparts of Lemmas 3.13 and 3.14 for the semi-norm $\|\cdot\|_{\langle p,\alpha\rangle,\gamma}$. Recall that q_1 is defined in Lemma 3.13.

Lemma 3.16. *For all positive numbers p,α,γ,q satisfying $p>3/2$, $d^{-1}\leq\alpha<d^{-5/(2p+2)}$ and $q<q_2:=\frac{\gamma}{\gamma+1}q_1$, there is a positive constant $c=c(p,\alpha,\gamma,q)$ such that*

$$\|g\|_{\log^q}\leq c\,\|g\|_{\langle p,\alpha\rangle,\gamma}\quad\text{ and }\quad\|g\|_{\langle p,\alpha\rangle,\gamma}\leq\|g\|_{\langle p,\alpha\rangle}$$

for every continuous function $g\colon\mathbb{P}^k\to\mathbb{R}$. Moreover, if $\chi: I\to\mathbb{R}$ is a Lipschitz function with Lipschitz constant κ on an interval $I\subset\mathbb{R}$ containing the image of g, then we have

$$\|\chi(g)\|_{\langle p,\alpha\rangle,\gamma}\leq\kappa\|g\|_{\langle p,\alpha\rangle,\gamma}.$$

Proof. Let us prove the first inequality. We can assume that $\|g\|_{\langle p,\alpha\rangle,\gamma}\leq 1$. Lemma 3.13 implies that $g_\epsilon^{(1)}$ has $\|\cdot\|_{\log^{q'}}$ semi-norm bounded by a constant times $(1/\epsilon)^{1/\gamma}$ when $q'<q_1$. Therefore, we have for $r>0$

$$m(g,r)\leq m(g_\epsilon^{(1)},r)+m(g_\epsilon^{(2)},r)\lesssim\frac{(1/\epsilon)^{1/\gamma}}{(\log^\star r)^{q'}}+\epsilon.$$

Choosing $\epsilon=(\log^\star r)^{-K}$ with $K=q'/(1+1/\gamma)$ gives

$$m(g,r)\lesssim(\log^\star r)^{-q'+K/\gamma}+(\log^\star r)^{-K}=2(\log^\star r)^{-q'/(1+1/\gamma)}.$$

The first assertion of the lemma follows by choosing q' close enough to q_1.

The second inequality follows from the definition of the semi-norm $\|\cdot\|_{\langle p,\alpha\rangle,\gamma}$, by taking $g_\epsilon^{(2)}=0$ in the decomposition $g=g_\epsilon^{(1)}+g_\epsilon^{(2)}$ for every ϵ.

We prove now the last assertion. Since we can approximate χ uniformly by smooth functions χ_n with $|\chi_n'|\leq 1$, we can assume for simplicity that χ is smooth. Define $h:=\chi(g)$ and recall that we are assuming that $\|g\|_{\langle p,\alpha\rangle,\gamma}\leq 1$. For every $0<\epsilon\leq 1$, we have the decomposition

$$g=g_\epsilon^{(1)}+g_\epsilon^{(2)}\quad\text{with}\quad\|g_\epsilon^{(1)}\|_{\langle p,\alpha\rangle}\leq(1/\epsilon)^{1/\gamma}\quad\text{and}\quad\|g_\epsilon^{(2)}\|_\infty\leq\epsilon.$$

Write

$$h=h_\epsilon^{(1)}+h_\epsilon^{(2)}\quad\text{with}\quad h_\epsilon^{(1)}:=\chi(g_\epsilon^{(1)})\quad\text{and}\quad h_\epsilon^{(2)}:=h-h_\epsilon^{(1)}.$$

We have

$$\|h_\epsilon^{(2)}\|_\infty = \|\chi(g) - \chi(g_\epsilon^{(1)})\|_\infty \lesssim \kappa\|g - g_\epsilon^{(1)}\|_\infty = \kappa\|g_\epsilon^{(2)}\|_\infty \leq \kappa\epsilon$$

and also

$$\|h_\epsilon^{(1)}\|_{\langle p,\alpha\rangle} = \|i\partial\chi(g_\epsilon^{(1)}) \wedge \overline{\partial}\chi(g_\epsilon^{(1)})\|_{p,\alpha}^{1/2} \leq \kappa\|i\partial g_\epsilon^{(1)} \wedge \overline{\partial} g_\epsilon^{(1)}\|_{p,\alpha}^{1/2} = \kappa\|g_\epsilon^{(1)}\|_{\langle p,\alpha\rangle} \leq \kappa(1/\epsilon)^{1/\gamma}.$$

It follows that $\|h\|_{\langle p,\alpha\rangle,\gamma} \leq \kappa$. This completes the proof of the lemma. □

Lemma 3.17. *For all positive numbers p, α, γ such that $d^{-1} \leq \alpha < 1$ we have*

$$\|gh\|_{\langle p,\alpha\rangle,\gamma} \leq 3\big(\|g\|_{\langle p,\alpha\rangle,\gamma} \|h\|_\infty + \|g\|_\infty \|h\|_{\langle p,\alpha\rangle,\gamma} \big)$$

for every continuous functions $g, h\colon \mathbb{P}^k \to \mathbb{R}$.

Proof. We can assume that $\|g\|_{\langle p,\alpha\rangle,\gamma} = \|h\|_{\langle p,\alpha\rangle,\gamma} = 1$. For every $0 < \epsilon \leq 1$ we need to find a decomposition $gh = L_\epsilon^{(1)} + L_\epsilon^{(2)}$ with $L_\epsilon^{(1)}, L_\epsilon^{(2)}$ such that

$$\|L_\epsilon^{(1)}\|_{\langle p,\alpha\rangle} \leq 3(\|g\|_\infty + \|h\|_\infty)(1/\epsilon)^{1/\gamma} \quad \text{and} \quad \|L_\epsilon^{(2)}\|_\infty \leq 3(\|g\|_\infty + \|h\|_\infty)\epsilon.$$

Case 1. Assume that $\|g\|_\infty \leq 3\epsilon$ and $\|h\|_\infty \leq 3\epsilon$. Choose $L_\epsilon^{(1)} = 0$ and $L_\epsilon^{(2)} = gh$. Clearly, these functions satisfy the desired estimates.

Case 2. Assume now that $\|g\|_\infty + \|h\|_\infty \geq 3\epsilon$. By the definition of the semi-norm $\|\cdot\|_{\langle p,\alpha\rangle,\gamma}$ we have the decompositions $g = g_\epsilon^{(1)} + g_\epsilon^{(2)}$ and $h = h_\epsilon^{(1)} + h_\epsilon^{(2)}$ with

$$\|g_\epsilon^{(1)}\|_{\langle p,\alpha\rangle} \leq (1/\epsilon)^{1/\gamma}, \quad \|g_\epsilon^{(2)}\|_\infty \leq \epsilon, \quad \|h_\epsilon^{(1)}\|_{\langle p,\alpha\rangle} \leq (1/\epsilon)^{1/\gamma}, \quad \|h_\epsilon^{(2)}\|_\infty \leq \epsilon.$$

Observe that $\|g_\epsilon^{(1)}\|_\infty \leq \|g\|_\infty + \epsilon$ and $\|h_\epsilon^{(1)}\|_\infty \leq \|h\|_\infty + \epsilon$, which imply that

$$\|g_\epsilon^{(1)}\|_\infty + \|h_\epsilon^{(1)}\|_\infty \leq 2(\|g\|_\infty + \|h\|_\infty).$$

Set

$$L_\epsilon^{(1)} := g_\epsilon^{(1)} h_\epsilon^{(1)} \qquad \text{and} \qquad L_\epsilon^{(2)} := g_\epsilon^{(1)} h_\epsilon^{(2)} + g_\epsilon^{(2)} h_\epsilon^{(1)} + g_\epsilon^{(2)} h_\epsilon^{(2)}.$$

The desired estimate for $\|L_\epsilon^{(1)}\|_{\langle p,\alpha\rangle}$ follows from Lemma 3.14 and the one for $\|L_\epsilon^{(2)}\|_\infty$ is obtained by a direct computation. This ends the proof of the lemma. □

3.7. **Approximations for Hölder and $\log^p$-continuous functions.** We end this section with the following technical lemmas. The first one will be used in Section 4.

Lemma 3.18. *For every $\log^p$-continuous function $g\colon \mathbb{P}^k \to \mathbb{R}$, $p > 0$, $s \geq 1$, and $0 < \epsilon \leq 1$, there exist continuous functions $g_\epsilon^{(1)}$ and $g_\epsilon^{(2)}$ such that*

$$g = g_\epsilon^{(1)} + g_\epsilon^{(2)}, \qquad \|g_\epsilon^{(1)}\|_{\mathcal{C}^s} \leq c\,\|g\|_\infty\, e^{(1/\epsilon)^{1/p}}, \qquad \text{and} \qquad \|g_\epsilon^{(2)}\|_\infty \leq c\,\|g\|_{\log^p} \epsilon,$$

where $c = c(p, s)$ is a positive constant independent of g and ϵ. In particular, for every $n \geq 1$ there exist $g_n^{(1)}$ of class $\mathcal{C}^2$ and $g_n^{(2)}$ continuous such that

$$g = g_n^{(1)} + g_n^{(2)}, \qquad \|g_n^{(1)}\|_{\mathcal{C}^2} \leq c\,\|g\|_\infty\, e^{\frac{1}{2}n^{2/p}}, \qquad \text{and} \qquad \|g_n^{(2)}\|_\infty \leq c\,\|g\|_{\log^p} n^{-2}.$$

Proof. Clearly, the second assertion is a consequence of the first one by taking $\epsilon = 2^p n^{-2}$ and replacing c by $2^p c$. We prove now the first assertion. Using a partition of unity, we can reduce the problem to the case where g is supported by the unit ball of an affine chart $\mathbb{C}^k \subset \mathbb{P}^k$.

Consider a smooth non-negative function χ with support in the unit ball of $\mathbb{C}^k$ whose integral with respect to the Lebesgue measure is 1. For $\nu > 0$, consider the function $\chi_\nu(z) := \nu^{-2k}\chi(z/\nu)$ which has integral 1 and tends to the Dirac mass at 0 when ν tends to 0. Define an approximation of g using the standard convolution operator $g_\nu := g * \chi_\nu$, and define $g_\epsilon^{(1)} := g_\nu$ and $g_\epsilon^{(2)} := g - g_\nu$. We consider $\nu := e^{-1/(M\epsilon)^{1/p}}$ for some constant $M > 0$ large enough. It remains to bound $\|g_\epsilon^{(1)}\|_{\mathcal{C}^s}$ and $\|g_\epsilon^{(2)}\|_\infty$.

By standard properties of the convolution we have, for some constant $\kappa > 0$,

$$\|g_\epsilon^{(2)}\|_\infty \lesssim m(g, \kappa\nu) \lesssim \|g\|_{\log^p} (\log^\star \nu)^{-p} \lesssim \|g\|_{\log^p} \epsilon$$

and, by definition of g_ν,

$$\|g_\epsilon^{(1)}\|_{\mathcal{C}^s} \lesssim \|g\|_\infty \|\chi_\nu\|_{\mathcal{C}^s} \operatorname{Leb}(\mathbb{B}^k_\nu) \lesssim \|g\|_\infty \nu^{-s} \lesssim \|g\|_\infty e^{(1/\epsilon)^{1/p}},$$

where we use the fact that M is large enough. This ends the proof of the lemma. □

The following lemma applied for $s = 1$ implies that $\|\cdot\|_{\langle p,\alpha\rangle,\gamma} \lesssim \|\cdot\|_{\mathcal{C}^\gamma}$ because $\|\cdot\|_{\langle p,\alpha\rangle} \lesssim \|\cdot\|_{\mathcal{C}^1}$, see Lemma 3.13.

Lemma 3.19. *Let $0 < \gamma \leq 1$ be a constant. Then, for every $\mathcal{C}^\gamma$ function $g\colon \mathbb{P}^k \to \mathbb{R}$, $s \geq 1$, and $0 < \epsilon \leq 1$, there exist a $\mathcal{C}^s$ function $g_\epsilon^{(1)}$ and a continuous function $g_\epsilon^{(2)}$ such that*

$$g = g_\epsilon^{(1)} + g_\epsilon^{(2)}, \qquad \|g_\epsilon^{(1)}\|_{\mathcal{C}^s} \leq c\, \|g\|_\infty (1/\epsilon)^{s/\gamma} \qquad \textit{and} \qquad \|g_\epsilon^{(2)}\|_\infty \leq c\, \|g\|_{\mathcal{C}^\gamma}\, \epsilon,$$

where $c = c(\gamma, s)$ is a positive constant independent of g and ϵ.

Proof. It is enough to follow the proof of Lemma 3.18 and take $\nu := \epsilon^{1/\gamma}$. We have in this case

$$\|g_\epsilon^{(1)}\|_{\mathcal{C}^s} \lesssim \|g\|_\infty \|\chi_\nu\|_{\mathcal{C}^s} \operatorname{Leb}(\mathbb{B}^k_\nu) \lesssim \|g\|_\infty \nu^{-s} \lesssim \|g\|_\infty \epsilon^{-s/\gamma}$$

and

$$\|g_\epsilon^{(2)}\|_\infty \lesssim m(g, \kappa\nu) \lesssim \|g\|_{\mathcal{C}^\gamma} (\kappa\nu)^\gamma \lesssim \|g\|_{\mathcal{C}^\gamma}\, \epsilon.$$

The lemma follows. □

4. Existence of the scaling ratio and equilibrium state

In this section we prove Theorem 1.1. We divide the exposition in two parts. In the first part (Sections 4.1 to 4.4) we prove the existence of a good scaling ratio λ using a new approach, see Theorem 4.1 below. In the second part (Sections 4.5 and 4.6), we deduce Theorem 1.1 from Theorem 4.1 using some classical arguments.

4.1. Main statement and first step of the proof.

Recall that the Perron-Frobenius operator $\mathcal{L}$ is defined as in (1.1). A direct computation gives

$$\mathcal{L}^n(g)(y) = \sum_{f^n(x)=y} e^{\phi(x)+\phi(f(x))+\cdots+\phi(f^{n-1}(x))} g(x).$$

The following result gives the scaling ratio λ.

Theorem 4.1. *Let f and ϕ be as in Theorem 1.1. There exist a number $\lambda > 0$ and a continuous function $\rho > 0$ on $\mathbb{P}^k$ such that for every continuous function $g\colon \mathbb{P}^k \to \mathbb{R}$ the sequence $\lambda^{-n}\mathcal{L}^n(g)$ is equicontinuous and converges uniformly to $c_g\rho$, where c_g is a constant depending linearly on g. Moreover, if g is strictly positive, then c_g is strictly positive and the sequence $\mathcal{L}^n(g)^{1/n}$ converges uniformly to λ as n tends to infinity.*

We will first study the case where g is equal to $\mathbb{1}$. The general case will be deduced from this particular case. Define $\mathbb{1}_n := \mathcal{L}^n(\mathbb{1})$. Denote by ρ_n^+ and ρ_n^- the maximum and the minimum of $\mathbb{1}_n$, respectively. Consider also the ratio $\theta_n := \rho_n^+/\rho_n^-$ and the function $\mathbb{1}_n^* := (\rho_n^-)^{-1}\mathbb{1}_n$. Observe that the last function satisfies $\min \mathbb{1}_n^* = 1$. The following result will be crucial for us.

Proposition 4.2. *Under the hypotheses of Theorem 4.1, the sequence $\{\theta_n\}$ is bounded and the sequence of functions $\{\mathbb{1}_n^*\}$ is uniformly bounded and equicontinuous.*

The proof of this result will use the technical tools that were presented in Sections 2 and 3. Before giving it, we need to first introduce some auxiliary objects.

By Lemma 3.18 applied to ϕ instead of g, we can find functions ϕ_n and ψ_n such that

$$\phi = \phi_n + \psi_n, \qquad \|\phi_n\|_{\mathcal{C}^2} \leq c\, \|\phi\|_\infty e^{\frac{1}{2}n^{2/q}}, \qquad \text{and} \qquad \|\psi_n\|_\infty \leq c\, \|\phi\|_{\log^q} n^{-2}. \tag{4.1}$$

Consider two integers $J \geq 0$ and $N \geq 0$, whose values will be specialised later. Define for $n \geq N+1$

$$\hat{\mathcal{L}}_n(g)(x) := \sum_{f^n(x)=y} e^{\phi_{n+J}(x)+\phi_{n+J-1}(f(x))+\cdots+\phi_{J+N+1}(f^{n-N-1}(x))} g(x). \tag{4.2}$$

This operator will be used to approximate $\mathcal{L}^n$. The gain here is the fact that the involved functions ϕ_m have controlled $\mathcal{C}^2$ norms. As above, we define

$$\hat{\mathbb{1}}_n := \hat{\mathcal{L}}_n \mathbb{1}, \quad \hat{\rho}_n^+ := \max \hat{\mathbb{1}}_n, \quad \hat{\rho}_n^- := \min \hat{\mathbb{1}}_n, \quad \hat{\theta}_n := \hat{\rho}_n^+/\hat{\rho}_n^-, \quad \text{and} \quad \hat{\mathbb{1}}_n^* := (\hat{\rho}_n^-)^{-1}\hat{\mathbb{1}}_n.$$

The following lemma allows us to reduce our problem to the study of the functions $\hat{\mathbb{1}}_n$.

Lemma 4.3. *There exists a positive constant $c = c(N)$ such that, for all $n > N \geq 0$ and J,*

$$c^{-1} \leq \rho_n^+/\hat{\rho}_n^+ \leq c \quad \text{and} \quad c^{-1} \leq \rho_n^-/\hat{\rho}_n^- \leq c.$$

In particular, the sequence $\{\hat{\theta}_n\}$ is bounded if and only if the sequence $\{\theta_n\}$ is bounded.

Proof. We have

$$\begin{aligned}\rho_n^+ = \max \mathbb{1}_n &= \max_y \sum_{f^n(x)=y} e^{\phi(x)+\phi(f(x))+\cdots+\phi(f^{n-1}(x))} \\ &= \max_y \sum_{f^n(x)=y} e^{\phi_{n+J}(x)+\cdots+\phi_{J+N+1}(f^{n-N-1}(x))} \cdot e^{\psi_{n+J}(x)+\cdots+\psi_{J+N+1}(f^{n-N-1}(x))} \\ &\qquad\qquad \cdot e^{\phi(f^{n-N}x))+\cdots+\phi(f^{n-1}(x))}\end{aligned}$$

and similarly for ρ_n^-. So, both $\rho_n^+/\hat{\rho}_n^+$ and $\rho_n^-/\hat{\rho}_n^-$ are bounded from above and below by $e^{N\max\phi}C_{n,N,J}$ and $e^{N\min\phi}/C_{n,N,J}$ respectively, where

$$C_{n,N,J} := e^{\|\psi_{n+J}\|_\infty + \|\psi_{n+J-1}\|_\infty + \cdots + \|\psi_{J+N+1}\|_\infty}.$$

It follows from the estimate on ψ_n given above that $C_{n,N,J}$ is bounded from above by a positive constant which does not depend on n, J and N. Therefore, both $\rho_n^+/\hat{\rho}_n^+$ and $\rho_n^-/\hat{\rho}_n^-$ are bounded from below and above by positive constants as in the statement. The lemma follows. $\square$

4.2. **An estimate for $dd^c\hat{\mathbb{1}}_n$.** Proposition 4.2 will be obtained using the following crucial estimate for $dd^c\hat{\mathbb{1}}_n$. We will see here the role of the estimate of the $\mathcal{C}^2$ norm of ϕ_n. Recall that $q > 2$, see Theorem 1.1. We also refer to Section 3.1 for notation.

Proposition 4.4. *There exists a sub-exponential function $\eta(t) = ct^3 e^{(t+J)^{2/q}}$ with a positive constant $c = c(\|\phi\|_{\log^q}, \|\phi\|_\infty)$ independent of n, J and N such that for all $n > N \geq 0$ we have*

$$\left|dd^c\hat{\mathbb{1}}_n\right| \leq \sum_{m=N+1}^{n-N} \eta(m) e^{m\max\phi} \hat{\rho}_{n-m}^+ d^{(k-1)m}\omega_m + \sum_{m=n_0}^{n} d^{kN}\eta(m)e^{(n-N)\max\phi} d^{(k-1)m}\omega_m,$$

where $n_0 := \max(n-N+1, N+1)$.

Recall that the function $\hat{\mathbb{1}}_n$ is given by

$$\hat{\mathbb{1}}_n(y) = \sum_{f^n(x)=y} e^{\phi_{n+J}(x)+\phi_{n+J-1}(f(x))+\cdots+\phi_{J+N+1}(f^{n-N-1}(x))}.$$

In order to estimate $dd^c\hat{\mathbb{1}}_n$, we will use a now classical construction due to Gromov [Gro03]. Define the manifold $\Gamma_n \subset (\mathbb{P}^k)^{n+1}$ by

$$\Gamma_n := \left\{(x, f(x), \ldots, f^n(x)) : x \in \mathbb{P}^k\right\},$$

which can also be seen as the graph of the map $(f, f^2, \ldots, f^n)$ in the product space $(\mathbb{P}^k)^{n+1}$. Consider the function $\mathbb{h}$ on $(\mathbb{P}^k)^{n+1}$ given by

$$\mathbb{h}(x_0, \ldots, x_n) := e^{\phi_{n+J}(x_0)+\phi_{n+J-1}(x_1)+\cdots+\phi_{J+N+1}(x_{n-N-1})}.$$

The function $\hat{\mathbb{1}}_n$ on $\mathbb{P}^k$ is equal to the push-forward of the function $\mathbb{h}_{|\Gamma_n}$ to the last factor $\mathbb{P}^k$ of $(\mathbb{P}^k)^{n+1}$. Indeed, denoting by π_n the restriction of the projection $x \mapsto x_n$ to Γ_n, we have

$$(\pi_n)_*(\mathbb{h})(y) = \sum_{(x_0,\ldots,x_n)\in\Gamma_n:\, x_n=y} \mathbb{h}(x) = \sum_{x\in f^{-n}(y)} e^{\phi_{n+J}(x)+\cdots+\phi_{J+N+1}(f^{n-N-1}(x))} = \hat{\mathbb{1}}_n(y).$$

Recall that, since $dd^c\hat{\mathbb{1}}_n$ is real, estimating $|dd^c\hat{\mathbb{1}}_n|$ means finding a good positive closed $(1,1)$-current S on $\mathbb{P}^k$ such that both $S \pm dd^c\hat{\mathbb{1}}_n$ are positive. According to the identities above, we have

$$dd^c\hat{\mathbb{1}}_n = (\pi_n)_*\big(dd^c\mathbb{h}\big).$$

Thus, we need to estimate $dd^c\mathbb{h}$ on $(\mathbb{P}^k)^{n+1}$ and Γ_n. We define $\omega^{(m)}$ as the pullback of the Fubini-Study form ω_{FS} to $(\mathbb{P}^k)^{n+1}$ by the projection $x \mapsto x_m$. Equivalently, $\omega^{(m)}$ is a $(1,1)$-form on $(\mathbb{P}^k)^{n+1}$ such that $\omega^{(m)}(x) = \omega_{\mathrm{FS}}(x_m)$.

Lemma 4.5. *There exists a sub-exponential function $\eta(t) = ct^3 e^{(t+J)^{2/q}}$ with a positive constant $c = c(\|\phi\|_{\log^q}, \|\phi\|_\infty)$ independent of $n, J,$ and N such that*

$$|dd^c\mathbb{h}| \le \mathbb{h} \sum_{m=0}^{n-N-1} \eta(n-m)\omega^{(m)}.$$

Proof. A direct computation gives

$$i\partial\bar\partial\mathbb{h} = \mathbb{h}\Big(\sum_{m=0}^{n-N-1} i\partial\overline{\partial}\phi_{n+J-m}(x_m) + \sum_{m,m'=0}^{n-N-1} i\partial\phi_{n+J-m}(x_m)\wedge\overline{\partial}\phi_{n+J-m'}(x_{m'})\Big).$$

For the first sum, observe that

$$|i\partial\overline{\partial}\phi_{n+J-m}(x_m)| \lesssim \|\phi_{n+J-m}\|_{\mathcal{C}^2}\omega^{(m)}(x) \lesssim e^{(n+J-m)^{2/q}}\omega^{(m)}(x).$$

For the second sum, consider $m' \le m \le n-N-1$. By using Cauchy-Schwarz's inequality, we have

$$\begin{aligned}
(4.3)\quad & |i\partial\phi_{n+J-m}(x_m)\wedge\overline{\partial}\phi_{n+J-m'}(x_{m'})| \\
&\le (m-m'+1)^{-2} i\partial\phi_{n+J-m}(x_m)\wedge\overline{\partial}\phi_{n+J-m}(x_m) \\
&\quad + (m-m'+1)^2 i\partial\phi_{n+J-m'}(x_{m'})\wedge\overline{\partial}\phi_{n+J-m'}(x_{m'}) \\
&\lesssim (m-m'+1)^{-2}\|\phi_{n+J-m}\|^2_{\mathcal{C}^1}\omega^{(m)}(x) + (m-m'+1)^2\|\phi_{n+J-m'}\|^2_{\mathcal{C}^1}\omega^{(m')}(x) \\
&\lesssim (m-m'+1)^{-2}e^{(n+J-m)^{2/q}}\omega^{(m)}(x) + (n-m'+1)^2 e^{(n+J-m')^{2/q}}\omega^{(m')}(x).
\end{aligned}$$

This and the fact that $\sum_{j=1}^\infty j^{-2}$ is finite imply that

$$\begin{aligned}
(4.4)\quad & \Big|\sum_{0\le m'\le m\le n-N-1} i\partial\phi_{n+J-m}(x_m)\wedge\overline{\partial}\phi_{n+J-m'}(x_{m'})\Big| \\
&\lesssim \sum_{m=0}^{n-N-1} e^{(n+J-m)^{2/q}}\omega^{(m)}(x) + \sum_{m'=0}^{n-N-1}(n-m'+1)^3 e^{(n+J-m')^{2/q}}\omega^{(m')}(x) \\
&\lesssim \sum_{m=0}^{n-N-1}(n-m)^3 e^{(n+J-m)^{2/q}}\omega^{(m)}(x).
\end{aligned}$$

We obtain by symmetry a similar estimate for the case where $m < m' \le n-N-1$.

Finally, combining all the above identities and estimates we get

$$|i\partial\bar\partial\mathbb{h}| \lesssim \mathbb{h}\sum_{m=0}^{n-N-1}(n-m)^3 e^{(n+J-m)^{2/q}}\omega^{(m)}.$$

The lemma follows. □

Proof of Proposition 4.4. We are only interested in the restriction of $\mathbb{h}$ to the graph Γ_n. We deduce from Lemma 4.5 that

$$(4.5)\qquad \big|dd^c\hat{\mathbb{1}}_n\big| = |(\pi_n)_* dd^c\mathbb{h}| \le \sum_{m=0}^{n-N-1}\eta(n-m)(\pi_n)_*\big(\mathbb{h}\omega^{(m)}\big).$$

We split the last sum into the two sums corresponding to $m < N$ and $m \geq N$. Note that when $n \leq 2N$, in the sum in (4.5) we always have $m < N$ and the first sum in the statement of the proposition vanishes. So, for simplicity, we assume that $n > 2N$ and we will see in the proof below that the arguments also work when $n \leq 2N$.

For $m < N$, using the definition of ϕ_m we have $\|\phi - \phi_m\|_\infty = \|\psi_m\|_\infty \leq c' m^{-2}$ and hence $\max \phi_m \leq \max \phi + c' m^{-2}$ for some positive constant c' which may depend on $\|\phi\|_{\log^q}$. It follows that $\mathbb{h} \lesssim e^{(n-N)\max \phi}$. Then, using the definition of Γ_n, we have for $m < N$

$$\begin{aligned}(\pi_n)_* \big(\mathbb{h}\omega^{(m)}\big) &\lesssim e^{(n-N)\max \phi} (\pi_n)_* \big(\omega^{(m)}\big) = e^{(n-N)\max \phi} d^{km} (f^{n-m})_* (\omega_{\mathrm{FS}}) \\ &= e^{(n-N)\max \phi} d^{km} d^{(k-1)(n-m)} \omega_{n-m}.\end{aligned}$$

Thus,

$$\begin{aligned}\sum_{m=0}^{N-1} \eta(n-m)(\pi_n)_* \big(\mathbb{h}\omega^{(m)}\big) &\lesssim \sum_{m=0}^{N-1} \eta(n-m) e^{(n-N)\max \phi} d^{km} d^{(k-1)(n-m)} \omega_{n-m} \\ &\leq \sum_{m=n-N+1}^{n} d^{kN} \eta(m) e^{(n-N)\max \phi} d^{(k-1)m} \omega_m. \qquad (4.6)\end{aligned}$$

The last expression is the second sum in the statement of the present proposition (this step also works for $n \leq 2N$ but in this case the above sums $\sum_0^{N-1}$ and $\sum_{n-N+1}^n$ are replaced by $\sum_0^{n-N-1}$ and $\sum_{N+1}^n$ respectively). In order to finish the proof, it is enough to have a similar estimate for $m \geq N$ (this step is superfluous when $n \leq 2N$, see (4.5)).

As above, using the definition of $\mathbb{h}$ and the estimates on $\max \phi_m$ and $\|\phi - \phi_m\|_\infty$, we have

$$\mathbb{h} \lesssim e^{(n-m)\max \phi} e^{\phi(x_0) + \phi(x_1) + \cdots + \phi(x_{m-N-1})} \lesssim e^{(n-m)\max \phi} \mathbb{h}'$$

with

$$\mathbb{h}' := e^{\phi_{m+J}(x_0) + \phi_{m+J-1}(x_1) + \cdots + \phi_{J+N+1}(x_{m-N-1})}.$$

Note that the sum in the definition of $\mathbb{h}'$ contains $m - N$ terms while the one of $\mathbb{h}$ contains $n - N$ terms. The specific choice of $\mathbb{h}'$ is convenient for our next computation as it is related to the function $\hat{\mathbb{1}}_m$.

Consider the map $\pi' : \Gamma_n \to (\mathbb{P}^k)^{n-m+1}$ defined by $\pi'(x) := x' := (x_m, \ldots, x_n)$. Denote by Γ' the image of Γ_n by π'. It is the graph of the map $(f, \ldots, f^{n-m})$ from $\mathbb{P}^k$ to $(\mathbb{P}^k)^{n-m}$. We also have for $x' \in \Gamma'$

$$\pi'^{-1}(x') = \big\{ \big(y, f(y), \ldots, f^{m-1}(y), x'\big) \quad \text{with} \quad y \in f^{-m}(x_m) \big\}.$$

So $\pi' : \Gamma_n \to \Gamma'$ is a ramified covering of degree d^{km}.

Consider the map $\pi'' : \Gamma' \to \mathbb{P}^k$ defined by $\pi''(x') := x_n$. We have, for $x_n \in \mathbb{P}^k$,

$$\pi''^{-1}(x_n) = \big\{ \big(z, f(z), \ldots, f^{n-m}(z)\big) \quad \text{with} \quad z \in f^{-n+m}(x_n) \big\}.$$

So $\pi'' : \Gamma' \to \mathbb{P}^k$ is a ramified covering of degree $d^{k(n-m)}$. We have $\pi_n = \pi'' \circ \pi'$. Observe that $\pi'_*(\mathbb{h}'\omega^{(m)})$ is a $(1,1)$-form on Γ' such that

$$\begin{aligned}\pi'_*(\mathbb{h}'\omega^{(m)})(x') &= \Big(\sum_{y \in f^{-m}(x_m)} e^{\phi_{m+J}(y) + \cdots + \phi_{J+N+1}(f^{m-N-1}(y))} \Big) \omega_{\mathrm{FS}}(x_m) \\ &\leq \hat{\rho}_m^+ \omega_{\mathrm{FS}}(x_m) =: \hat{\rho}_m^+ \omega'(x'),\end{aligned}$$

where we define ω' as the pull-back of ω_{FS} to Γ' by the map $x' \mapsto x_m$. We also have

$$\pi''_*(\omega')(x_n) = \sum_{x_m \in f^{-n+m}(x_n)} \omega_{\mathrm{FS}}(x_m) = (f^{n-m})_*(\omega_{\mathrm{FS}})(x_n) = d^{(k-1)(n-m)} \omega_{n-m}(x_n).$$

Thus,

$$(\pi_n)_*(\mathbb{h}\omega^{(m)}) \lesssim e^{(n-m)\max \phi} \pi''_* \pi'_*(\mathbb{h}'\omega^{(m)}) \leq e^{(n-m)\max \phi} \hat{\rho}_m^+ d^{(k-1)(n-m)} \omega_{n-m}$$

and

$$\sum_{m=N}^{n-N-1} \eta(n-m)(\pi_n)_*\big(\mathbb{h}\omega^{(m)}\big) \lesssim \sum_{m=N}^{n-N-1} \eta(n-m)e^{(n-m)\max\phi}\hat\rho^+_m d^{(k-1)(n-m)}\omega_{n-m}$$

$$= \sum_{m=N+1}^{n-N} \eta(m)e^{m\max\phi}\hat\rho^+_{n-m}d^{(k-1)m}\omega_m. \tag{4.7}$$

Finally, we deduce the proposition from (4.5), (4.6), and (4.7) by multiplying η with a large enough constant. □

4.3. **Proof of Proposition 4.2.** We are working under the hypotheses of Theorem 4.1. We will obtain Proposition 4.2 using Lemmas 4.6 and 4.7 below.

Lemma 4.6. *Under the hypotheses of Theorem 4.1, given an integer $J \geq 0$, we have $\hat\theta_n \leq d^{kN}$ for all $n > N$, with N large enough. In particular, the sequences (θ_n) and $(\hat\theta_n)$ are bounded for all $J \geq 0$ and $N \geq 0$.*

Proof. Observe that the last assertion is a consequence of the first one. Indeed, we can first fix J and N satisfying the first assertion of the lemma. Then, by Lemma 4.3, the sequence (θ_n) is bounded. Applying again Lemma 4.3 for arbitrary J and N gives that the sequence $(\hat\theta_n)$ is also bounded. We prove now the first assertion in the lemma with J fixed and N large enough.

Observe that, by the definition of $\hat\rho^\pm_n, \hat\theta_n$, and $\Omega(\cdot)$, for every $K \geq 1$ the two inequalities $\hat\theta_n \leq K$ and $\Omega(\hat{\mathbb{1}}_n) \leq (K-1)\hat\rho^-_n$ are equivalent. Hence, in order to get the first assertion in the lemma, it is enough to show that $\Omega(\hat{\mathbb{1}}_n)/\hat\rho^-_n \leq d^{kN}/2$. The constants that we use below are independent of N and n. Fix a constant δ such that $e^{\Omega(\phi)} < \delta < d$. By the estimate on $\|\psi_n\|_\infty$ in (4.1), for every j sufficiently large, we have $\Omega(\phi_j) \leq \Omega(\phi) + \Omega(\psi_j) < \log\delta$. Since we assume that N is large enough, the last inequality holds for all $j \geq N$.

We use Proposition 4.4 and Corollary 2.5 in order to estimate $\Omega(\hat{\mathbb{1}}_n)$ in terms of $\Omega(u_m)$. Recall that u_m is the dynamical potential of ω_m. We also use Lemma 3.2, which gives $\Omega(u_m) \lesssim d^m\delta'^{-m}$ for any δ' such that $\delta < \delta' < d$. More precisely, we obtain from those results that

$$\Omega(\hat{\mathbb{1}}_n) \lesssim \sum_{m=N+1}^{n-N} \eta(m)e^{m\max\phi}d^{km}\delta'^{-m}\hat\rho^+_{n-m} + \sum_{m=\max(n-N+1,N+1)}^{n} d^{kN}\eta(m)e^{(n-N)\max\phi}d^{km}\delta'^{-m}.$$

Since $\delta < \delta'$ and N is large, the fact that η is sub-exponential and independent of n and N implies that

$$\Omega(\hat{\mathbb{1}}_n) \lesssim \sum_{m=N+1}^{n-N} e^{m\max\phi}d^{km}\delta^{-m}\hat\rho^+_{n-m} + \sum_{m=\max(n-N+1,N+1)}^{n} d^{kN}e^{(n-N)\max\phi}d^{km}\delta^{-m}. \tag{4.8}$$

We now distinguish two cases.

Case 1. Assume that $N < n \leq 2N$. In this case, the first sum in (4.8) is empty. We thus deduce from (4.8) that

$$\Omega(\hat{\mathbb{1}}_n) \lesssim d^{kN}e^{(n-N)\max\phi}\Big(\frac{d^{k(N+1)}}{\delta^{N+1}} + \cdots + \frac{d^{kn}}{\delta^n}\Big) \lesssim d^{kN}e^{(n-N)\max\phi}\frac{d^{kn}}{\delta^n}.$$

On the other hand, by the definitions of $\hat\rho^-_n$ we have the following general estimates (with $n \geq N$ in the first inequality and $n-m \geq N$ in the second one)

$$\hat\rho^-_n \gtrsim d^{kn}e^{(n-N)\min\phi} \quad \text{and} \quad \hat\rho^-_n \gtrsim d^{km}e^{m\min\phi}\hat\rho^-_{n-m}. \tag{4.9}$$

The first inequality and the above estimate of $\Omega(\hat{\mathbb{1}}_n)$ imply that

$$\frac{\Omega(\hat{\mathbb{1}}_n)}{\hat\rho^-_n} \lesssim d^{kN}\frac{e^{(n-N)\Omega(\phi)}}{\delta^n} \leq d^{kN}\frac{e^{n\Omega(\phi)}}{\delta^n}.$$

Hence, $\Omega(\hat{\mathbb{1}}_n)/\hat{\rho}_n^- \leq d^{kN}/2$ because N is chosen large enough and $\delta > e^{\Omega(\phi)}$. The lemma in this case follows.

Case 2. Assume now that $n > 2N$. By induction on n and the previous case, we can assume that $\Omega(\hat{\mathbb{1}}_m)/\hat{\rho}_m^- \leq d^{kN}/2$, which implies $\hat{\rho}_m^+ \leq d^{kN}\hat{\rho}_m^-$, for all $m < n$. We need to prove the same inequality for $m = n$. From (4.8) and the induction hypothesis, we have

$$\begin{aligned}\Omega(\hat{\mathbb{1}}_n) &\lesssim d^{kN}\sum_{m=N+1}^{n-N} e^{m\max\phi}d^{km}\delta^{-m}\hat{\rho}_{n-m}^- + d^{kN}\sum_{m=n-N+1}^{n} e^{(n-N)\max\phi}d^{km}\delta^{-m} \\ &\lesssim d^{kN}\sum_{m=N+1}^{n-N} e^{m\max\phi}d^{km}\delta^{-m}\hat{\rho}_{n-m}^- + d^{kN}e^{(n-N)\max\phi}d^{kn}\delta^{-n}.\end{aligned}$$

This and the second inequality in (4.9) imply that

$$\Omega(\hat{\mathbb{1}}_n) \lesssim d^{kN}\sum_{m=N+1}^{n-N} e^{m\Omega(\phi)}\delta^{-m}\hat{\rho}_n^- + d^{kN}e^{(n-N)\max\phi}d^{kn}\delta^{-n}.$$

Then, by the first inequality in (4.9) and using that $\delta > e^{\Omega(\phi)}$ and $n > 2N$, we obtain

$$\frac{\Omega(\hat{\mathbb{1}}_n)}{\hat{\rho}_n^-} \lesssim d^{kN}\sum_{m=N+1}^{n-N} e^{m\Omega(\phi)}\delta^{-m} + d^{kN}e^{(n-N)\Omega(\phi)}\delta^{-n} \lesssim d^{kN}e^{N\Omega(\phi)}\delta^{-N}.$$

Recall that all the constants involved in our computations do not depend on n and N. Since N is chosen large enough, we obtain that $\Omega(\hat{\mathbb{1}}_n)/\hat{\rho}_n^- \leq d^{kN}/2$. This ends the proof of the lemma. □

Lemma 4.7. *Under the hypotheses of Theorem 4.1, for all $J \geq 0$, $N \geq 0$, and $p > 0$, the sequence $\|\hat{\mathbb{1}}_n^*\|_{\log^p}$ is bounded. In particular, the sequence of functions $\hat{\mathbb{1}}_n^*$ is equicontinuous.*

Proof. We only need to consider $n > 2N$, and the implicit constants below may depend on N. We will use Lemma 3.3 and need to estimate $dd^c\hat{\mathbb{1}}_n^*$. By Lemma 4.6 the sequence $(\hat{\theta}_n)$ is bounded. This and Proposition 4.4 imply that

$$\left|dd^c\hat{\mathbb{1}}_n^*\right| \lesssim \frac{1}{\hat{\rho}_n^-}\Big(\sum_{m=N+1}^{n-N}\eta(m)e^{m\max\phi}\hat{\rho}_{n-m}^- d^{(k-1)m}\omega_m + \sum_{m=n-N+1}^{n}\eta(m)e^{(n-N)\max\phi}d^{(k-1)m}\omega_m\Big).$$

Then, using the two inequalities in (4.9), we obtain

$$\begin{aligned}\left|dd^c\hat{\mathbb{1}}_n^*\right| &\lesssim \sum_{m=N+1}^{n-N}\eta(m)e^{m\Omega(\phi)}d^{-m}\omega_m + \sum_{m=n-N+1}^{n}\eta(m)e^{(n-N)\Omega(\phi)}d^{(k-1)m-kn}\omega_m \\ &\lesssim \sum_{m=0}^{\infty}\eta(m)e^{m\Omega(\phi)}d^{-m}\omega_m.\end{aligned}$$

Finally, since η is sub-exponential and $e^{\Omega(\phi)} < d$, Lemma 3.3 implies the result. □

End of the proof of Proposition 4.2. By Lemma 4.6, we already know that the sequence (θ_n) is bounded. Since $\min\mathbb{1}_n^* = 1$, we have $\max\mathbb{1}_n^* = \theta_n$, hence the sequence $(\mathbb{1}_n^*)$ is uniformly bounded. In order to show that this sequence is equicontinuous, it is enough to approximate it uniformly by an equicontinuous sequence.

Take $N = 0$. Fix an arbitrary constant $0 < \epsilon < 1$. Since $\|\phi - \phi_m\|_\infty \lesssim m^{-2}$ by (4.1), we can choose an integer J large enough so that for every $n \geq 0$ we have

$$(1-\epsilon)\hat{\mathbb{1}}_n \leq \mathbb{1}_n \leq (1+\epsilon)\hat{\mathbb{1}}_n.$$

This implies

$$\frac{1-\epsilon}{1+\epsilon}\,\hat{\mathbb{1}}_n^* \leq \mathbb{1}_n^* \leq \frac{1+\epsilon}{1-\epsilon}\,\hat{\mathbb{1}}_n^*.$$

Therefore, $|\mathbb{1}_n^* - \hat{\mathbb{1}}_n^*|$ is bounded uniformly by a constant times ϵ. By Lemma 4.7, the sequence $(\hat{\mathbb{1}}_n^*)$ is equicontinuous. We easily deduce that the sequence $(\mathbb{1}_n^*)$ is equicontinuous as well. □

Remark 4.8. The method of the proof of Proposition 4.2 developed above is based on deducing a bound on $\Omega(\mathbb{1}_n^*)$ and $\|\mathbb{1}_n^*\|_{\log^p}$ from estimates on $dd^c\mathbb{1}_n^*$, by means of Proposition 2.6, Corollaries 2.7 and 2.8, and Lemma 3.3. We could reach the same goal by estimating $i\partial\mathbb{1}_n^* \wedge \bar\partial\mathbb{1}_n^*$ and using Proposition 2.11, Corollary 2.12, and Remark 3.4.

4.4. **Proof of Theorem 4.1.** We first define the scaling ratio λ. By definition of ρ_n^+, we easily see that the sequence (ρ_n^+) is sub-multiplicative, that is, $\rho_{n+m}^+ \leq \rho_m^+\rho_n^+$ for all $m, n \geq 0$. It follows that the first limit in the following line exists

$$\lambda := \lim_{n\to\infty} \left(\rho_n^+\right)^{1/n} = \lim_{n\to\infty} \left(\rho_n^-\right)^{1/n},$$

where the last identity is due to the fact that (θ_n) is bounded, see Lemma 4.6. We have the following lemma.

Lemma 4.9. *The sequences $(\lambda^{-n}\rho_n^+)$ and $(\lambda^{-n}\rho_n^-)$ are both bounded above and below by positive constants. In particular, the sequence $\big(\lambda^{-n}\mathbb{1}_n\big)$ is uniformly bounded and equicontinuous.*

Proof. It is clear that the second assertion is a consequence of the first one and Proposition 4.2. We prove now the first assertion. Since the sequence ρ_n^+ is sub-multiplicative, it is well-known that $\inf_n(\rho_n^+)^{1/n}$ is equal to λ. Hence, we have $\lambda^{-n}\rho_n^+ \geq 1$. Since θ_n is bounded, we have $\rho_n^+ \lesssim \rho_n^-$. It follows that both $\lambda^{-n}\rho_n^\pm$ are bounded from below by positive constants. Similarly, the sequence ρ_n^- is super-multiplicative, i.e., $\rho_{n+m}^- \geq \rho_m^-\rho_n^-$ for all $m, n \geq 0$, and we deduce that that both $\lambda^{-n}\rho_n^\pm$ are bounded from above by positive constants. The lemma follows. □

We can extend the above result to all continuous test functions.

Lemma 4.10. *Let $\mathcal{F}$ be a uniformly bounded and equicontinuous family of real-valued functions on $\mathbb{P}^k$. Then the family*

$$\mathcal{F}_{\mathbb{N}} := \{\lambda^{-n}\mathcal{L}^n(g) \ : \ g \in \mathcal{F}, n \geq 0\}$$

is also uniformly bounded and equicontinuous.

Proof. By Lemma 4.9, the family $\mathcal{F}_{\mathbb{N}}$ is uniformly bounded. We prove now that it is equicontinuous. Given any constant $\epsilon > 0$, using a convolution, we can find for every $g \in \mathcal{F}$ a smooth function g' such that $\|g - g'\|_\infty \leq \epsilon$ and $\|g'\|_{\mathcal{C}^2}$ is bounded by a constant depending on ϵ. Denote by $\mathcal{F}'$ the family of these g'. Observe that

$$|\lambda^{-n}\mathcal{L}^n(g) - \lambda^{-n}\mathcal{L}^n(g')| = |\lambda^{-n}\mathcal{L}^n(g - g')| \leq \epsilon\lambda^{-n}\mathbb{1}_n \leq \epsilon\lambda^{-n}\rho_n^+$$

and the last expression is bounded by a constant times ϵ. Therefore, in order to prove the lemma, it is enough to show that the family $\mathcal{F}'_{\mathbb{N}}$, defined in a similar way as for $\mathcal{F}_{\mathbb{N}}$, is equicontinuous. For simplicity, we replace $\mathcal{F}$ by $\mathcal{F}'$ and assume that $\|g\|_{\mathcal{C}^2}$ is bounded by a constant for $g \in \mathcal{F}$. The constants involved in the computation below do not depend on $g \in \mathcal{F}$.

We continue to use the notation introduced above. Consider an arbitrary constant $\epsilon > 0$. Take $N = 0$ and choose J large enough depending on ϵ. From the definitions of $\mathcal{L}$ and $\hat{\mathcal{L}}_n$ (see (4.2)) and the fact that $\|\phi - \phi_m\|_\infty \lesssim m^{-2}$ we obtain that

$$|\lambda^{-n}\mathcal{L}^n(g)(x) - \lambda^{-n}\hat{\mathcal{L}}_n(g)(x)| \leq \epsilon\lambda^{-n} \sum_{f^n(x)=y} e^{\phi(x)+\phi(f(x))+\cdots+\phi(f^{n-1}(x))}|g(x)|.$$

This and Lemma 4.9 imply that

$$\|\lambda^{-n}\mathcal{L}^n(g) - \lambda^{-n}\hat{\mathcal{L}}_n(g)\|_\infty \leq \epsilon\lambda^{-n}\rho_n^+\|g\|_\infty \lesssim \epsilon.$$

So, in order to prove that the family $\lambda^{-n}\mathcal{L}^n(g)$ is equicontinuous, it is enough to show the same property for the family $\lambda^{-n}\hat{\mathcal{L}}_n(g)$.

We will use the same idea as in Proposition 4.4 and Lemma 4.5. Instead of the function $\mathbb{h}$, we need to consider the following slightly different function (recall that $N=0$)

$$\mathbb{H}(x_0,\ldots,x_n) := e^{\phi_{n+J}(x_0)+\phi_{n+J-1}(x_1)+\cdots+\phi_{J+1}(x_{n-1})} g(x_0) = \mathbb{h}(x_0,\ldots,x_n) g(x_0).$$

We have

$$i\partial\overline{\partial}\mathbb{H} = (i\partial\overline{\partial}\mathbb{h}) g(x_0) + \mathbb{h}(i\partial\overline{\partial} g(x_0)) + i\partial\mathbb{h}\wedge\overline{\partial} g(x_0) - i\overline{\partial}\mathbb{h}\wedge\partial g(x_0).$$

Applying Cauchy-Schwarz's inequality to the last two terms, and since g has a bounded $\mathcal{C}^2$ norm, we obtain

$$\begin{aligned} |i\partial\overline{\partial}\mathbb{H}| &\leq |(i\partial\overline{\partial}\mathbb{h})g(x_0)| + |\mathbb{h}(i\partial\overline{\partial}g(x_0))| + i\mathbb{h}^{-1}\partial\mathbb{h}\wedge\overline{\partial}\mathbb{h} + i\mathbb{h}\partial g(x_0)\wedge\overline{\partial}g(x_0) \\ &\lesssim |i\partial\overline{\partial}\mathbb{h}| + \mathbb{h}\omega_{\rm FS}(x_0) + i\mathbb{h}^{-1}\partial\mathbb{h}\wedge\overline{\partial}\mathbb{h} + \mathbb{h}\omega_{\rm FS}(x_0) \\ &\lesssim |i\partial\overline{\partial}\mathbb{h}| + \mathbb{h}\omega_{\rm FS}(x_0) + i\mathbb{h}^{-1}\partial\mathbb{h}\wedge\overline{\partial}\mathbb{h}. \end{aligned}$$

We claim that the last sum satisfies

$$|i\partial\overline{\partial}\mathbb{h}| + \mathbb{h}\omega_{\rm FS}(x_0) + i\mathbb{h}^{-1}\partial\mathbb{h}\wedge\overline{\partial}\mathbb{h} \lesssim \mathbb{h}\sum_{m=0}^{n-1}\eta(n-m)\omega^{(m)}.$$

Lemma 4.5 shows that the first term $|i\partial\overline{\partial}\mathbb{h}|$ of the LHS is bounded by the RHS. The second term clearly satisfies the same property (consider $m=0$ in the above sum). For the last term, by Cauchy-Schwarz's inequality and using a computation as in the proof of Lemma 4.5, we have (recall that $N=0$)

$$i\mathbb{h}^{-1}\partial\mathbb{h}\wedge\overline{\partial}\mathbb{h} = \mathbb{h}\sum_{m,m'=0}^{n-1} i\partial\phi_{n+J-m}(x_m)\wedge\overline{\partial}\phi_{n+J-m'}(x_{m'}) \lesssim \mathbb{h}\sum_{m=0}^{n-1}\eta(n-m)\omega^{(m)}.$$

This implies the claim and gives a bound for $|i\partial\overline{\partial}\mathbb{H}|$.

Since $\hat{\mathcal{L}}_n(g) = (\pi_n)_*(\mathbb{H})$, we obtain as in the proof of Proposition 4.4 that

$$|dd^c\lambda^{-n}\hat{\mathcal{L}}_n(g)| \lesssim \lambda^{-n}\sum_{m=1}^{n}\eta(m)e^{m\max\phi}\hat{\rho}^{+}_{n-m}d^{(k-1)m}\omega_m.$$

By Lemmas 4.3 and 4.9 we have $\hat{\rho}^{\pm}_{n-m}\lesssim\rho^{\pm}_{n-m}\lesssim\lambda^{n-m}$. Therefore, we obtain

$$|dd^c\lambda^{-n}\hat{\mathcal{L}}_n(g)| \lesssim \sum_{m=1}^{n}\eta(m)e^{m\max\phi}\lambda^{-m}d^{(k-1)m}\omega_m.$$

Finally, since $\lambda\geq d^k e^{\min\phi}$ by definition of λ, the last estimate implies that

$$|dd^c\lambda^{-n}\hat{\mathcal{L}}_n(g)| \lesssim \sum_{m=1}^{n}\eta(m)e^{m\Omega(\phi)}d^{-m}\omega_m.$$

Lemma 3.3 and the fact that $d>e^{\Omega(\phi)}$ imply the result. □

We now construct the density function ρ on $\mathbb{P}^k$. Recall that the sequence $\lambda^{-n}\mathbb{1}_n$ is uniformly bounded and equicontinuous. Therefore, the Cesaro sums

$$\widetilde{\mathbb{1}}_n := \frac{1}{n}\sum_{j=0}^{n-1}\lambda^{-j}\mathbb{1}_j$$

also form a uniformly bounded and equicontinuous sequence of functions. It follows that there is a subsequence of $\widetilde{\mathbb{1}}_n$ which converges uniformly to a continuous function ρ. Observe that $\rho\geq\inf_n\lambda^{-n}\rho_n^-$. Hence, by Lemma 4.9, the function ρ is strictly positive. A direct computation gives

$$\lambda^{-1}\mathcal{L}(\widetilde{\mathbb{1}}_n) - \widetilde{\mathbb{1}}_n = \frac{1}{n}(\lambda^{-n}\mathbb{1}_n - \mathbb{1}_0).$$

Since $\lambda^{-n}\mathbb{1}_n$ is bounded uniformly in n, the last expression tends uniformly to 0 when n tends to infinity. We then deduce from the definition of ρ that $\lambda^{-1}\mathcal{L}(\rho)=\rho$.

End of the proof of Theorem 4.1. Observe that we only need to show that $\lambda^{-n}\mathcal{L}^n(g)$ converges to $c_g\rho$ for some constant c_g. The remaining part of the theorem is then clear. Let $\mathcal{G}$ denote the family of all limit functions of subsequences of $\lambda^{-n}\mathcal{L}^n(g)$. By Lemma 4.10, the sequence $\lambda^{-n}\mathcal{L}^n(g)$ is uniformly bounded and equicontinuous. Therefore, by Arzelà-Ascoli theorem, $\mathcal{G}$ is a uniformly bounded and equicontinuous family of functions which is compact for the uniform topology. Observe also that $\mathcal{G}$ is invariant under the action of $\lambda^{-1}\mathcal{L}$. Define

$$M := \max\{l(a)/\rho(a) \ : \ l \in \mathcal{G}, a \in \mathbb{P}^k\}.$$

We first prove the following properties.

Claim 1. We have $\max_{\mathbb{P}^k}(l/\rho) = M$ for every $l \in \mathcal{G}$.

Assume by contradiction that there is a sequence $\lambda^{-n_j}\mathcal{L}^{n_j}(g)$ which converges uniformly to a function $l \in \mathcal{G}$ such that $l \leq (M-2\epsilon)\rho$ for some constant $\epsilon > 0$. Then, for j large enough, we have $\lambda^{-n_j}\mathcal{L}^{n_j}(g) \leq (M-\epsilon)\rho$. Fix such an index j. For $n > n_j$ we have

$$\lambda^{-n}\mathcal{L}^n(g) = \lambda^{-n+n_j}\mathcal{L}^{n-n_j}(\lambda^{-n_j}\mathcal{L}^{n_j}(g)) \leq (M-\epsilon)\lambda^{-n+n_j}\mathcal{L}^{n-n_j}(\rho) = (M-\epsilon)\rho.$$

Since this is true for every $n > n_j$, we get a contradiction with the definition of M. This ends the proof of Claim 1.

Claim 2. We have $l/\rho = M$ on the small Julia set $\mathrm{supp}(\mu)$ for every $l \in \mathcal{G}$.

Consider an arbitrary function $l \in \mathcal{G}$ and define $l_n := \lambda^{-n}\mathcal{L}^n(l) \in \mathcal{G}$. By Claim 1, there is a point $a_n \in \mathbb{P}^k$ such that $l_n(a_n) = M\rho(a_n)$. By definition of M, we have $l \leq M\rho$ and hence

$$M\rho(a_n) = l_n(a_n) = \lambda^{-n}\mathcal{L}^n(l)(a_n) \leq \lambda^{-n}\mathcal{L}^n(M\rho)(a_n) = M\rho(a_n).$$

So the inequality in the last line is actually an equality. This and the definition of $\mathcal{L}$ imply that $l/\rho = M$ on $f^{-n}(a_n)$. Observe that when n tends to infinity, the limit of $f^{-n}(a_n)$ contains $\mathrm{supp}(\mu)$, see, e.g., [DS10b, Cor. 1.4]. By continuity, we obtain $l/\rho = M$ on $\mathrm{supp}(\mu)$. This ends the proof of Claim 2.

Applying the above claims to the function $-g$ instead of g, we obtain that l/ρ is equal on $\mathrm{supp}(\mu)$ to $\min_{\mathbb{P}^k}(l/\rho)$. We can now conclude that $l = M\rho$ on $\mathbb{P}^k$ for every $l \in \mathcal{G}$. Define $c_g := M$. We obtain that $\lambda^{-n}\mathcal{L}^n(g)$ converges uniformly to $c_g\rho$. This completes the proof of the theorem. □

4.5. **Equidistribution of preimages and mixing properties.** We have seen that the operator $\mathcal{L}$ acts on the space of continuous functions $g\colon \mathbb{P}^k \to \mathbb{R}$. It is also positive, i.e., $\mathcal{L}(g) \geq 0$ when $g \geq 0$. Therefore, $\mathcal{L}$ induces by duality a linear operator $\mathcal{L}^*$ acting on the space of measures and preserving the cone of positive measures.

Proposition 4.11. *Under the assumptions of Theorem 1.1, there exists a unique conformal measure associated with ϕ, that is, there exists a unique probability measure m_ϕ which is an eigenvector of $\mathcal{L}^*$. We also have $\mathcal{L}^*(m_\phi) = \lambda m_\phi$, $\mathrm{supp}(m_\phi) = \mathrm{supp}(\mu)$, and if ν is a positive measure, $\lambda^{-n}(\mathcal{L}^n)^*(\nu)$ converges to $\langle \nu, \rho\rangle m_\phi$ when n tends to infinity. Moreover, if $\mathcal{F}$ is a uniformly bounded and equicontinuous family of functions on $\mathbb{P}^k$, then $\lambda^{-n}\mathcal{L}^n(g) - c_g\rho$ converges to 0 when n goes to infinity, uniformly on $g \in \mathcal{F}$, where $c_g := \langle m_\phi, g\rangle$.*

Proof. For any probability measure m_ϕ as in the first assertion, there is a constant $\lambda' > 0$ such that $\mathcal{L}^*(m_\phi) = \lambda' m_\phi$. It follows that, for every continuous function g,

$$\langle m_\phi, g\rangle = \lim_{n\to\infty} \langle \lambda'^{-n}(\mathcal{L}^n)^*(m_\phi), g\rangle = \lim_{n\to\infty} \langle m_\phi, \lambda'^{-n}\mathcal{L}^n(g)\rangle.$$

We necessarily have $\lambda' = \lambda$ because we know from the end of the proof of Theorem 4.1 that $\lambda^{-n}\mathcal{L}^n(g)$ converges uniformly to $c_g\rho$ and c_g is not always 0. We conclude that $\langle m_\phi, g\rangle = c_g\langle m_\phi, \rho\rangle$. Since $\langle m_\phi, g\rangle = c_g = 1$ when $g = \mathbb{1}$ (because m_ϕ is a probability measure) we deduce that $\langle m_\phi, \rho\rangle = 1$ and hence $\langle m_\phi, g\rangle = c_g$ for every continuous function g. This gives the uniqueness of m_ϕ.

Consider now an arbitrary probability measure ν on $\mathbb{P}^k$. We have

$$\langle \lambda^{-n}(\mathcal{L}^n)^*(\nu), g\rangle = \langle \nu, \lambda^{-n}\mathcal{L}^n(g)\rangle \to \langle \nu, c_g\rho\rangle = \langle \nu, \rho\rangle\langle m_\phi, g\rangle.$$

It follows that $\lambda^{-n}(\mathcal{L}^n)^*(\nu)$ converges to $\langle \nu, \rho\rangle m_\phi$. If ν is supported by $\mathrm{supp}(\mu)$ and g vanishes on $\mathrm{supp}(\mu)$, by definition of $\mathcal{L}$, the function $\mathcal{L}^n(g)$ also vanishes on $\mathrm{supp}(\mu)$ and the last computation implies that $\langle m_\phi, g\rangle = 0$. Equivalently, the measure m_ϕ is supported by $\mathrm{supp}(\mu)$.

In order to show that $\mathrm{supp}(m_\phi) = \mathrm{supp}(\mu)$, we assume by contradiction that there is a continuous function $g \geq 0$ on $\mathbb{P}^k$ such that $g > 0$ on some open subset U of $\mathrm{supp}(\mu)$ and $\langle m_\phi, g\rangle = 0$. The $\lambda^{-1}\mathcal{L}^*$-invariance of m_ϕ implies that $\langle m_\phi, \mathcal{L}^n g\rangle = \lambda^n\langle m_\phi, g\rangle = 0$ and the definition of $\mathcal{L}$ implies that $\mathcal{L}^n(g) > 0$ on $f^n(U)$. It follows that m_ϕ has no mass on $f^n(U)$ and hence on $\cup_{n\geq 0} f^n(U)$. On the other hand, we have for every $x \in \mathbb{P}^k$ that $d^{-kn}(f^n)^*(\delta_x)$ converges to μ, see, e.g., [DS10b, Cor. 1.4]. Therefore, $f^{-n}(\delta_x) \cap U \neq \varnothing$ for some n or equivalently $x \in \cup_{n\geq 0} f^n(U)$. So we have $\cup_{n\geq 0} f^n(U) = \mathbb{P}^k$. This contradicts the fact that m_ϕ has no mass on this union. So we have $\mathrm{supp}(m_\phi) = \mathrm{supp}(\mu)$ as desired.

For the last assertion of the proposition, we can replace g with $g - c_g\rho$ in order to assume that $c_g = 0$ for $g \in \mathcal{F}$. By Lemma 4.10, the family $\mathcal{F}_\mathbb{N}$ is uniformly bounded and equicontinuous. So the limit of the sequence of sets $\lambda^{-n}\mathcal{L}^n(\mathcal{F})$ is a compact, uniformly bounded and equicontinuous family of functions that we denote by $\mathcal{F}_\infty$. This family is invariant by $\lambda^{-1}\mathcal{L}$ and we also have $c_g = 0$ for $g \in \mathcal{F}_\infty$. We want to show that it contains only the function 0.

Define

$$M := \max\{l(a)/\rho(a) \ : \ l \in \mathcal{F}_\infty, a \in \mathbb{P}^k\}.$$

Choose a function $l \in \mathcal{F}_\infty$ and a point a such that $l(a)/\rho(a) = M$. There are an increasing sequence of integers (n_j) and a sequence $(g_j) \subset \mathcal{F}$ such that $\lambda^{-n_j}\mathcal{L}^{n_j}(g_j)$ converges uniformly to l. For every $n \geq 0$, choose a limit function l_{-n} of the sequence $\lambda^{-n_j+n}\mathcal{L}^{n_j-n}(g_j)$. We have $l = \lambda^{-n}\mathcal{L}^n(l_{-n})$ and $l_{-n} \in \mathcal{F}_\infty$.

As in the end of the proof of Theorem 4.1, we obtain that $l_{-n}/\rho = M$ on the set $f^{-n}(a)$ and if $l_{-\infty}$ is a limit of the sequence l_{-n} then $l_{-\infty}$ belongs to $\mathcal{F}_\infty$ and $l_{-\infty}/\rho = M$ on the small Julia set $\mathrm{supp}(\mu)$. Since m_ϕ is supported by the small Julia set and $\langle m_\phi, g\rangle = c_g = 0$ for $g \in \mathcal{F}_\infty$, we conclude that $M = 0$. Using the same argument for $-g$ with $g \in \mathcal{F}$, we obtain that the minimal value of the functions in $\mathcal{F}_\infty$ is also 0. So $\mathcal{F}_\infty$ contains only the function 0. This ends the proof of the proposition. □

Proposition 4.11 in particular gives the following equidistribution result for the (weighted) preimages of a given point.

Corollary 4.12. *Under the assumptions of Theorem 1.1, for every $x \in \mathbb{P}^k$ the points in $f^{-n}(x)$, with suitable weights, are equidistributed with respect to the conformal measure m_ϕ when n tends to infinity. More precisely, if δ_a denotes the Dirac mass at a, then*

$$\lim_{n\to\infty} \lambda^{-n} \sum_{f^n(a)=x} e^{\phi(a)+\cdots+\phi(f^{n-1}(a))}\delta_a = \rho(x)m_\phi$$

for every $x \in \mathbb{P}^k$.

Proof. Denote by μ_n the measure in the LHS of the last identity. Let g be any continuous function on $\mathbb{P}^k$. We have

$$\langle \mu_n, g\rangle = \lambda^{-n} \sum_{f^n(a)=x} e^{\phi(a)+\cdots+\phi(f^{n-1}(a))}g(a) = \lambda^{-n}(\mathcal{L}^n g)(x).$$

The last expression converges to $c_g\rho(x) = \rho(x)\langle m_\phi, g\rangle$. The result follows. □

For our convenience, define the operator L by $L(g) := (\lambda\rho)^{-1}\mathcal{L}(\rho g)$. Define also the positive measure μ_ϕ by $\mu_\phi := \rho m_\phi$. We have the following lemma.

Lemma 4.13. *For any continuous function $g\colon \mathbb{P}^k \to \mathbb{R}$, the sequence $L^n(g)$ converges uniformly to the constant $c_{\rho g} = \langle \mu_\phi, g\rangle = \langle m_\phi, \rho g\rangle$. We also have that μ_ϕ is an f-invariant probability measure such that* $\mathrm{supp}(\mu_\phi) = \mathrm{supp}(\mu)$.

Proof. Define $g' := \rho g$. We have $c_{g'} = \langle m_\phi, \rho g\rangle = \langle \mu_\phi, g\rangle$. The first assertion is a direct consequence of the fact that $\lambda^{-n}\mathcal{L}^n(g')$ converges uniformly to $c_{g'}\rho$.

For the second assertion, we have seen in the proof of Proposition 4.11 that $\langle m_\phi, \rho\rangle = 1$. It follows that μ_ϕ is a probability measure. Moreover, we obtain from the $\lambda^{-1}\mathcal{L}^*$-invariance of m_ϕ that

$$\langle \mu_\phi, g\circ f\rangle = \langle m_\phi, \rho(g\circ f)\rangle = \langle \lambda^{-1}\mathcal{L}^*(m_\phi), \rho(g\circ f)\rangle = \langle m_\phi, \lambda^{-1}\mathcal{L}(\rho(g\circ f))\rangle = \langle \mu_\phi, L(g\circ f)\rangle.$$

Using that $\lambda^{-1}\mathcal{L}(\rho) = \rho$ and the definition of $\mathcal{L}$, we can easily check that $L(g\circ f) = g$. So the previous identities imply that $\langle \mu_\phi, g\circ f\rangle = \langle \mu_\phi, g\rangle$. Hence, μ_ϕ is an invariant measure. The assertion on the support of μ_ϕ is clear because $\mathrm{supp}(m_\phi) = \mathrm{supp}(\mu)$ by Proposition 4.11 and ρ is strictly positive. $\square$

The operator $\mathcal{L}$ can also be extended to a continuous operator on $L^2(\mu_\phi)$ and $L^2(m_\phi)$. Since $\mu_\phi = \rho m_\phi$ and ρ is positive and continuous, these two spaces are actually the same and the corresponding norms are equivalent.

Lemma 4.14. *Under the assumptions of Theorem 1.1, the operator $\mathcal{L}$ extends to a linear continuous operator on $L^2(m_\phi)$ whose norm is bounded by $\lambda e^{\frac{1}{2}\Omega(\phi)}$. Moreover, there exists a positive constant c such that $\|\lambda^{-n}\mathcal{L}^n\|_{L^2(m_\phi)} \leq c$ for all $n \geq 0$.*

Proof. By Cauchy-Schwarz's inequality and using the $\lambda^{-1}\mathcal{L}^*$-invariance of m_ϕ, we have

$$\big\langle m_\phi, |\mathcal{L}^n g|^2\big\rangle \leq \big\langle m_\phi, (\mathcal{L}^n\mathbb{1})\cdot(\mathcal{L}^n|g|^2)\big\rangle \leq \rho_n^+\big\langle m_\phi, \mathcal{L}^n|g|^2\big\rangle = \rho_n^+\lambda^n\big\langle m_\phi, |g|^2\big\rangle$$

for every $g \in L^2(m_\phi)$ and $n \geq 0$. The second assertion of the lemma follows because $\rho_n^+ \lesssim \lambda^n$.

For the first assertion, take $n = 1$. From the definition of ρ_n^+ and λ, we have $\rho_1^+ \leq d^k e^{\max\phi}$ and $\lambda \geq d^k e^{\min\phi}$. The above inequality implies that

$$\big\langle m_\phi, |\mathcal{L}g|^2\big\rangle \leq e^{\Omega(\phi)}\lambda^2\big\langle m_\phi, |g|^2\big\rangle.$$

The first assertion in the lemma follows. $\square$

Proposition 4.15. *Under the assumptions of Theorem 1.1, the measure $\mu_\phi = \rho m_\phi$ is K-mixing and mixing of all orders.*

Proof. We start with the second property. Let $\{g_0, \dots, g_r\}$ be any finite family of continuous test functions on $\mathbb{P}^k$. We need to show that, for $0 = n_0 \leq n_1 \leq \cdots \leq n_r$,

$$\big\langle \mu_\phi, g_0(g_1\circ f^{n_1})\dots(g_r\circ f^{n_r})\big\rangle - \prod_{j=0}^r \langle \mu_\phi, g_j\rangle \to 0$$

when $n := \inf_{0\leq j<r}(n_{j+1} - n_j)$ tends to infinity. This property is clearly true for $r = 0$. Take $r \geq 1$. By induction, we can assume that the above convergence holds for the case of $r-1$ test functions. We prove now the same property for r test functions.

By the f_*-invariance of μ_ϕ and the induction hypothesis, we have

$$\big\langle \mu_\phi, (g_1\circ f^{n_1})\dots(g_r\circ f^{n_r})\big\rangle = \big\langle \mu_\phi, g_1(g_2\circ f^{n_2-n_1})\dots(g_r\circ f^{n_r-n_1})\big\rangle \to \prod_{j=1}^r \langle \mu_\phi, g_j\rangle.$$

So the desired property holds when g_0 is a constant function. Therefore, we can subtract from g_0 a constant and assume that $\langle \mu_\phi, g_0\rangle = 0$, which implies that the product $\prod_{j=0}^n \langle \mu_\phi, g_j\rangle$ vanishes. Using that $\lambda^{-1}\mathcal{L}(\rho) = \rho$, the $\lambda^{-1}\mathcal{L}^*$-invariance of m_ϕ, and the definition of $\mathcal{L}$, we can easily check by induction that for all functions g, l we have

$$\langle \mu_\phi, g\rangle = \langle \mu_\phi, L^n(g)\rangle \qquad \text{and} \qquad L^n(g(l\circ f^n)) = L^n(g)l.$$

We then deduce that

$$\begin{aligned}\big\langle \mu_\phi, g_0(g_1\circ f^{n_1})\dots(g_r\circ f^{n_r})\big\rangle &= \big\langle \mu_\phi, L^{n_1}\big(g_0(g_1\circ f^{n_1})\dots(g_r\circ f^{n_r})\big)\big\rangle \\ &= \big\langle \mu_\phi, L^{n_1}(g_0)g_1\dots(g_r\circ f^{n_r-n_1})\big\rangle.\end{aligned}$$

By Lemma 4.13, the sequence $L^{n_1}(g_0)$ converges uniformly to 0 as n_1 tends to ∞. So the last integral tends to 0 because the function $g_1 \ldots (g_r \circ f^{n_r - n_1})$ is bounded. We then conclude that μ_ϕ is mixing of all orders.

We prove now that μ_ϕ is K-mixing, that is, that given $g \in L^2(\mu_\phi)$, when n tends to infinity the difference

$$\big\langle \mu_\phi, g(l \circ f^n)\big\rangle - \langle \mu_\phi, g\rangle\langle \mu_\phi, l\rangle$$

tends to 0 uniformly on test functions l whose $L^2(\mu_\phi)$-norm is bounded by a constant. As above, we can assume that $\langle \mu_\phi, g\rangle = 0$. We can also assume that the $L^2(\mu_\phi)$-norms of g and l are bounded by 1. Fix an arbitrary constant $\epsilon > 0$. It is enough to show the existence of an integer $N = N(\epsilon)$ independent of l such that $|\langle \mu_\phi, g(l \circ f^n)\rangle| \leq 2\epsilon$ for $n \geq N$.

Choose a continuous function g' such that $\langle \mu_\phi, g'\rangle = 0$ and $\|g - g'\|_{L^2(\mu_\phi)} \leq \epsilon$. Using the invariance of μ_ϕ we have

$$\begin{aligned}
|\langle \mu_\phi, g(l \circ f^n)\rangle - \langle \mu_\phi, g'(l \circ f^n)\rangle| &= |\langle \mu_\phi, (g - g')(l \circ f^n)\rangle| \leq \|g - g'\|_{L^2(\mu_\phi)} \|l \circ f^n\|_{L^2(\mu_\phi)} \\
&= \|g - g'\|_{L^2(\mu_\phi)} \|l\|_{L^2(\mu_\phi)} \leq \epsilon.
\end{aligned}$$

It remains to show that $|\langle \mu_\phi, g'(l \circ f^n)\rangle| \leq \epsilon$ when $n \geq N$ for some N large enough. As above, we have

$$|\langle \mu_\phi, g'(l \circ f^n)\rangle| = |\langle \mu_\phi, L^n(g')l\rangle| \leq \|L^n(g')\|_\infty.$$

Lemma 4.13 and the identity $\langle \mu_\phi, g'\rangle = 0$ imply the result. □

For positive real numbers q, M, and Ω with $q > 2$ and $\Omega < \log d$, consider the following set of weights

$$\mathcal{P}(q, M, \Omega) := \big\{\phi \colon \mathbb{P}^k \to \mathbb{R} \ : \ \|\phi\|_{\log^q} \leq M, \ \Omega(\phi) \leq \Omega\big\}$$

and the uniform topology induced by the sup norm. Observe that this family is equicontinuous. In the two lemmas below, we study the dependence on $\phi \in \mathcal{P}(q, M, \Omega)$ of the objects introduced in this section. Therefore, we will use the index ϕ or parameter ϕ for objects which depend on ϕ, e.g., we will write $\lambda_\phi, \mathcal{L}_\phi, \rho_\phi, \mathbb{1}_n(\phi)$ instead of $\lambda, \mathcal{L}, \rho$ and $\mathbb{1}_n$.

Lemma 4.16. *Let q, M, and Ω be positive real numbers such that $q > 2$ and $\Omega < \log d$. The maps $\phi \mapsto \lambda_\phi$, $\phi \mapsto m_\phi$, $\phi \mapsto \mu_\phi$, and $\phi \mapsto \rho_\phi$ are continuous on $\phi \in \mathcal{P}(q, M, \Omega)$ with respect to the standard topology on $\mathbb{R}$, the weak topology on measures, and the uniform topology on functions. In particular, ρ_ϕ is bounded from above and below by positive constants which are independent of $\phi \in \mathcal{P}(q, M, \Omega)$. Moreover, $\|\lambda_\phi^{-n}\mathcal{L}_\phi^n\|_\infty$ is bounded by a constant which is independent of n and of $\phi \in \mathcal{P}(q, M, \Omega)$.*

Proof. Fix q, M, and Ω as above. Observe that when we add to ϕ a constant c the scaling ratio λ_ϕ and the operator $\mathcal{L}_\phi$ are both changed by a factor e^c. It follows that the operator $\lambda_\phi^{-1}\mathcal{L}_\phi$, the measures m_ϕ, μ_ϕ, and the density function ρ_ϕ do not change. So, for simplicity, it is enough to prove the lemma for ϕ in the family

$$\mathcal{P}_0(q, M, \Omega) := \big\{\phi \colon \mathbb{P}^k \to \mathbb{R} \ : \ \min \phi = 0, \ \|\phi\|_{\log^q} \leq M, \ \Omega(\phi) \leq \Omega\big\}.$$

Notice that this family is compact for the uniform topology.

Consider two weights ϕ and ϕ' in this space. From the definition of λ_ϕ and $\lambda_{\phi'}$, we have $e^{-\|\phi - \phi'\|_\infty} \leq \lambda_\phi/\lambda_{\phi'} \leq e^{\|\phi - \phi'\|_\infty}$. It follows that $\phi \mapsto \lambda_\phi$ is continuous. When $\phi' \to \phi$, any limit value of $m_{\phi'}$ is a probability measure invariant by $\lambda_\phi^{-1}\mathcal{L}_\phi^*$ thanks to the invariance of $m_{\phi'}$ by $\lambda_{\phi'}^{-1}\mathcal{L}_{\phi'}^*$. Since m_ϕ is the only probability measure which is invariant by $\lambda_\phi^{-1}\mathcal{L}_\phi^*$, this limit value must be m_ϕ. Thus, $\phi \mapsto m_{\phi'}$ is continuous.

We deduce from the proof of Proposition 4.2 that $\theta_n(\phi) = \rho_n^+(\phi)/\rho_n^-(\phi)$ is bounded by a constant independent of n and ϕ. Moreover, the family of functions

$$\big\{\mathbb{1}_n^*(\phi) \quad \text{with} \quad n \geq 0 \quad \text{and} \quad \phi \in \mathcal{P}_0(q, M, \Omega)\big\}$$

is uniformly bounded and equicontinuous. Recall that $\mathbb{1}_n^*(\phi) = (\rho_n^-(\phi))^{-1}\mathbb{1}_n(\phi)$ and $\rho_n^-(\phi) \leq \lambda_\phi^n \leq \rho_n^+(\phi)$, see the proof of Lemma 4.9. It follows that $\lambda_\phi^{-n}\mathbb{1}_n(\phi)$ belongs to a uniformly bounded and equicontinuous family of functions.

From the definition of ρ_ϕ and $\rho_{\phi'}$, we also see that these functions belong to a uniformly bounded and equicontinuous family of functions. When $\phi' \to \phi$, if ρ' is any limit of $\rho_{\phi'}$, then ρ' is continuous and invariant by $\lambda_\phi^{-1}\mathcal{L}_\phi$ because $\rho_{\phi'}$ satisfies a similar property. It follows from Theorem 4.1 that $\rho' = c\rho_\phi$ for some constant c. On the other hand, since $\mu_{\phi'} = \rho_{\phi'}m_{\phi'}$ is a probability measure, any limit of $\rho_{\phi'}m_{\phi'}$ is a probability measure. Thus, $\rho' m_\phi = c\mu_\phi$ is a probability measure and hence $c = 1$. We conclude that $\rho_{\phi'} \to \rho_\phi$ and also $\mu_{\phi'} \to \mu_\phi$. In other words, the maps $\phi \mapsto \mu_\phi$ and $\phi \mapsto \rho_\phi$ are continuous. Since ρ_ϕ is strictly positive and the family $\mathcal{P}_0(q, M, \Omega)$ is compact, we deduce that ρ_ϕ is bounded from above and below by positive constants independent of ϕ.

The last assertion in the lemma is also clear because $\|\lambda_\phi^{-n}\mathcal{L}_\phi^n\|_\infty = \lambda_\phi^{-n}\|\mathbb{1}_n(\phi)\|_\infty \leq \theta_n(\phi)$. This ends the proof of the lemma. □

Lemma 4.17. *Let $q, M,$ and Ω be positive real numbers such that $q > 2$ and $\Omega < \log d$. Let $\mathcal{F}$ be a uniformly bounded and equicontinuous family of real-valued functions on $\mathbb{P}^k$. Then the family*

$$\big\{\lambda_\phi^{-n}\mathcal{L}_\phi^n(g) \ : \ n \geq 0,\ \phi \in \mathcal{P}(p, M, \Omega),\ g \in \mathcal{F}\big\}$$

is equicontinuous. Moreover, $\big\|\lambda_\phi^{-n}\mathcal{L}_\phi^n(g) - \langle m_\phi, g\rangle\big\|_\infty$ tends to 0 uniformly on $\phi \in \mathcal{P}(p, M, \Omega)$ and $g \in \mathcal{F}$ when n goes to infinity.

Proof. As in Lemma 4.16, we can assume that $\phi \in \mathcal{P}_0(p, M, \Omega)$. The first assertion is clear from the proof of Lemma 4.10. We prove now the second assertion. By Lemma 4.16, m_ϕ belongs to a compact family of probability measures. It follows that $|\langle m_\phi, g\rangle|$ is bounded by a constant independent of ϕ and g. It follows that the family

$$\mathcal{F}'_{\mathbb{N}} := \big\{\lambda_\phi^{-n}\mathcal{L}_\phi^n(g) - \langle m_\phi, g\rangle \ : \ n \geq 0,\ \phi \in \mathcal{P}_0(p, M, \Omega),\ g \in \mathcal{F}\big\}$$

is uniformly bounded and equicontinuous. Denote by $\mathcal{F}'_\infty$ the set of all functions l' obtained as the limit of a sequence

$$h_j := \lambda_{\phi_j}^{-n_j}\mathcal{L}_{\phi_j}^{n_j}(g_j) - \langle m_{\phi_j}, g_j\rangle$$

in $\mathcal{F}'_{\mathbb{N}}$ with $n_j \to \infty$. By taking a subsequence, we can assume that ϕ_j converges uniformly to some function $\phi \in \mathcal{P}_0(p, M, \Omega)$. Since $\langle m_{\phi_j}, h_j\rangle = 0$, we also obtain that $\langle m_\phi, l'\rangle = 0$ by the continuity of $\phi \mapsto m_\phi$. Now, as in the end of the proof of Proposition 4.11, we obtain that $l' = \lambda_\phi^{-n}\mathcal{L}_\phi^n(l'_{-n})$ for some $l'_{-n} \in \mathcal{F}'_\infty$ and then deduce that $l' = 0$. The lemma follows. □

4.6. **Pressure and uniqueness of the equilibrium state.** Using the results in the previous section, to prove the next proposition we only need to follow the arguments in [UZ13, Sections 6 and 7] and [PU10, Section 5.6].

Proposition 4.18. *The probability measure μ_ϕ is a unique equilibrium state associate to ϕ. Moreover, the pressure $P(\phi)$ is equal to $\log \lambda$.*

Proof. We follow the approach in [PU10, Th. 5.6.5]. To simplify the notation, set $S_n(g) := \sum_{j=0}^{n-1} g \circ f^j$ for any function $g\colon \mathbb{P}^k \to \mathbb{R}$. Recall that, given $\phi'\colon \mathbb{P}^k \to \mathbb{R}$ with $\|\phi'\|_{\log^q} < \infty$ and $\Omega(\phi') < \log d$, we denote by $\lambda_{\phi'}, \rho_{\phi'}$ the objects associated to $\mathcal{L}_{\phi'}$.

Claim 1. We have $\mathrm{Ent}_f(\mu_{\phi'}) + \big\langle \mu_{\phi'}, \phi'\big\rangle = P(\phi') = \log \lambda_{\phi'}$ for all $\phi'\colon \mathbb{P}^k \to \mathbb{R}$ such that $\|\phi'\|_{\log^q} < \infty$ and $\Omega(\phi') < \log d$.

Proof of Claim 1. The proof of the inequality $P(\phi') \leq \log \lambda_{\phi'}$ is an adaptation of Gromov's proof of the fact that the topological entropy of f is bounded above by $k \log d$, see [Gro03]. We refer to [UZ13, Th. 6.1] for the complete details. To complete the proof, it is enough to show that $\mathrm{Ent}_f(\mu_{\phi'}) + \big\langle \mu_{\phi'}, \phi'\big\rangle \geq \log \lambda_{\phi'}$.

It follows from [Par69] that $\mathrm{Ent}_f(\mu_{\phi'}) \geq \langle \mu_{\phi'}, \log J_{\mu_{\phi'}} \rangle$, where $J_{\mu_{\phi'}}$ is defined as the Radon-Nikodym derivative of $f^*\mu_{\phi'}$ with respect to $\mu_{\phi'}$ (when this derivative exists). In our setting, it follows from a straightforward computation that $J_{\mu_{\phi'}}$ is well defined and given by

$$J_{\mu_{\phi'}} = \lambda_{\phi'} \rho_{\phi'}^{-1} e^{-\phi'} (\rho_{\phi'} \circ f).$$

Indeed, denoting by J' the RHS in the above expression, for every continuous function $g\colon \mathbb{P}^k \to \mathbb{R}$, we have

$$\begin{aligned}\langle \mu_{\phi'}, J'g \rangle &= \langle \lambda m_{\phi'}, e^{-\phi'}(\rho_{\phi'} \circ f) g \rangle = \langle \mathcal{L}_{\phi'}^* m_{\phi'}, e^{-\phi'}(\rho_{\phi'} \circ f) g \rangle = \langle m_{\phi'}, \mathcal{L}_{\phi'}(e^{-\phi'}(\rho_{\phi'} \circ f) g) \rangle \\ &= \langle m_{\phi'}, \rho_{\phi'} \mathcal{L}_{\phi'}(e^{-\phi'} g) \rangle = \langle \mu_{\phi'}, f_* g \rangle = \langle f^* \mu_{\phi'}, g \rangle,\end{aligned}$$

which proves that $J' = J_{\mu_{\phi'}}$. We then have, using the f_*-invariance of $\mu_{\phi'}$,

$$\mathrm{Ent}_f(\mu_{\phi'}) + \langle \mu_{\phi'}, \phi' \rangle \geq \langle \mu_{\phi'}, \log J_{\mu_{\phi'}} \rangle + \langle \mu_{\phi'}, \phi' \rangle = \langle \mu_{\phi'}, \log(\rho_{\phi'} \circ f) - \log \rho_{\phi'} \rangle + \log \lambda_{\phi'} = \log \lambda_{\phi'}$$

and the proof is complete. □

Claim 2. Let M and Ω be positive real numbers such that $\Omega < \log d$, and $g\colon \mathbb{P}^k \to \mathbb{R}$ a continuous function. Then, for every $y \in \mathbb{P}^k$, we have

$$\frac{1}{n} \frac{\sum_{f^n(x)=y} S_n(g)(x) e^{S_n(\phi')(x)}}{\mathcal{L}_{\phi'}^n \mathbb{1}(y)} \to \langle \mu_{\phi'}, g \rangle \tag{4.10}$$

where the convergence is uniform on $\phi' \in \mathcal{P}(q, M, \Omega)$.

Proof of Claim 2. Observe that the LHS of (4.10) is equal to

$$\frac{1}{n} \frac{\lambda_{\phi'}^{-n} \sum_{f^n(x)=y} S_n(g)(x) e^{S_n(\phi')(x)}}{\lambda_{\phi'}^{-n} \mathcal{L}_{\phi'}^n \mathbb{1}(y)}.$$

The denominator of the last quotient converges to $\rho_{\phi'}(y)$ and the numerator satisfies

$$\lambda_{\phi'}^{-n} \sum_{f^n(x)=y} S_n(g)(x) e^{S_n(\phi')(x)} = \lambda_{\phi'}^{-n} \sum_{j=0}^{n-1} \mathcal{L}_{\phi'}^n (g \circ f^j)(y) = \lambda_{\phi'}^{-j} \sum_{j=0}^{n-1} \lambda_{\phi'}^{j-n} \mathcal{L}_{\phi'}^{n-j} (g \cdot \mathcal{L}_{\phi'}^j \mathbb{1})(y). \tag{4.11}$$

It follows from Lemma 4.17 that

$$\lambda_{\phi'}^{j-n} \mathcal{L}_{\phi'}^{n-j} (g \cdot \mathcal{L}_{\phi'}^j \mathbb{1}) \to \langle m_{\phi'}, g \cdot \mathcal{L}_{\phi'}^j \mathbb{1} \rangle \rho_{\phi'} \tag{4.12}$$

as $n - j \to \infty$, where the convergence is uniform on $\phi' \in \mathcal{P}(q', M, \Omega)$. We deduce from (4.11), (4.12), and the fact that $\lambda_{\phi'}^{-j} \mathcal{L}_{\phi'}^j \mathbb{1} \to \rho_{\phi'}$ as $j \to \infty$ that, as $n \to \infty$, the LHS in (4.10) tends to

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \lambda_{\phi'}^{-j} \langle m_{\phi'}, g \cdot \mathcal{L}_{\phi'}^j \mathbb{1} \rangle = \langle m_{\phi'}, g \cdot \rho_{\phi'} \rangle = \langle \mu_{\phi'}, g \rangle.$$

The proof is complete. □

Claim 3. For every $\psi\colon \mathbb{P}^k \to \mathbb{R}$ such that $\|\psi\|_{\log^q} < \infty$ the function $t \mapsto P(\phi + t\psi)$ is differentiable in a neighbourhood of 0.

Proof of Claim 3. Fix $y \in \mathbb{P}^k$ and set

$$P_n(t) := \frac{1}{n} \log \mathcal{L}_{\phi+t\psi}^n \mathbb{1}(y) \quad \text{and} \quad Q_n(t) := \frac{d}{dt} P_n(t) = \frac{1}{n} \frac{\sum_{f^n(x)=y} S_n(\psi)(x) e^{S_n(\phi+t\psi)(x)}}{\mathcal{L}_{\phi+t\psi}^n \mathbb{1}(y)}.$$

Notice that $\Omega(\phi+t\psi) < \log d$ for t sufficiently small. A direct computation and Claim 2 (applied with $\phi+t\psi, \psi$ instead of ϕ', g) imply that $Q_n(t) \to \langle \mu_{\phi+t\psi}, \psi \rangle$ as $n \to \infty$, locally uniformly with respect to t. We also have $P_n(t) \to \log \lambda_{\phi+t\psi} = P(\phi + t\psi)$, where the convergence follows from Lemma 4.9 and the equality from Claim 1 applied with ϕ' instead of $\phi+t\psi$. We deduce that the pressure function P, in a neighbourhood of $t = 0$, is the uniform limit of the $\mathcal{C}^1$ functions $P_n(t)$, whose derivatives $Q_n(t)$ are also uniformly convergent. Thus, the function P is differentiable in a neighbourhood of $t = 0$, with derivative at t equal to $\langle \mu_{\phi+t\psi}, \psi \rangle$. □

It follows from Claim 1 that μ_ϕ is an equilibrium state. By [PU10, Cor. 3.6.7], the fact that the pressure function $t \mapsto P(\phi + t\psi)$ is differentiable at $t = 0$ with respect to a dense set of continuous functions ψ implies the uniqueness of the equilibrium state for the weight ϕ. Since this property holds by Claim 3 for all ψ such that $\|\psi\|_{\log^q} < \infty$, the proof is complete. □

We will prove in Section 6 that, when ϕ and ψ are Hölder continuous, the pressure function $P(t)$ defined above is actually analytic, see Proposition 6.8. We conclude this section with the following properties of the equilibrium state μ_ϕ that we will use in the next section.

Proposition 4.19. *Under the assumptions of Theorem 1.1, the metric entropy* $\mathrm{Ent}_f(\mu_\phi)$ *of* μ_ϕ *is strictly larger than* $(k-1)\log d$*. In particular,* μ_ϕ *has no mass on proper analytic subsets of* $\mathbb{P}^k$*, its Lyapunov exponents are strictly positive and at least equal to* $\frac{1}{2}(\mathrm{Ent}_f(\mu_\phi) - (k-1)\log d)$*, and the function* $\log|\mathrm{Jac}\, Df|$ *is integrable with respect to* μ_ϕ*. Moreover, the Hausdorff dimension of* μ_ϕ *satisfies*

$$\dim_H(\mu_\phi) \geq \frac{(k-1)\log d}{\lambda_1} + \frac{\mathrm{Ent}_f(\mu_\phi) - (k-1)\log d}{\lambda_k}.$$

Proof. Since μ_ϕ maximizes the pressure and $\mathrm{Ent}_f(\mu) = k \log d$, we have

$$\mathrm{Ent}_f(\mu_\phi) + \langle \mu_\phi, \phi \rangle \geq \mathrm{Ent}_f(\mu) + \langle \mu, \phi \rangle \geq k \log d + \min \phi.$$

Since by assumption we have $\Omega(\phi) < \log d$, it follows that

$$\mathrm{Ent}_f(\mu_\phi) \geq k \log d + \min \phi - \langle \mu_\phi, \phi \rangle \geq k \log d - \Omega(\phi) > (k-1)\log d.$$

The Lyapunov exponents of every ergodic invariant probability measure satisfying this property are bounded below as in the statement, and in particular the function $\log|\mathrm{Jac}\,|$ is integrable with respect to it, see de Thélin [DT08] and Dupont [Dup12]. The bound on the Hausdorff dimension of μ_ϕ is then a consequence of [Dup11].

Let now X be a proper analytic subset of $\mathbb{P}^k$. Assume by contradiction that $m := \mu_\phi(X) > 0$. We choose such an X which is irreducible and of minimal dimension p. So, for all $n \geq 0$, $f^n(X)$ is also an irreducible analytic set of dimension p. We have

$$\mu_\phi(f^n(X)) = \mu_\phi(f^{-n}(f^n(X))) \geq \mu_\phi(X) = m.$$

It follows that $\mu_\phi(f^n(X) \cap f^{n'}(X)) > 0$ for some $n' > n \geq 0$. The minimality of the dimension p implies that $f^n(X) = f^{n'}(X)$.

Replacing X, f, and ϕ by $f^n(X), f^{n'-n}$, and $\phi + \cdots + \phi \circ f^{n'-n-1}$ we can assume that X is invariant and $\mu_\phi(X) > 0$. Since μ_ϕ is mixing, it is ergodic. We then deduce that $\mu_\phi(X) = 1$. Therefore, the metric entropy of μ_ϕ is smaller than the topological entropy of f on X. But this is a contradiction because the last one is at most equal to $p \log d$, see [DS10a, Th. 1.108 and Ex. 1.122]. The result follows. □

4.7. **Equidistribution of periodic points and end of the proof of Theorem 1.1.** Because of Proposition 4.11, Corollary 4.12, Lemma 4.13, and Propositions 4.15 and 4.18, to prove Theorem 1.1 it only remains to establish the equidistribution of (weighted) repelling periodic points of period n with respect to μ_ϕ, as $n \to \infty$.

Theorem 4.20. *Let* $f\colon \mathbb{P}^k \to \mathbb{P}^k$ *be a holomorphic endomorphism of* $\mathbb{P}^k$ *of algebraic degree* $d \geq 2$ *and satisfying Assumption* **(A)***. Let* $\phi\colon \mathbb{P}^k \to \mathbb{R}$ *satisfy* $\|\phi\|_{\log^q} < \infty$ *for some* $q > 2$ *and* $\Omega(\phi) < \log d$*. Let* μ_ϕ *be the unique equilibrium state associated to* ϕ*, and* λ *the scaling ratio. Then for every* $n \in \mathbb{N}$ *there exists a set* P'_n *of repelling periodic points of period* n *in the small Julia set such that*

$$\lim_{n\to\infty} \lambda^{-n} \sum_{y \in P'_n} e^{\phi(y)+\phi(f(y))+\cdots+\phi(f^{n-1}(y))} \delta_y = \mu_\phi. \tag{4.13}$$

Note that a related equidistribution property for Hölder continuous weights was proved by Comman-River-Letelier [CRL11] for (hyperbolic and) topologically Collect-Eckmann rational maps on $\mathbb{P}^1$.

To prove Theorem 4.20, we follow a now classical strategy due to Briend-Duval [BD99] for the measure of maximal entropy (which corresponds to the case $\phi \equiv 0$). We employ a trick due to X. Buff which simplifies the original proof. An extra difficulty with respect to the case $\phi \equiv 0$ is due to the fact that there is no a priori upper bound for the mass of the left hand side of (4.13) when P'_n is replaced by the set of all repelling periodic points of period n.

Given any point $x \in \mathbb{P}^k$ we denote by $\mu_{x,n}$ the measure

$$\mu_{x,n} := \lambda^{-n} \rho(x)^{-1} \sum_{f^n(a)=x} e^{\phi(a)+\phi(f(a))+\cdots+\phi(f^{n-1}(a))} \rho(a) \delta_a.$$

It follows from Corollary 4.12 that, for every continuous function $g \colon \mathbb{P}^k \to \mathbb{R}$, we have

$$\langle \mu_{x,n}, g \rangle = \lambda^{-n} \rho(x)^{-1} \sum_{f^n(a)=x} e^{\phi(a)+\phi(f(a))+\cdots+\phi(f^{n-1}(a))} \rho(a) g(a) \to \rho(x)^{-1} \langle \rho(x) m_\phi, \rho g \rangle = \langle \mu_\phi, g \rangle$$

as $n \to \infty$. This means that, for all $x \in \mathbb{P}^k$, we have $\mu_{x,n} \to \mu_\phi$ as $n \to \infty$.

We denote by $0 < L_1 \leq \cdots \leq L_k$ the Lyapunov exponents of μ_ϕ, see Proposition 4.19. We fix in what follows a constant $0 < L_0 < L_1$. Given $x \in X$, a ball B of center x, and $n \in \mathbb{N}$, we say that $g \colon B \to B'$ is an *m-good inverse branch of f of order n on B* if

$$g \circ f^n = \mathrm{id}_{|B'} \quad \text{and} \quad \operatorname{diam} f^l(B') \leq e^{-m-(n-l)L_0} \text{ for all } 0 \leq l \leq n.$$

Notice that the definition in particular implies that $\operatorname{diam}(B) \leq e^{-m}$. We denote by $\mu^{(m)}_{B,n}$ the measure

$$\mu^{(m)}_{B,n} := \lambda^{-n} \rho(x)^{-1} \sum_{a=g(x)} e^{\phi(a)+\phi(f(a))+\cdots+\phi(f^{n-1}(a))} \rho(a) \delta_a,$$

where the sum is taken on the m-good inverse branches g of f of order n on B. Since we have $\mu^{(m)}_{B,n} \leq \mu_{x,n}$ for all $n \geq 0$, it follows that any limit value μ'_B of the sequence $\{\mu^{(m)}_{B,n}\}$ satisfies $\mu'_B \leq \mu_\phi$. In particular, we have $\|\mu'_B\| \leq 1$.

Given $m > 1$ we say that a ball B centred at x is *m-nice* if

(i) $\inf_B \rho > (1 - 1/m) \sup_B \rho$;

(ii) $\|\mu^{(m)}_{B,n}\| \geq 1 - 1/m$ for every n sufficiently large.

Observe that the second condition implies that $\operatorname{diam}(B) \leq e^{-m}$ for every m-nice ball B. Moreover, we have $\|\mu'_B\| \geq 1 - 1/m$ for every limit value μ'_B of the sequence $\mu^{(m)}_{B,n}$.

Lemma 4.21. *For μ_ϕ-almost every $x \in \mathbb{P}^k$, every sufficiently small ball centred at x is m-nice.*

The proof of Lemma 4.21 is elementary but makes uses of the natural extension of the system $(\mathbb{P}^k, f, \mu_\phi)$, see for instance [CFS12, Sec. 10.4]. We denote by X_0, C_f, PC_f the small Julia set, the critical set and the postcritical set $PC_f := \cup_{n\geq 0} f^n(C_f)$ of f, respectively. We also set $X := X_0 \setminus \cup_{m\in\mathbb{N}} f^{-m}(PC_f)$. By Proposition 4.19 we have $\mu_\phi(f^{-m}(PC_f)) = 0$ for every $m \in \mathbb{N}$, hence $\mu(X) = 1$. We denote by $\hat{X}$ the set

$$\hat{X} := \{\hat{x} := (x_n)_{n\in\mathbb{Z}} \colon x_n \in X, f(x_n) = x_{n+1}\},$$

by $\pi_n : \hat{x} \mapsto x_n$ the natural projection from $\hat{X}$ to X and by $\hat{f} \colon \hat{X} \to \hat{X}$ the map

$$\hat{f}(\dots, x_{-1}, x_0, x_1, \dots) := (\dots, f(x_{-1}), f(x_0), f(x_1), \dots) = (\dots, x_0, x_1, x_2, \dots).$$

Observe that $\pi_n \circ \hat{f} = f \circ \pi_n$ for all $n \in \mathbb{Z}$. Let us consider on $\hat{X}$ the σ-algebra $\hat{\mathcal{B}}$ generated by all *cylinders*, i.e., the sets of the form

$$A_{n,B} := \pi_n^{-1}(B) = \{\hat{x} \colon x_n \in B\} \text{ for } n \leq 0 \text{ and } B \subseteq \mathbb{P}^k \text{ a Borel set}$$

and set

$$\hat{\mu}_\phi(A_{n,B}) := \mu_\phi(B) \text{ for all } A_{n,B} \text{ as above.}$$

It follows from the invariance of μ_ϕ and the fact that $x_n \in B$ if and only if $x_{n-m} \in f^{-m}(B)$ (with $m \geq 0$) that $\hat{\mu}_\phi$ is well defined on the collection of the sets $A_{n,B}$ and

$$\hat{\mu}_\phi(A_{n,B}) = \hat{\mu}_\phi(A_{n-m,B}) \text{ for all } m \geq 0.$$

Similarly, for every $m > 0$ and Borel sets $B_0, B_{-1}, \ldots, B_{-m} \subseteq \mathbb{P}^k$ we then have

$$\begin{aligned}\hat{\mu}_\phi(\{\hat{x}\colon x_0 \in B_0, x_{-1} \in B_{-1}, \ldots, x_{-m} \in B_{-m}\}) \\ = \hat{\mu}_\phi(\{\hat{x}\colon x_{-m} \in f^{-m}(B_0) \cap f^{-(m-1)}(B_{-1}) \cap \cdots \cap B_{-m}\}) \\ = \mu_\phi(f^{-m}(B_0) \cap f^{-(m-1)}(B_{-1}) \cap \cdots \cap B_{-m}).\end{aligned}$$

We then extend $\hat{\mu}_\phi$ to a probability measure, still denoted by $\hat{\mu}_\phi$, on $\hat{\mathcal{B}}$. Observe that $\hat{\mu}_\phi$ is $\hat{f}$-invariant by construction and satisfies $(\pi_0)_*\hat{\mu}_\phi = \mu_\phi$.

For $n > 0$ we denote by $f_{\hat{x}}^{-n}$ the inverse branch of f^n defined in a neighbourhood of x_0 and such that $f_{\hat{x}}^{-n}(x_0) = x_{-n}$. This branch exists for all $x_0 \in X$. We have the following lemma.

Lemma 4.22. *For every $0 < L < L_1$ there exist two measurable functions $\eta_L\colon \hat{X} \to (0,1]$ and $S_L\colon \hat{X} \to (1,+\infty)$ such that, for $\hat{\mu}_\phi$-almost every $\hat{x} \in \hat{X}$, the map $f_{\hat{x}}^{-n}$ is defined on $\mathbb{B}_{\mathbb{P}^k}(x_0, \eta_L(\hat{x}))$ with $\mathrm{Lip}(f_{\hat{x}}^{-n}) \leq S_L(\hat{x})e^{-nL}$ for every $n \in \mathbb{N}$.*

Sketch of proof. The statement is a consequence of Proposition 4.19. A direct proof in the case $\phi = 0$ is given in [BD99, Sec. 2] and [BDM08, Thm. 1.4(3)]. The case $n = 1$ comes from a (quantitative) application of the inverse mapping theorem, which is then iterated to get functions η_L and S_L valid for all n. The main point in the proof is an application of the Birkhoff ergodic theorem to the function $\log|\operatorname{Jac} Df|$. This function is integrable with respect to the measure of maximal entropy μ_0, which has continuous potentials, because of the Chern-Levine-Nirenberg inequality [CLN69]. Since this function is integrable with respect to μ_ϕ by Proposition 4.19, the same proof applies in our setting. □

Proof of Lemma 4.21. Since ρ is continuous and strictly positive, we only need to check that, for μ_ϕ-almost every $x \in \mathbb{P}^k$, every sufficiently small ball B centred at x satisfies $\|\mu_{B,n}^{(m)}\| \geq 1 - 1/m$ for every n sufficiently large.

Let us consider the disintegration of the measure $\hat{\mu}_\phi$ with respect to μ_ϕ and the projection π_0. We denote by $\hat{\mu}_\phi^x$ the conditional measure on $\{x_0 = x\}$. The measure $\hat{\mu}_\phi^x$ is uniquely defined for μ_ϕ-almost all $x \in X$ and characterized by the identity

$$\langle \hat{\mu}_\phi, g \rangle = \langle \mu_\phi, u(x) \rangle, \text{ where } u(x) := \langle \hat{\mu}_\phi^x, g \rangle$$

for all bounded measurable functions $g\colon \hat{X} \to \mathbb{R}$. Since $(\pi_0)_*\hat{\mu}_\phi = \mu_\phi$, $\hat{\mu}_\phi^x$ is a probability measure for μ_ϕ-almost every x.

We will need a more explicit description of the conditional measures $\hat{\mu}_\phi^x$. For $n > 0$ and $x \in X$ we consider the measure $\hat{\mu}_n^x$ on $\hat{X}$ defined as follows. First, let us consider the projection $\hat{X} \to X^{n+1}$ given by

$$\hat{\pi}^n := (\pi_{-n}, \ldots, \pi_{-1}, \pi_0).$$

For every element $(y_{-n}, \ldots, y_0) \in X^{n+1}$ we choose a representative $\hat{z} \in \hat{X}$ such that $z_j = y_j$ for all $-n \leq j \leq 0$. For any given y_0 and any $n > 0$ we then have d^{kn} distinct such representatives, and we denote by $\hat{Z}_n$ their collection. We then set

$$\hat{\mu}_n^x := \lambda^{-n}\rho(x)^{-1} \sum_{\hat{z} \in \hat{Z}_n\colon z_0 = x} e^{\phi(z_{-n}) + \phi(z_{-n+1}) + \cdots + \phi(z_{-1})} \rho(z_{-n}) \delta_{\hat{z}}.$$

Since this is a finite sum, the measures $\hat{\mu}_n^x$ are well defined on $\hat{X}$.

Claim. We have $\lim_{n\to\infty} \hat{\mu}_n^x = \hat{\mu}_\phi^x$ for μ_ϕ-almost every $x \in X$.

Proof. It is enough to check the assertion on the cylinders $A_{-i,B}$ for $i \geq 0$ and $B \subseteq \mathbb{P}^k$ a Borel set. It is clear that, for all $n > 0$, we have $\hat{\mu}_n^x(A_{0,B}) = \delta_x(B)$, which implies that

$$\int \hat{\mu}_n^x(A_{0,B})\mu_\phi(x) = \int \delta_x(B)\mu_\phi(x) = \mu_\phi(B).$$

Moreover, for all $n > i$, using the invariance of ρ by $\lambda^{-1}\mathcal{L}$ we have

$$\begin{aligned}
\hat{\mu}_n^x(A_{-i,B}) &= \hat{\mu}_n^x(A_{-i,B} \cap \pi_0^{-1}(x)) \\
&= \lambda^{-n}\rho(x)^{-1} \sum_{\hat{z} \in \hat{Z}_n :\, z_0 = x} e^{\phi(z_{-n})+\phi(z_{-n+1})+\cdots+\phi(z_{-1})}\rho(z_{-n})\delta_{\hat{z}}(A_{-i,B}) \\
&= \lambda^{-n}\rho(x)^{-1} \sum_{\hat{z} \in \hat{Z}_i :\, z_0 = x} (\mathcal{L}^{n-i}\rho)(z_{-i})e^{\phi(z_{-i})+\phi(z_{-i+1})+\cdots+\phi(z_{-1})}\delta_{\hat{z}}(A_{-i,B}) \\
&= \lambda^{-i}\rho(x)^{-1} \sum_{\hat{z} \in \hat{Z}_i :\, z_0 = x} \rho(z_{-i})e^{\phi(z_{-i})+\phi(z_{-i+1})+\cdots+\phi(z_{-1})}\delta_{\hat{z}}(A_{-i,B}) \\
&= \hat{\mu}_i^x(A_{-i,B}).
\end{aligned}$$

In order to conclude it is enough to prove that

$$\int \hat{\mu}_i^x(A_{-i,B})\mu_\phi(x) = \mu_\phi(B) \text{ for all } i > 0.$$

We have

$$\begin{aligned}
\int \hat{\mu}_i^x(A_{-i,B})\mu_\phi(x) &= \int \Big(\lambda^{-i}\rho(x)^{-1} \sum_{\hat{z} \in \hat{Z}_i :\, z_0 = x} e^{\phi(z_{-i})+\phi(z_{-i+1})+\cdots+\phi(z_{-1})}\rho(z_{-i})\delta_{\hat{z}}(A_{-i,B})\Big)\mu_\phi(x) \\
&= \int \Big(\lambda^{-i}\rho(x)^{-1} \sum_{f^i(a)=x} e^{\phi(a)+\phi(f(a))+\cdots+\phi(f^{i-1}(a))}\rho(a)\mathbb{1}_B(a)\Big)\mu_\phi(x) \\
&= \Big\langle \mu_\phi, \lambda^{-i}\rho^{-1}f^i_*(e^{\phi+\phi\circ f+\cdots+\phi\circ f^{i-1}}\rho\mathbb{1}_B)\Big\rangle \\
&= \Big\langle \frac{\rho e^{\phi+\phi\circ f+\cdots+\phi\circ f^{i-1}}}{\lambda^i(\rho\circ f^i)}(f^i)^*\mu_\phi, \mathbb{1}_B\Big\rangle = \mu_\phi(B),
\end{aligned}$$

where in the last step we used the fact that the Jacobian of μ_ϕ (i.e., the Radon-Nidokym derivative $\frac{f^*\mu_\phi}{\mu_\phi}$) is given by $\lambda\rho^{-1}e^{-\phi}(\rho\circ f)$, which implies that

$$(f^i)^*\mu_\phi = \lambda^i\rho^{-1}e^{-\sum_{j=0}^{i-1}\phi\circ f^j}(\rho\circ f^i)\mu_\phi.$$

This completes the proof of the Claim. $\square$

Let us now fix an integer $m > 0$, a constant $L_0 < L < L_1$, and a second positive integer γ. For every integer $N > 0$ we set

$$\hat{X}_N := \big\{\hat{x} \in \hat{X} : \eta_L(\hat{x}) \geq N^{-1} \text{ and } S_L(\hat{x}) \leq N\big\}.$$

Observe that $\hat{\mu}_\phi(\hat{X}_N) \to 1$ as $N \to \infty$. In particular, there exists $N_0 = N_0(m,\gamma)$ such that, for every $N > N_0$, we have $\hat{\mu}_\phi(\hat{X}_N) > 1 - 1/(2m^{\gamma+1})$. It follows by Markov inequality that there exists a subset $X_\gamma \subset X$ with $\mu_\phi(X_\gamma) > 1 - 1/m^\gamma$ such that, for all $N > N_0$,

$$\hat{\mu}_\phi^x(\hat{X}_N \cap \{x_0 = x\}) > 1 - 1/(2m) \text{ for all } x \in X_\gamma.$$

It is enough to prove the property in the lemma for all $x \in X_\gamma$. Let us fix one such x. By Lemma 4.22 and the definition of $\hat{X}_N$, for every $\hat{x} \in \hat{X}_N$ and $n \geq 0$ the inverse branch $f_{\hat{x}}^{-n}$ is defined on the ball $B_{\mathbb{P}^k}(x_0, N^{-1})$ with $\mathrm{Lip}(f_{\hat{x}}^{-n}) \leq Ne^{-nL}$. In particular, $\mathrm{diam}(f_{\hat{x}}^{-n}(B_{\mathbb{P}^k}(x_0, e^{-m}/(2N)))) \leq e^{-m-nL_0}$ for all $n \geq 0$. It follows that all inverse branches on $B_{\mathbb{P}^k}(x, e^{-m}/(2N))$ corresponding to elements $\hat{x} \in \hat{X}_N \cap \{x_0 = x\}$ are m-good for all n.

The Claim above implies that

$$\hat{\mu}_n^x(\hat{X}_N \cap \{x_0 = x\}) > 1 - 1/m \text{ for all } n \text{ large enough.}$$

This precisely means that, for all n sufficiently large, we have $\|\mu_{B,n}^{(m)}\| > 1 - 1/m$, where $B = B_{\mathbb{P}^k}(x, e^{-m}/(2N))$. This implies that such a ball B is m-nice. The proof is complete. □

Lemma 4.23. *There exists a positive constant $C = C(L_0, q)$ such that, for all $n \in \mathbb{N}, m > 0$, and every m-good inverse branch $g\colon B \to B'$ of f of order n on a ball B, and for all sequences of points $\{x_l\}, \{y_l\}$ with $0 \le l \le n-1$ and $x_l, y_l \in f^l(B')$ we have*

$$\sum_{l=0}^{n-1} |\phi(x_l) - \phi(y_l)| \le C m^{-(q-1)}.$$

Proof. Since g is m-good, we have $\mathrm{dist}(x_l, y_l) \le e^{-m-(n-l)L_0}$ for all $0 \le l \le n-1$. Hence,

$$\sum_{l=0}^{n-1} |\phi(x_l) - \phi(y_l)| \le \sum_{l=0}^{n-1} \|\phi\|_{\log^q} |\log^\star \mathrm{dist}(x_l, y_l)|^{-q} \le \|\phi\|_{\log^q} \sum_{l=1}^{\infty} |1 + m + lL_0|^{-q} \lesssim m^{-(q-1)},$$

where the implicit constant depends on L_0, q and we used the assumption that $q > 2$. □

Lemma 4.24. *Let $\mathcal{U}$ be a finite collection of disjoint open subsets of $\mathbb{P}^k$. For every $m > 0$ there exists $n(m, \mathcal{U}) > m$ and, for every $n \ge n(m, \mathcal{U})$, a set $Q_{m,n}$ of repelling periodic points of period n in the intersection of the union of the sets in $\mathcal{U}$ with the small Julia set such that, for all $U \in \mathcal{U}$,*

$$(1 - 1/m)\mu_\phi(U) \le \lambda^{-n} \sum_{y \in Q_{m,n} \cap U} e^{\phi(y) + \phi(f(y)) + \cdots + \phi(f^{n-1}(y))} \le (1 + 1/m)\mu_\phi(U).$$

Proof. We can assume that $\mathcal{U}$ consists of a single open set U, the general case follows by taking $n(m, \mathcal{U})$ to be the maximum of the $n(m, U)$, for $U \in \mathcal{U}$. We can also assume that $\mu_\phi(U) > 0$ because otherwise we can choose $n(m, U) = m + 1$ and $Q_{m,n} = \varnothing$. Fix integers $m_2 \gg m_1 \gg m$. By Lemma 4.21, for μ_ϕ-almost every point a, every ball of sufficiently small radius centred at a is m_2-nice. Hence, we can find a finite family of disjoint m_2-nice balls $B_i \Subset U$, such that $\mu_\phi(U \setminus \cup B_i) < \mu_\phi(U)/m_2$. It is then enough to prove the lemma for each B_i instead of U. More precisely, let $B = B_{\mathbb{P}^k}(a, r)$ be an m_2-nice ball. It is enough to find an $n(m_2) > m_2$ and, for all $n \ge n(m_2)$, a set Q of repelling periodic points of period n in $B \cap \mathrm{supp}(\mu_\phi)$ such that

$$(1 - 1/m_1)\mu_\phi(B) \le \lambda^{-n} \sum_{y \in Q} e^{\phi(y) + \phi(f(y)) + \cdots + \phi(f^{n-1}(y))} \le (1 + 1/m_1)\mu_\phi(B). \tag{4.14}$$

We fix in what follows an integer $m_3 \gg m_2/\mu_\phi(B)$ and a second ball $B^\star = B_{\mathbb{P}^k}(a, r^\star)$, with $r^\star < r$, such that $\mu_\phi(B^\star) > (1 - 1/m_2)\mu_\phi(B)$. Choose a finite family of disjoint m_3-nice balls D_i with the property that $\mu_\phi(\cup D_i) > 1 - 1/m_3$. We set $D := \cup D_i$ and let b_i be the center of D_i. We also fix balls $D_i^\star \Subset D_i$ centred at b_i and such that $\mu_\phi(\cup D_i^\star) > 1 - 1/m_3$ and set $D^\star := \cup D_i^\star$.

Claim 1. There is an integer $M_1 = M_1(m_2, B, B^\star, D_i)$ such that, for all $N \ge M_1$, we have

$$(1 - 4/m_2)\mu_\phi(B) \le \mu_{D_i,N}^{(m_3)}(B^\star) \le (1 + 4/m_2)\mu_\phi(B) \quad \text{for all } i. \tag{4.15}$$

Proof. Since the balls D_i are m_3-nice and $m_3 \gg m_2/\mu_\phi(B)$, for every i we have

$$\|\mu_{D_i,N}^{(m_3)}\| \ge (1 - \mu_\phi(B)/m_2) \text{ for all } N \text{ large enough.}$$

Hence, since $\mu_{D_i,N}^{(m_3)} \le \mu_{b_i,N}$ and $\|\mu_{b_i,N}\| \le 1 + o(1)$, we have $\|\mu_{b_i,N} - \mu_{D_i,N}^{(m_3)}\| \le \mu_\phi(B)/m_2 + o(1)$. Therefore, in order to prove the claim it is enough to show that

$$(1 - 2/m_2)\mu_\phi(B) \le \mu_{b_i,N}(B^\star) \le (1 + 2/m_2)\mu_\phi(B)$$

for all i and all N large enough. This is a consequence of Corollary 4.12 and of the inequality $\mu_\phi(B^\star) > (1 - 1/m_2)\mu_\phi(B)$. □

Similarly, we also have the following.

Claim 2. There is an integer $M_2 = M_2(m_2, B, D^\star)$ such that, for all $N \geq M_2$, we have

$$1 - 4/m_2 \leq \mu_{B,N}^{(m_2)}(D^\star) \leq 1 + 4/m_2. \tag{4.16}$$

Proof. Since the ball B is m_2-nice, we have

$$\|\mu_{B,N}^{(m_2)}\| \geq (1 - 1/m_2) \text{ for all } N \text{ large enough.}$$

Hence, by the fact that $\mu_{B,N}^{(m_2)} \leq \mu_{a,N}$ and $\|\mu_{a,N}\| \leq 1 + o(1)$, in order to prove the claim it is enough to show that

$$1 - 2/m_2 \leq \mu_{a,N}(D^\star) \leq 1 + 2/m_2$$

for all N large enough. This is again a consequence of Corollary 4.12 and of the inequality $\mu_\phi(D^\star) > (1 - 1/m_3)$. □

For every N_1 sufficiently large, every point in the support of $\mathbb{1}_{B^\star}\mu_{D_i,N_1}^{(m_3)}$ corresponds to an m_3-good inverse branch of f of order N_1 mapping D_i to a relatively compact subset of B. Similarly, for every N_2 sufficiently large every point in the support of $\mathbb{1}_{D^\star}\mu_{B,N_2}^{(m_2)}$ corresponds to an m_2-good inverse branch of f of order N_2 mapping B to a relatively compact subset of D. Composing such inverse branches we get inverse branches g_j of $f^{N_1+N_2}$ defined on B whose images are relatively compact in B. In what follows, we only consider these inverse branches g_j. We also write g_j as $g_j^{(1)} \circ g_j^{(2)}$, where $g_j^{(2)}$ is the corresponding inverse branch of f^{N_2} on B (whose image is then in D) and $g_j^{(1)}$ is the corresponding inverse branch of f^{N_1} on $g_j^{(2)}(B)$. We also set $i = i(j)$, where $g_j^{(2)}(B) \subset D_i$.

Each inverse branch g_j as above contracts the Kobayashi metric of B, and thus admits a unique fixed point y_j, which is attracting for g_j and hence repelling for $f^{N_1+N_2}$. Up to possibly increasing the integers M_1 and M_2 given by the Claims above, we can assume that the above properties hold for $N_1 = M_1$ and $N_2 = M_2$. We set $n(m) := M_1(m_2) + M_2(m_2)$ for a fixed choice of sufficiently large m_1, m_2, m_3 and, for all $n \geq n(m)$, we define the set Q as the union of all such fixed points constructed as above with $N_1 = M_1(m_2)$ and $N_2 = n - N_1 \geq M_2(m_2)$. The points in Q are then repelling periodic points of period $n = N_1 + N_2$ for f. Observe that, for all j and all $z \in B$, since $g_j(B) \Subset B$ we have $g_j^l(z) \to y_j$ as $l \to \infty$. Since B intersects the small Julia set, by taking z in the small Julia set we see that y_j belongs to the small Julia set. To conclude, we need to prove (4.14) for this choice of Q. We set

$$\mu_n := \lambda^{-n} \sum_{y \in Q} e^{\phi(y)+\phi(f(y))+\cdots+\phi(f^{n-1}(y))}\delta_y = \sum_j e^{\phi(y)+\phi(f(y))+\cdots+\phi(f^{n-1}(y))}\delta_{y_j}$$

and

$$\tilde{\mu}_n := \lambda^{-n} \sum_j \Big(e^{\phi(g_j^{(1)}(b_{i(j)}))+\phi(f\circ g_j^{(1)}(b_{i(j)}))+\cdots+\phi(f^{N_1-1}\circ g_j^{(1)}(b_{i(j)}))} \frac{\rho(g_j^{(1)}(b_{i(j)}))}{\rho(b_{i(j)})} \cdot$$

$$\cdot\, e^{\phi(g_j^{(2)}(a))+\phi(f\circ g_j^{(2)}(a))+\cdots+\phi(f^{N_2-1}\circ g_j^{(2)}(a))} \frac{\rho(g_j^{(2)}(a))}{\rho(a)} \delta_{g_j(a)} \Big).$$

Observe that there is a correspondence between the terms in μ_n and those in $\tilde{\mu}_n$. Moreover, since all the balls B and D_i are m_2-nice, we have

$$|\rho(g_j^{(1)}(b_{i(j)}))/\rho(a) - 1| \lesssim m_2^{-1} \text{ and } |\rho(g_j^{(2)}(a))/\rho(b_{i(j)}) - 1| \lesssim m_2^{-1} \text{ for all } i \text{ and } j.$$

It follows from these inequalities and Lemma 4.23 that $|\mu_n(B) - \tilde{\mu}_n(B)| \lesssim \tilde{\mu}_n(B) m_2^{-1}$. Hence, in order to conclude it is enough to prove that

$$(1 - 1/(2m_1))\mu_\phi(B) \leq \tilde{\mu}_n(B) \leq (1 + 1/(2m_1))\mu_\phi(B)$$

because m_2 is chosen large enough. By construction, we have

$$\tilde{\mu}_n(B) = \sum_i \mu^{(m_2)}_{B,N_2}(D^\star_i) \cdot \mu^{(m_3)}_{D_i,N_1}(B^\star).$$

By Claim 1, this implies that

$$(1 - 4/m_2)\mu_\phi(B) \sum_i \mu^{(m_2)}_{B,N_2}(D^\star_i) \leq \tilde{\mu}_n(B) \leq (1 + 4/m_2)\mu_\phi(B) \sum_i \mu^{(m_2)}_{B,N_2}(D^\star_i).$$

The assertion then follows from Claim 2 and the fact that $\sum_i \mu^{(m_2)}_{B,N_2}(D^\star_i) = \mu^{(m_2)}_{B,N_2}(D^\star)$, by taking m_2 large enough. □

We can now conclude the proof of Theorem 4.20. As mentioned at the beginning of the section, this also completes the proof of Theorem 1.1.

End of the proof of Theorem 4.20. For every $i \in \mathbb{N}$ we construct a finite family of disjoint open sets $\mathcal{U}_i := \{U_{i,j}\}_{1\leq j\leq J_i}$ with the following properties:

(i) $\mu_\phi(\cup_{1\leq j\leq J_i} U_{i,j}) = 1$;
(ii) for all $1 \leq j \leq J_i$ we have $\mathrm{diam}(U_{i,j}) < 1/i$;
(iii) for all $i \geq 2$ and $1 \leq j \leq J_i$ there exists $1 \leq j' \leq J_{i-1}$ such that $U_{i,j} \subset U_{i-1,j'}$.

We can construct these sets using local coordinates and generic real hyperplanes which are parallel to the coordinate hyperplanes. Observe also that, by the first condition, we have $\mu_\phi(\partial U_{i,j}) = 0$ for all i and $1 \leq j \leq J_i$.

For every n, we define $i_n := \max\{m \leq n : n \geq n(m, \mathcal{U}_m)\}$, where $n(m, \mathcal{U}_m)$ is given by Lemma 4.24. Observe that $i_n \to \infty$ as $n \to \infty$. We define $P'_n \subset \cup_j U_{i_n,j}$ as the union of the sets of repelling periodic points of period n in the small Julia set obtained by applying Lemma 4.24 to the collection $\mathcal{U}_{i_n}$ instead of $\mathcal{U}$, and set

$$\mu'_n := \lambda^{-n} \sum_{y\in P'_n} e^{\phi(y)+\phi(f(y))+\cdots+\phi(f^{n-1}(y))}\delta_y.$$

By Properties (i) and (ii) of the open sets $U_{i,j}$ and Lemma 4.24, any limit μ' of the sequence $\{\mu'_n\}$ has mass 1. So, since $\mu_\phi(\cup_j U_{i_n,j}) = 1$ for all n and $\mathrm{diam}(U_{i,j}) < 1/i$ for all i, it is enough to prove that

$$\liminf_{n\to\infty} \mu'_n(U_{i^\star,j^\star}) \geq \mu_\phi(U_{i^\star,j^\star}) \text{ for all } i^\star \in \mathbb{N} \text{ and } 1 \leq j^\star \leq J_{i^\star}. \tag{4.17}$$

Indeed, given any open set $A \subseteq \mathbb{P}^k$, we can write A as a countable union of compact sets of the form $\bar{U}_{i,j} \Subset A$, overlapping only on their boundaries. We then see that (4.17) implies that $\mu_\phi(A) \leq \mu'(A)$ for every open set A, and the facts that $\|\mu_\phi\| = \|\mu'\|$ and $\mu_\phi(\partial U_{i,j}) = 0$ for all i, j imply that $\mu_\phi = \mu'$.

We can then fix $i^\star, j^\star$ as in (4.17) and a positive number ϵ, and it is enough to prove that

$$\mu'_n(U_{i^\star,j^\star}) \geq \mu_\phi(U_{i^\star,j^\star}) - \epsilon \quad \text{for all } n \text{ sufficiently large.}$$

We only consider in what follows integers n such that $i_n > i^\star$ and the sets $U_{i_n,j}$ which are contained in $U_{i^\star,j^\star}$. For all such n, we have $\mu_\phi(U_{i^\star,j^\star}) = \sum_j \mu_\phi(U_{i_n,j})$ and $\mu'_n(U_{i^\star,j^\star}) = \sum_j \mu'_n(U_{i_n,j})$. It follows by the definition of μ'_n and Lemma 4.24 that

$$\left|\mu'_n(U_{i^\star,j^\star}) - \mu_\phi(U_{i^\star,j^\star})\right| \leq \sum_j \left|\mu'_n(U_{i_n,j}) - \mu_\phi(U_{i_n,j})\right| \leq i_n^{-1} \sum_j \mu_\phi(U_{i_n,j}) = i_n^{-1}\mu_\phi(U).$$

The assertion follows. □

Remark 4.25. One could improve the argument in the proof of Lemma 4.24 to obtain that P'_n can be taken to be a subset of the repelling periodic points with a good control of the eigenvalues, see for instance [BDM08, BD19]. This implies that, setting $\Sigma_j := L_{k-j+1} + \cdots + L_k$, we have

$$\Sigma_j = \lim_{n\to\infty} \lambda^{-n} \sum_{y\in P'_n} \frac{1}{n} e^{\phi(y)+\phi(f(y))+\cdots+\phi(f^{n-1}(y))} \log \left\| \bigwedge^j Df^n_y \right\|$$

and, in particular,

$$\Sigma_k = \sum_{j=1}^k L_j = \lim_{n\to\infty} \lambda^{-n} \sum_{y\in P'_n} e^{\phi(y)+\phi(f(y))+\cdots+\phi(f^{n-1}(y))} \log|\operatorname{Jac} Df_y|.$$

Here, $Df^n_x\colon T_x\mathbb{P}^k \to T_{f^n(x)}\mathbb{P}^k$ denotes the differential of f^n at x. This is a linear map from the complex tangent space of $\mathbb{P}^k$ at x to the one at $f^n(x)$. It induces the natural linear map $\bigwedge^j Df^n_x$ from the exterior power $\bigwedge^j T_x\mathbb{P}^k$ to $\bigwedge^j T_{f^n(x)}\mathbb{P}^k$.

5. Spectral gap for the transfer operator

In this section we prove our main Theorem 1.2. We already defined in the previous section the scaling ratio λ, the density function ρ as an eigenfunction for the operator $\mathcal{L} = \mathcal{L}_\phi$, and the probability measures m_ϕ and μ_ϕ, all under the hypothesis that $\|\phi\|_{\log^q} < \infty$ for some $q > 2$. The semi-norms $\|\cdot\|_{\langle p,\alpha\rangle}$ and $\|\cdot\|_{\langle p,\alpha\rangle,\gamma}$ were introduced in Sections 3.5 and 3.6, respectively.

5.1. Main result and first step of the proof.

The following is the main result of this section. We will use it in order to prove Theorem 1.2 with a suitable norm and another value of β.

Theorem 5.1. *Let $f, \phi, \lambda, m_\phi, \rho$ be as in Theorem 1.1 and $\mathcal{L}$ the Perron-Frobenius operator associated to ϕ. Let $p, \alpha, \gamma, A, \Omega$ be positive constants and q_2 as in Lemma 3.16 such that $p > 3/2$, $d^{-1} \le \alpha < d^{-5/(2p+2)}$, $\Omega < \log(d\alpha)$, and $q_2 > 2$. Assume that $\|\phi\|_{\langle p,\alpha\rangle,\gamma} \le A$ and $\Omega(\phi) \le \Omega$. Then we have*

$$\|\lambda^{-n}\mathcal{L}^n\|_{\langle p,\alpha\rangle,\gamma} \le c, \quad \|\rho\|_{\langle p,\alpha\rangle,\gamma} \le c, \quad \text{and} \quad \|1/\rho\|_{\langle p,\alpha\rangle,\gamma} \le c$$

for some positive constant $c = c(p, \alpha, \gamma, A, \Omega)$ independent of ϕ and n. Moreover, for every constant $0 < \beta < 1$ there is a positive integer $N = N(p, \alpha, \gamma, A, \Omega, \beta)$ independent of ϕ such that

$$\big\|\lambda^{-N}\mathcal{L}^N g\big\|_{\langle p,\alpha\rangle,\gamma} \le \beta \, \|g\|_{\langle p,\alpha\rangle,\gamma} \tag{5.1}$$

for every function $g\colon \mathbb{P}^k \to \mathbb{R}$ with $\langle m_\phi, g\rangle = 0$. Furthermore, we can take $\beta = \delta^{-N}$ for every constant $0 < \delta < (d\alpha)^{\gamma/(2\gamma+2)}$, provided that A is small enough.

Notice that Lemma 3.16 and the assumption $q_2 > 2$ imply that $\|\phi\|_{\log^q} < \infty$ for some $q > 2$. Hence, the scaling ratio λ, the density function ρ, and the measures m_ϕ and μ_ϕ are well defined by Theorem 1.1. Notice also that $q_2 > 2$ implies that the condition $\alpha < d^{-5/(2p+2)}$ is automatically satisfied.

As it was the case for Theorem 4.1, the proof of Theorem 5.1 will be reduced to a comparison between suitable currents and their norms. Theorem 5.1 will then follow from some interpolation techniques (see Section 5.3). A crucial estimate that we will need here is the following.

Proposition 5.2. *Let f be as in Theorem 1.1. Take $0 < \alpha < 1$ and $p > 0$. Given n functions $\phi^{(j)}\colon \mathbb{P}^k \to \mathbb{R}$ for $j = 1, \dots, n$, set*

$$\Phi_m := \alpha^{-m} d^{(k-1)m} e^{\sum_{j=1}^m \max(\phi^{(j)})} \quad \text{and} \quad \mathcal{L}_{m,n} := \mathcal{L}_{\phi^{(m)}} \circ \cdots \circ \mathcal{L}_{\phi^{(n)}}.$$

Then there exists a positive constant $c = c(p, \alpha)$, independent of $\phi^{(j)}$, such that

$$\|\mathcal{L}_{1,n} g\|_{\langle p,\alpha\rangle} \le c\, \|\mathcal{L}_{1,n}\mathbb{1}\|_\infty^{1/2} \Phi_n^{1/2} \|g\|_{\langle p,\alpha\rangle} + c \sum_{m=1}^n \big\|\phi^{(m)}\big\|_{\langle p,\alpha\rangle} m^{3/2} \Phi_m^{1/2} \|\mathcal{L}_{1,m}\mathbb{1}\|_\infty^{1/2} \|\mathcal{L}_{m+1,n} g\|_\infty$$

for every function $g\colon \mathbb{P}^k \to \mathbb{R}$.

Proof. By Definition 3.12 of the semi-norm $\|\cdot\|_{\langle p,\alpha\rangle}$, we need to bound the current $i\partial\mathcal{L}_{1,n}g \wedge \bar\partial\mathcal{L}_{1,n}g$. We will use here the map π_n introduced in Section 4.2. We have, using a direct computation,

$$\partial\big(e^{\phi^{(n)}(x_0)+\cdots+\phi^{(1)}(x_{n-1})} g(x_0)\big) = \Theta_1 + \Theta_2$$

with

$$\Theta_1 := \mathbb{h}''\partial g(x_0), \quad \Theta_2 := \mathbb{h}''g(x_0)\sum_{m=0}^{n-1}\partial\phi^{(n-m)}(x_m), \quad \text{and} \quad \mathbb{h}'' := e^{\phi^{(n)}(x_0)+\cdots+\phi^{(1)}(x_{n-1})}.$$

Using Cauchy-Schwarz's inequality, we obtain

$$\begin{aligned} i\partial\mathcal{L}_{1,n}g \wedge \bar\partial\mathcal{L}_{1,n}g &= i(\pi_n)_*(\Theta_1+\Theta_2)\wedge(\pi_n)_*(\overline{\Theta}_1+\overline{\Theta}_2) \\ &\leq 2i(\pi_n)_*(\Theta_1)\wedge(\pi_n)_*(\overline{\Theta}_1)+2i(\pi_n)_*(\Theta_2)\wedge(\pi_n)_*(\overline{\Theta}_2). \end{aligned}$$

We need to bound the norm $\|\cdot\|_{p,\alpha}$ of the two terms in the last sum by the square of the RHS of the inequality in the proposition.

For the first term, using again Cauchy-Schwarz's inequality, the definition of Φ_m as in the statement, and Lemma 3.11, we get (notice that $(\pi_n)_*(\mathbb{h}'') = \mathcal{L}_{1,n}\mathbb{1}$)

$$\begin{aligned} \big\|i(\pi_n)_*(\Theta_1)\wedge(\pi_n)_*(\overline{\Theta}_1)\big\|_{p,\alpha} &\leq \big\|(\pi_n)_*(\mathbb{h}'')(\pi_n)_*\big(\mathbb{h}''i\partial g(x_0)\wedge\overline{\partial}g(x_0)\big)\big\|_{p,\alpha} \\ &\leq \|\mathcal{L}_{1,n}\mathbb{1}\|_\infty e^{\sum_{j=1}^n \max\phi^{(j)}}\big\|(f^n)_*(i\partial g\wedge\bar\partial g)\big\|_{p,\alpha} \\ &\leq \|\mathcal{L}_{1,n}\mathbb{1}\|_\infty \Phi_n \big\|i\partial g\wedge\bar\partial g\big\|_{p,\alpha}. \end{aligned}$$

This gives the desired estimate for the first term.

For the second term, observe that $i(\pi_n)_*(\Theta_2)\wedge(\pi_n)_*(\overline{\Theta}_2)$ is equal to

$$\begin{aligned} &\sum_{0\leq m,m'<n} i(\pi_n)_*\big(\mathbb{h}''g(x_0)\partial\phi^{(n-m)}(x_m)\big)\wedge(\pi_n)_*\big(\mathbb{h}''g(x_0)\overline{\partial}\phi^{(n-m')}(x_{m'})\big) \\ &\leq 2\sum_{0\leq m'\leq m<n}\big|i(\pi_n)_*\big(\mathbb{h}''g(x_0)\partial\phi^{(n-m)}(x_m)\big)\wedge(\pi_n)_*\big(\mathbb{h}''g(x_0)\overline{\partial}\phi^{(n-m')}(x_{m'})\big)\big|. \end{aligned}$$

Using Cauchy-Schwarz's inequality as in (4.3) and (4.4) we can bound the current in the absolute value signs by

$$\begin{aligned} &(m-m'+1)^{-2}i(\pi_n)_*\big(\mathbb{h}''g(x_0)\partial\phi^{(n-m)}(x_m)\big)\wedge(\pi_n)_*\big(\mathbb{h}''g(x_0)\overline{\partial}\phi^{(n-m)}(x_m)\big) \\ &\quad +(m-m'+1)^2 i(\pi_n)_*\big(\mathbb{h}''g(x_0)\partial\phi^{(n-m')}(x'_m)\big)\wedge(\pi_n)_*\big(\mathbb{h}''g(x_0)\overline{\partial}\phi^{(n-m')}(x'_m)\big) \end{aligned}$$

and deduce that $i(\pi_n)_*(\Theta_2)\wedge(\pi_n)_*(\overline{\Theta}_2)$ is bounded by a constant times

$$\sum_{0\leq m<n}(n-m)^3 i(\pi_n)_*\big(\mathbb{h}''g(x_0)\partial\phi^{(n-m)}(x_m)\big)\wedge(\pi_n)_*\big(\mathbb{h}''g(x_0)\overline{\partial}\phi^{(n-m)}(x_m)\big).$$

Therefore, in order to get the proposition, setting $\eta := (\pi_n)_*\big(\mathbb{h}''g(x_0)\partial\phi^{(n-m)}(x_m)\big)$ we only need to show that

$$\|i\eta\wedge\overline{\eta}\|_{p,\alpha} \leq \big\|\phi^{(n-m)}\big\|^2_{\langle p,\alpha\rangle}\Phi_{n-m}\|\mathcal{L}_{1,n-m}\mathbb{1}\|_\infty\|\mathcal{L}_{n-m+1,n}g\|^2_\infty.$$

Now, using π'' as in the proof of Proposition 4.4, we see that

$$\eta = \pi''_*\big(\mathcal{L}_{n-m+1,n}g(x_m)\mathbb{h}_m\partial\phi^{(n-m)}(x_m)\big) \quad \text{with} \quad \mathbb{h}_m := e^{\phi^{(n-m)}(x_m)+\cdots+\phi^{(1)}(x_{n-1})}.$$

It follows from Cauchy-Schwarz's inequality that

$$\begin{aligned} i\eta\wedge\overline{\eta} &\leq \|\mathcal{L}_{n-m+1,n}g\|^2_\infty\pi''_*(\mathbb{h}_m)\pi''_*(\mathbb{h}_m i\partial\phi^{(n-m)}(x_m)\wedge\overline{\partial}\phi^{(n-m)}(x_m)) \\ &\leq \|\mathcal{L}_{n-m+1,n}g\|^2_\infty\|\pi''_*(\mathbb{h}_m)\|_\infty\|\mathbb{h}_m\|_\infty(f^{n-m})_*(i\partial\phi^{(n-m)}\wedge\overline{\partial}\phi^{(n-m)}). \end{aligned}$$

Thus, by Lemma 3.11 and the definition of π'', we get

$$\|i\eta\wedge\overline{\eta}\|_{p,\alpha} \leq \|\mathcal{L}_{n-m+1,n}g\|^2_\infty\|\mathcal{L}_{1,n-m}\mathbb{1}\|_\infty\Phi_{n-m}\|i\partial\phi^{(n-m)}\wedge\overline{\partial}\phi^{(n-m)}\|_{p,\alpha}.$$

This ends the proof of the proposition. □

5.2. **Proof of Theorem 5.1.** We will need the following elementary lemma.

Lemma 5.3. *Let $\phi, \phi^{(j)}, \psi \colon \mathbb{P}^k \to \mathbb{R}$ be such that $\sum_{j=1}^{\infty} \big\|\phi - \phi^{(j)}\big\|_\infty \leq a$ for some positive constant a. Define $\vartheta : \mathbb{R}^+ \to \mathbb{R}^+$ by $\vartheta(t) := t^{-1}(e^t - 1)$ and $\vartheta(0) = 1$, which is a smooth increasing function. Then we have*

(i) $\|\mathcal{L}_\phi - \mathcal{L}_\psi\|_\infty \leq \vartheta(\|\phi - \psi\|_\infty) \|\mathcal{L}_\phi\|_\infty \|\phi - \psi\|_\infty$;
(ii) *for every $n \geq 1$,* $\big\|\mathcal{L}^n - \mathcal{L}_{\phi^{(1)}} \circ \cdots \circ \mathcal{L}_{\phi^{(n)}}\big\|_\infty \leq \vartheta(a) a \|\mathcal{L}^n\|_\infty$.

Proof. Observe that for $\phi, \psi \colon \mathbb{P}^k \to \mathbb{R}$ we have, using the definition of $\mathcal{L}_\phi$ and $\mathcal{L}_\psi$,

$$\|\mathcal{L}_\phi(g) - \mathcal{L}_\psi(g)\|_\infty \leq \big\|(1 - e^{\psi - \phi})\|g\|_\infty\big\|_\infty \|\mathcal{L}_\phi(\mathbb{1})\|_\infty = \big\|1 - e^{\psi-\phi}\big\|_\infty \|\mathcal{L}_\phi\|_\infty \|g\|_\infty .$$

The first item in the lemma follows. For the second item, notice that

$$\Big\|\big(\phi + \phi \circ f + \cdots + \phi \circ f^{n-1}\big) - \Big(\phi^{(n)} + \phi^{(n-1)} \circ f + \cdots + \phi^{(1)} \circ f^{n-1}\Big)\Big\|_\infty \leq \sum_{j=1}^{n} \big\|\phi - \phi^{(j)}\big\|_\infty .$$

Therefore, using this estimate and the expansions of $\mathcal{L}^n(g)$ and $\mathcal{L}_{\phi^{(1)}} \circ \cdots \circ \mathcal{L}_{\phi^{(n)}}(g)$, we obtain the result in the same way as the first item. □

We continue the proof of Theorem 5.1. We first prove the following result.

Proposition 5.4. *Under the hypotheses of Theorem 5.1, there exists a positive integer $N_0 = N_0(p, \alpha, \gamma, A, \Omega, \beta)$ independent of ϕ and g and such that (5.1) holds for all $N \geq N_0$.*

As in Lemmas 4.16 and 4.17, we can assume that ϕ belongs to the family of weights

$$\mathcal{Q}_0 := \big\{\phi \colon \mathbb{P}^k \to \mathbb{R} \ : \ \min \phi = 0,\ \|\phi\|_{\langle p,\alpha\rangle,\gamma} \leq A,\ \Omega(\phi) \leq \Omega\big\}.$$

Observe that we can apply Lemmas 4.16 and 4.17 because, by Lemma 3.16 and the assumptions on α and p, the family $\mathcal{Q}_0$ is contained in $\mathcal{P}_0(q, M, \Omega)$ for suitable $q > 2$ and M. Observe also that $\|\phi\|_\infty = \Omega(\phi) \leq \Omega$ and $\|\phi\|_\infty \lesssim \|\phi\|_{\langle p,\alpha\rangle,\gamma} \leq A$.

Consider two constants $K \geq 1$ and $K' \geq 1$ whose values depend on β and will be specialised later. We will not fix A but we assume $A \leq A_0$ for some fixed constant $A_0 > 0$. For a large part of this section we can take $A = A_0$, but at the end of the proof of Theorem 5.1 we will consider $A \to 0$. This is the reason why we will keep the constant A in the estimates below. Note that the constants hidden in the signs $\lesssim$ below are independent of the parameters A, β, K, K', n and also of the constant $0 < \epsilon \leq 1$ and the integer j that we consider now.

Since $\|\phi\|_{\langle p,\alpha\rangle,\gamma} \leq A$, for every $j \geq 1$ there are functions $\phi^{(j)}$ and $\psi^{(j)}$ such that

$$\phi = \phi^{(j)} + \psi^{(j)}, \quad \big\|\phi^{(j)}\big\|_{\langle p,\alpha\rangle} \leq A(Kj^2)^{1/\gamma}(1/\epsilon)^{1/\gamma}, \quad \text{and} \quad \big\|\psi^{(j)}\|_\infty \leq AK^{-1}j^{-2}\epsilon.$$

Observe that $\|\phi^{(j)}\|_\infty$ is bounded by a constant since $\|\phi\|_\infty$ is bounded by a constant.

We can assume for simplicity that $\|g\|_{\langle p,\alpha\rangle,\gamma} \leq 1$, which implies that $\Omega(g)$ is bounded by a constant. Since $\langle m_\phi, g\rangle = 0$ by hypothesis, we deduce that $\|g\|_\infty$ is bounded by a constant. By the definition of the semi-norm $\|\cdot\|_{\langle p,\alpha\rangle,\gamma}$, we can find two functions $g_\epsilon^{(1)}$ and $g_\epsilon^{(2)}$ satisfying

$$g = g_\epsilon^{(1)} + g_\epsilon^{(2)}, \quad \big\|g_\epsilon^{(1)}\big\|_{\langle p,\alpha\rangle} \leq K'^{1/\gamma}(1/\epsilon)^{1/\gamma}, \quad \big\|g_\epsilon^{(2)}\big\|_\infty \leq 2K'^{-1}\epsilon, \quad \langle m_\phi, g_\epsilon^{(1)}\rangle = \langle m_\phi, g_\epsilon^{(2)}\rangle = 0.$$

Notice that without the condition $\langle m_\phi, g_\epsilon^{(2)}\rangle = 0$ we would not need the coefficient 2 in the above estimate of $\|g_\epsilon^{(2)}\|_\infty$. We obtain this condition by adding to $g_\epsilon^{(2)}$ a suitable constant and subtracting the same constant from $g_\epsilon^{(1)}$. The condition $\langle m_\phi, g_\epsilon^{(1)}\rangle = 0$ is deduced from the hypothesis $\langle m_\phi, g\rangle = 0$ when we have $\langle m_\phi, g_\epsilon^{(2)}\rangle = 0$. Since $\|g\|_\infty$ is bounded by a constant, $\|g_\epsilon^{(1)}\|_\infty$ is also bounded by a constant.

Define as above $\mathcal{L}_{m,n} := \mathcal{L}_{\phi^{(m)}} \circ \cdots \circ \mathcal{L}_{\phi^{(n)}}$ and write

$$\lambda^{-n}\mathcal{L}^n g = \lambda^{-n}\mathcal{L}_{1,n} g_\epsilon^{(1)} + \lambda^{-n}\big(\mathcal{L}^n g_\epsilon^{(1)} - \mathcal{L}_{1,n} g_\epsilon^{(1)}\big) + \lambda^{-n}\mathcal{L}^n g_\epsilon^{(2)} =: G_{n,\epsilon}^{(a)} + G_{n,\epsilon}^{(b)} + G_{n,\epsilon}^{(c)}. \tag{5.2}$$

Lemma 5.5. *When K and K' are large enough, we have for every $n \geq 1$*

$$\|G^{(b)}_{n,\epsilon}\|_\infty \leq \frac{1}{2}\beta\epsilon \quad \text{and} \quad \|G^{(c)}_{n,\epsilon}\|_\infty \leq \frac{1}{2}\beta\epsilon.$$

Proof. The above estimate on $\|\psi^{(j)}\|_\infty$ implies that $\sum \|\psi^{(j)}\|_\infty \lesssim AK^{-1}\epsilon$. Therefore, using Lemma 5.3 and the fact that the sequence $\lambda^{-n}\mathcal{L}^n\mathbb{1}$ is bounded uniformly on n and ϕ (see Lemma 4.16), we get

$$\|G^{(b)}_{n,\epsilon}\|_\infty \lesssim AK^{-1}\epsilon\|g^{(1)}_\epsilon\|_\infty \lesssim AK^{-1}\epsilon \leq A_0K^{-1}\epsilon$$

because $\|g^{(1)}_\epsilon\|_\infty$ is bounded by a constant. So we get the first estimate in the lemma when K is large enough (depending on A_0 and β).

For the second estimate, using again that the sequence $\lambda^{-n}\mathcal{L}^n\mathbb{1}$ is uniformly bounded, we obtain

$$\|G^{(c)}_{n,\epsilon}\|_\infty \lesssim \|g^{(2)}_\epsilon\|_\infty \leq K'^{-1}\epsilon.$$

The result follows provided that K' is large enough. □

Lemma 5.6. *When $K \geq 1$ and K' are fixed, there is a constant $0 < \epsilon_0 \leq 1$ independent of ϕ and g such that, for all n large enough and $0 < \epsilon \leq \epsilon_0$,*

$$\|G^{(a)}_{n,\epsilon}\|_{\langle p,\alpha\rangle} \leq \beta(1/\epsilon)^{1/\gamma}.$$

Proof. Fix an $\epsilon_0 > 0$ small enough. We will apply Proposition 5.2. First, using the estimates $\sum \|\psi^{(j)}\|_\infty \lesssim AK^{-1}\epsilon \leq A_0$ and $\lambda \geq d^k e^{\min\phi}$, we obtain

$$\Phi_m \lesssim \alpha^{-m}d^{(k-1)m}e^{m\max\phi} \lesssim \alpha^{-m}d^{-m}\lambda^m e^{m\Omega(\phi)} \leq \alpha^{-m}d^{-m}\lambda^m e^{m\Omega}.$$

By Lemmas 5.3 and 4.16, we have

$$\|\mathcal{L}_{1,m}\mathbb{1}\|_\infty \lesssim \|\mathcal{L}^m\mathbb{1}\|_\infty \lesssim \lambda^m \quad \text{and} \quad \|\mathcal{L}_{m+1,n}\mathbb{1}\|_\infty \lesssim \|\mathcal{L}^{n-m}\mathbb{1}\|_\infty \lesssim \lambda^{n-m}$$

and also, again by Lemma 5.3,

$$\begin{aligned}
\|\mathcal{L}_{m+1,n}g^{(1)}_\epsilon\|_\infty &\lesssim \|\mathcal{L}^{n-m}g^{(1)}_\epsilon\|_\infty + \|\mathcal{L}_{m+1,n}g^{(1)}_\epsilon - \mathcal{L}^{n-m}g^{(1)}_\epsilon\|_\infty \\
&\lesssim \|\mathcal{L}^{n-m}g^{(1)}_\epsilon\|_\infty + \|\mathcal{L}^{n-m}\mathbb{1}\|_\infty\|g^{(1)}_\epsilon\|_\infty AK^{-1}\epsilon \\
&\lesssim \|\mathcal{L}^{n-m}g\|_\infty + \|\mathcal{L}^{n-m}g^{(2)}_\epsilon\|_\infty + \lambda^{n-m}AK^{-1}\epsilon \\
&\lesssim \|\mathcal{L}^{n-m}g\|_\infty + \lambda^{n-m}K'^{-1}\epsilon + \lambda^{n-m}AK^{-1}\epsilon.
\end{aligned}$$

This, Proposition 5.2, and the estimates in the definitions of $g^{(1)}_\epsilon$ and $\phi^{(j)}$ allow us to bound $\|G^{(a)}_{n,\epsilon}\|_{\langle p,\alpha\rangle}$ by a constant times

(5.3)
$$\Big[K'^{1/\gamma}\big(\frac{e^\Omega}{d\alpha}\big)^{n/2} + \sum_{m=1}^{n} AK^{1/\gamma}m^{2/\gamma+3/2}\big(\frac{e^\Omega}{d\alpha}\big)^{m/2}\big(\|\lambda^{-n+m}\mathcal{L}^{n-m}g\|_\infty + (K'^{-1} + AK^{-1})\epsilon_0\big)\Big]\epsilon^{1/\gamma}.$$

Recall that g belongs to a uniformly bounded and equicontinuous family of functions, see Lemma 3.16. It follows from Proposition 4.11 and Lemma 4.17 that $\|\lambda^{-n+m}\mathcal{L}^{n-m}g\|_\infty$ tends to 0, uniformly on ϕ and g, when $n-m$ tends to infinity. This and the fact that $e^\Omega < d\alpha$ imply that, when n tends to infinity, the sum between brackets in (5.3) converges to

$$\sum_{m=1}^{\infty} AK^{1/\gamma}m^{2/\gamma+3/2}\big(\frac{e^\Omega}{d\alpha}\big)^{m/2}(K'^{-1} + AK^{-1})\epsilon_0.$$

This sum is smaller than β because ϵ_0 is chosen small enough. Therefore, we get the estimate in the lemma for n large enough, independently of ϕ and g, because the last convergence is uniform on ϕ and g. □

Proof of Proposition 5.4. Take N large enough, independent of ϕ. It suffices to show that we can write

$$\lambda^{-N}\mathcal{L}^N g = G^{(1)}_{N,\epsilon} + G^{(2)}_{N,\epsilon} \quad \text{with} \quad \big\|G^{(1)}_{N,\epsilon}\big\|_{\langle p,\alpha\rangle} \leq \beta(1/\epsilon)^{1/\gamma} \quad \text{and} \quad \big\|G^{(2)}_{N,\epsilon}\big\|_\infty \leq \beta\epsilon.$$

We apply Lemmas 5.5 and 5.6 to $n := N$. When $\epsilon \leq \epsilon_0$, it is enough to choose $G^{(1)}_{N,\epsilon} := G^{(a)}_{N,\epsilon}$ and $G^{(2)}_{N,\epsilon} := G^{(b)}_{N,\epsilon} + G^{(c)}_{N,\epsilon}$. Assume now that $\epsilon_0 \leq \epsilon \leq 1$ and choose $G^{(1)}_{N,\epsilon} := 0$ and $G^{(2)}_{N,\epsilon} := \lambda^{-N}\mathcal{L}^N g$. With N large enough, we have $\|G^{(2)}_{N,\epsilon}\|_\infty \leq \beta\epsilon_0 \leq \beta\epsilon$ because $\|\lambda^{-n}\mathcal{L}^n g\|_\infty$ tends to 0 uniformly on ϕ and g when n goes to infinity, see Lemma 4.17. Thus, we have the desired decomposition of $\lambda^{-N}\mathcal{L}^N g$ and hence the property (5.1) for all N large enough. □

Proposition 5.7. *Under the hypotheses of Theorem 5.1, there is a positive constant $c = c(p,\alpha,\gamma,A,\Omega)$ independent of ϕ and n such that*

$$\|\lambda^{-n}\mathcal{L}^n\|_{\langle p,\alpha\rangle,\gamma} \leq c, \quad \|\rho\|_{\langle p,\alpha\rangle,\gamma} \leq c, \quad \textit{and} \quad \|1/\rho\|_{\langle p,\alpha\rangle,\gamma} \leq c.$$

Proof. We prove the first inequality. We will use the above computations for $K = K' = \epsilon_0 = 1$. Consider any function $g\colon \mathbb{P}^k \to \mathbb{R}$ such that $\|g\|_{\langle p,\alpha\rangle,\gamma} \leq 1$. We do not assume that $\langle m_\phi, g\rangle = 0$. As before, for any $0 < \epsilon \leq 1$ we can write

$$g = g^{(1)}_\epsilon + g^{(2)}_\epsilon \quad \text{with} \quad \big\|g^{(1)}_\epsilon\big\|_{\langle p,\alpha\rangle} \leq (1/\epsilon)^{1/\gamma} \quad \text{and} \quad \big\|g^{(2)}_\epsilon\big\|_\infty \leq \epsilon.$$

We also consider as above the decomposition

$$\lambda^{-n}\mathcal{L}^n g = G^{(1)}_{n,\epsilon} + G^{(2)}_{n,\epsilon} \quad \text{with} \quad G^{(1)}_{n,\epsilon} := G^{(a)}_{n,\epsilon} \quad \text{and} \quad G^{(2)}_{n,\epsilon} := G^{(b)}_{n,\epsilon} + G^{(c)}_{n,\epsilon},$$

see (5.2). The computations in Lemmas 5.5 and 5.6 give $\|G^{(2)}_{n,\epsilon}\|_\infty \lesssim \epsilon$ and $\|G^{(1)}_{n,\epsilon}\|_{\langle p,\alpha\rangle} \lesssim \epsilon^{1/\gamma}$. We use here the fact that $\|\lambda^{-n+m}\mathcal{L}^{n-m}g\|_\infty$ is bounded by a constant, see Lemma 4.17. Therefore, $\|\lambda^{-n}\mathcal{L}^n g\|_{\langle p,\alpha\rangle,\gamma}$ is bounded by a constant. Thus, the first inequality in the proposition holds.

Consider now the second inequality. Observe that

$$\rho = \lim_{n\to\infty} \lambda^{-n}\mathcal{L}^n\mathbb{1} = \mathbb{1} + \sum_{n=0}^{\infty} \lambda^{-n}\mathcal{L}^n g \quad \text{with} \quad g := \lambda^{-1}\mathcal{L}\mathbb{1} - \mathbb{1}.$$

The $\lambda^{-1}\mathcal{L}^*$-invariance of m_ϕ implies that $\langle m_\phi, g\rangle = 0$. Therefore, Proposition 5.4 and the first inequality in the present proposition imply that $\|\lambda^{-n}\mathcal{L}^n g\|_{\langle p,\alpha\rangle,\gamma} \lesssim \beta^{n/N}$. We deduce that $\|\rho\|_{\langle p,\alpha\rangle,\gamma}$ is bounded by a constant.

For the last inequality in the lemma, observe that ρ is bounded from above and below by positive constants which are independent of ϕ, see Lemma 4.16. The result is then a consequence of Lemma 3.16 applied to the function $\chi(t) := 1/t$. The proof is complete. □

End of the proof of Theorem 5.1. By Propositions 5.4 and 5.7, it only remains to prove the last assertion in this theorem. We continue to use the computations in Lemmas 5.5 and 5.6 and take $K = 1$, $K' = \delta'^N$, and $\epsilon_0 = 1$ for some δ' and d' such that $\delta < \delta' = d'^{\gamma/(2\gamma+2)}$ and $d' < d\alpha$. As above, we consider the decomposition

$$\lambda^{-N}\mathcal{L}^N g = G^{(1)}_{N,\epsilon} + G^{(2)}_{N,\epsilon} \quad \text{with} \quad G^{(1)}_{N,\epsilon} := G^{(a)}_{N,\epsilon} \quad \text{and} \quad G^{(2)}_{N,\epsilon} := G^{(b)}_{N,\epsilon} + G^{(c)}_{N,\epsilon}.$$

Take $A \to 0$, which also implies that $\Omega(\phi) \to 0$. So we can fix an Ω as small as needed and assume that $d' < e^{-\Omega}d\alpha$. Then, the estimates in Lemmas 5.5 and 5.6 give

$$\|G^{(1)}_{N,\epsilon}\|_{\langle p,\alpha\rangle} \leq c\big(K'^{1/\gamma}d'^{-N/2} + O(A)\big)\epsilon^{1/\gamma} \quad \text{and} \quad \|G^{(2)}_{N,\epsilon}\|_\infty \leq c(K'^{-1} + O(A))\epsilon,$$

where c is a positive constant. With N large enough and A small enough, since $\delta < \delta'$ and $K' = \delta'^N$, we get

$$\|G^{(2)}_{N,\epsilon}\|_\infty \leq \delta^{-N}\epsilon \quad \text{and} \quad \|G^{(1)}_{N,\epsilon}\|_{\langle p,\alpha\rangle} \leq \delta^{-N}\epsilon^{1/\gamma}.$$

In other words, we can take $\beta = \delta^{-N}$. This completes the proof of the theorem. □

5.3. **Proof of Theorem 1.2.** The statement is a consequence of Theorem 5.1, namely, of the estimate (5.1). Note that the constant β in Theorem 1.2 is not the one in (5.1). Given ϕ as in the statement, we first choose α sufficiently close to 1 so that $\Omega(\phi) < \log(\alpha d)$. Then, we choose p large enough so that $q < q_2$, where q and γ are as in the statement and q_2 is defined in Lemma 3.16 (this also implies that $\alpha < d^{-5/(2p+2)}$ since $q > 2$). Recall that the semi-norm $\|\cdot\|_{\langle p,\alpha\rangle,\gamma}$ is almost a norm. Define

$$\|\cdot\|_{\diamond_1} := \|\cdot\|_\infty + \|\cdot\|_{\langle p,\alpha\rangle,\gamma}.$$

This is now a norm, which is independent of ϕ. By Lemmas 3.16 and 3.19, we have

$$\|\cdot\|_\infty + \|\cdot\|_{\log^q} \lesssim \|\cdot\|_{\diamond_1} \lesssim \|\cdot\|_{\mathcal{C}^\gamma}.$$

By Lemma 4.16, the quantities $\|\lambda^{-n}\mathcal{L}^n\|_\infty$, $\|\rho\|_\infty$, and $\|1/\rho\|_\infty$ are bounded by a constant when $\|\phi\|_{\diamond_1} \leq A$. By Theorem 5.1, $\|\lambda^{-n}\mathcal{L}^n\|_{\langle p,\alpha\rangle,\gamma}$, $\|\rho\|_{\langle p,\alpha\rangle,\gamma}$, and $\|1/\rho\|_{\langle p,\alpha\rangle,\gamma}$ are also bounded by a constant. We deduce that $\|\lambda^{-n}\mathcal{L}^n\|_{\diamond_1}$, $\|\rho\|_{\diamond_1}$, and $\|1/\rho\|_{\diamond_1}$ satisfy the same property.

Let N and β_0 be as in Theorem 5.1 (we write β_0 instead of β to distinguish it from the constant that we use now for Theorem 1.2). Fix a constant β such that $\beta_0^{1/N} < \beta < 1$ and consider the following norms

$$\|g\|_\diamond := |c_g| + \|g'\|_{\langle p,\alpha\rangle,\gamma} \quad \text{and} \quad \|g\|_{\diamond_2} := |c_g| + \sum_{n=0}^{\infty} \beta^{-n} \left\|\lambda^{-n}\mathcal{L}^n g'\right\|_{\langle p,\alpha\rangle,\gamma}$$

for every function $g\colon \mathbb{P}^k \to \mathbb{R}$, where $c_g := \langle m_\phi, g\rangle$ and $g' := g - c_g\rho$.

Lemma 5.8. *We have $\|gh\|_{\diamond_1} \leq 3\|g\|_{\diamond_1}\|h\|_{\diamond_1}$ for all functions $g,h\colon \mathbb{P}^k \to \mathbb{R}$. Moreover, both of the norms $\|\cdot\|_\diamond$ and $\|\cdot\|_{\diamond_2}$ are equivalent to $\|\cdot\|_{\diamond_1}$.*

Proof. The first assertion is a direct consequence of Lemma 3.17. We prove now the second assertion. Since m_ϕ is a probability measure and $\|\rho\|_{\langle p,\alpha\rangle,\gamma}$ is bounded, we have

$$\|g\|_\diamond = |c_g| + \|g - c_g\rho\|_{\langle p,\alpha\rangle,\gamma} \leq |c_g| + \|g\|_{\langle p,\alpha\rangle,\gamma} + |c_g|\|\rho\|_{\langle p,\alpha\rangle,\gamma} \lesssim \|g\|_\infty + \|g\|_{\langle p,\alpha\rangle,\gamma} = \|g\|_{\diamond_1}.$$

Conversely, assume that $\|g\|_\diamond \leq 1$, then $|c_g| \leq 1$ and $\|g'\|_{\langle p,\alpha\rangle,\gamma} \leq 1$. It follows that $\|g\|_{\langle p,\alpha\rangle,\gamma}$ is bounded by a constant because it is bounded by $\|g'\|_{\langle p,\alpha\rangle,\gamma} + |c_g|\,\|\rho\|_{\langle p,\alpha\rangle,\gamma}$. By Lemma 3.16, $\Omega(g)$ is also bounded by a constant. This and the inequality $|\langle m_\phi, g\rangle| = |c_g| \leq 1$ imply that $\|g\|_\infty$ is bounded by a constant. We deduce that $\|\cdot\|_\diamond$ is equivalent to $\|\cdot\|_{\diamond_1}$.

Observe that $\|\cdot\|_\diamond \leq \|\cdot\|_{\diamond_2}$. To complete the proof, it is enough to show that $\|g\|_{\diamond_2} \lesssim \|g\|_\diamond$ for every function g. Recall that ρ is invariant by $\lambda^{-1}\mathcal{L}$ and $\langle m_\phi, \rho\rangle = 1$. Therefore, we have $\langle m_\phi, g'\rangle = 0$. Theorem 5.1 and Proposition 5.7 imply that $\|\lambda^{-n}\mathcal{L}^n g'\|_{\langle p,\alpha\rangle,\gamma} \lesssim \beta_0^{n/N}\|g'\|_{\langle p,\alpha\rangle,\gamma}$. Hence

$$\|g\|_{\diamond_2} \lesssim |c_g| + \|g'\|_{\langle p,\alpha\rangle,\gamma} \sum_{n=0}^{\infty} \beta^{-n}\beta_0^{n/N} \lesssim |c_g| + \|g'\|_{\langle p,\alpha\rangle,\gamma} = \|g\|_\diamond.$$

The last infinite sum is finite because $\beta > \beta_0^{1/N}$. This ends the proof of the lemma. □

Consider now a function g with $c_g = \langle m_\phi, g\rangle = 0$, which implies $g = g'$. We also have $\langle m_\phi, \lambda^{-1}\mathcal{L}g\rangle = 0$ because m_ϕ is invariant by $\lambda^{-1}\mathcal{L}^*$. From the definition of $\|\cdot\|_{\diamond_2}$, we get

$$\left\|\lambda^{-1}\mathcal{L}g\right\|_{\diamond_2} = \beta(\|g\|_{\diamond_2} - \|g\|_\diamond) \leq \beta\,\|g\|_{\diamond_2}.$$

This is the desired contraction. Finally, the last assertion in Theorem 1.2 is a direct consequence of the last assertion in Theorem 5.1 by taking α close enough to 1. The proof of Theorem 1.2 is now complete.

5.4. **Spectral gap in the limit case.** The semi-norm $\|\cdot\|_{\langle p,\alpha\rangle}$ can be seen as the limit of the semi-norm $\|\cdot\|_{\langle p,\alpha\rangle,\gamma}$ as γ goes to infinity. In order to complete our study, we will prove here a spectral gap with respect to this limit norm. The following is an analogue of Theorem 5.1.

Theorem 5.9. *Let $f, \phi, \lambda, m_\phi, \rho$ be as in Theorem 1.1 and $\mathcal{L}$ the Perron-Frobenius operator associated to ϕ. Let p, α, A, Ω be positive constants and q_1 as in Lemma 3.13 such that $p > 3/2$, $d^{-1} \leq \alpha < d^{-5/(2p+2)}$, $\Omega < \log(d\alpha)$, and $q_1 > 2$. Assume that $\|\phi\|_{\langle p,\alpha\rangle} \leq A$ and $\Omega(\phi) \leq \Omega$. Then we have*

$$\|\lambda^{-n}\mathcal{L}^n\|_{\langle p,\alpha\rangle} \leq c, \quad \|\rho\|_{\langle p,\alpha\rangle} \leq c, \quad \text{and} \quad \|1/\rho\|_{\langle p,\alpha\rangle} \leq c$$

for some positive constant $c = c(p, \alpha, A, \Omega)$ independent of ϕ and n. Moreover, for every constant $0 < \beta < 1$ there is a positive integer $N = N(p, \alpha, A, \Omega, \beta)$ independent of ϕ such that

$$\left\|\lambda^{-N}\mathcal{L}^N g\right\|_{\langle p,\alpha\rangle} \leq \beta \|g\|_{\langle p,\alpha\rangle}$$

for every function $g\colon \mathbb{P}^k \to \mathbb{R}$ with $\langle m_\phi, g\rangle = 0$. Furthermore, we can take $\beta = \delta^{-N}$ for every constant $0 < \delta < (d\alpha)^{1/2}$, provided that A is small enough.

Notice that Lemma 3.13 and the assumption $q_1 > 2$ imply that $\|\phi\|_{\log^q}$ is finite for some $q > 2$. Hence, the scaling ratio λ, the density function ρ, and the measures m_ϕ and μ_ϕ are well defined by Theorem 1.1. Notice also that $q_1 > 2$ implies that the condition $\alpha < d^{-5/(2p+2)}$ is automatically satisfied.

Proof. The proof follows the same lines as the one of Theorem 5.1. It is however simpler because the definition of the semi-norm $\|\cdot\|_{\langle p,\alpha\rangle}$ is simpler than the one of $\|\cdot\|_{\langle p,\alpha\rangle,\gamma}$. In particular, we do not need any decomposition of $\lambda^{-n}\mathcal{L}^n g$. Applying directly Proposition 5.2 with $\phi^{(j)} := \phi$ for all $j \geq 1$ and recalling that $\|\mathbb{1}_n\|_\infty \lesssim \lambda^n$ we obtain

$$\left\|\lambda^{-n}\mathcal{L}^n g\right\|_{\langle p,\alpha\rangle} \lesssim \|g\|_{\langle p,\alpha\rangle} \big(\frac{e^\Omega}{d\alpha}\big)^{n/2} + \|\phi\|_{\langle p,\alpha\rangle} \sum_{m=1}^{n} m^{3/2} \big(\frac{e^\Omega}{d\alpha}\big)^{m/2} \|\lambda^{-n+m}\mathcal{L}^{n-m} g\|_\infty.$$

With this estimate, the rest of the proof is the same as that of Theorem 5.1. □

As in the last section, we obtain the following counterpart of Theorem 1.2 as a consequence of the last result.

Theorem 5.10. *Let $f, p, \alpha, A, \Omega, \phi, \rho, \lambda, m_\phi$ and $\mathcal{L}$ be as in Theorem 5.9. Then there is an explicit norm $\|\cdot\|_{\diamond_0}$, depending on f, p, α, ϕ and equivalent to $\|\cdot\|_\infty + \|\cdot\|_{\langle p,\alpha\rangle}$, such that when $\|\phi\|_{\langle p,\alpha\rangle} \leq A$ and $\Omega(\phi) \leq \Omega$ we have*

$$\|\lambda^{-n}\mathcal{L}^n\|_{\langle p,\alpha\rangle} \leq c, \quad \|\rho\|_{\langle p,\alpha\rangle} \leq c, \quad \|1/\rho\|_{\langle p,\alpha\rangle} \leq c, \quad \text{and} \quad \left\|\lambda^{-1}\mathcal{L}g\right\|_{\diamond_0} \leq \beta \|g\|_{\diamond_0}$$

for every $g\colon \mathbb{P}^k \to \mathbb{R}$ with $\langle m_\phi, g\rangle = 0$, and for some positive constants $c = c(f, p, \alpha, A, \Omega)$ and $\beta = \beta(f, p, \alpha, A, \Omega)$ with $\beta < 1$, both independent of ϕ, n, and g. Furthermore, given any constant $0 < \delta < (d\alpha)^{1/2}$, when A is small enough, the norm $\|\cdot\|_{\diamond_0}$ can be chosen so that we can take $\beta = 1/\delta$.

Note that Lipschitz functions have finite $\|\cdot\|_{\langle p,\alpha\rangle}$ semi-norm (this follows from Lemma 3.13, since Lipschitz functions can be uniformly approximated by $\mathcal{C}^1$ ones whose norm is dominated by the Lipschitz constant, see also the proof of Lemma 3.16). So the last theorem can be applied to Lipschitz functions. For such functions we can take any p large enough and α close to 1. The rate of contraction is then almost equal to $d^{-1/2}$ when A is small enough (i.e., when ϕ is close to a constant function). This rate is likely optimal as it corresponds to known results obtained in the setting of zero weight, see [DS10a].

6. Statistical properties of equilibrium states

In this section we prove Theorem 1.3. The definitions of the statistical properties that we consider are all recalled in Appendix A, as well as criteria ensuring their validity in an abstract setting. We work under the hypotheses of Theorems 1.2 and 1.3 and with the equivalent norms $\|\cdot\|_{\diamond_1}$ and $\|\cdot\|_{\diamond_2}$ as in Theorem 1.2, see Section 5.3. Recall also that the operator L is defined by $L(g) := (\lambda\rho)^{-1}\mathcal{L}(\rho g)$.

6.1. **Speed of equidistribution for preimages of points.** The following result gives a quantitative version of Corollary 4.12. Because of Lemma 3.19 and of the definition of the norm $\|\cdot\|_{\diamond_1}$, it applies in particular to Hölder continuous test functions.

Theorem 6.1. *Under the hypotheses of Theorem 1.2, for every $x \in \mathbb{P}^k$, as n tends to infinity the points in $f^{-n}(x)$, with suitable weights, are equidistributed exponentially fast with respect to the conformal measure m_ϕ. More precisely, we have*

$$\Big|\Big\langle \lambda^{-n} \sum_{f^n(a)=x} e^{\phi(a)+\cdots+\phi(f^{n-1}(a))}\delta_a - \rho(x)m_\phi, g\Big\rangle\Big| \leq c\beta^n \|g\|_{\diamond_1},$$

for all $g\colon \mathbb{P}^k \to \mathbb{R}$, where $0 < \beta < 1$ is the constant in Theorem 1.2 and c is a positive constant independent of x and g.

Proof. For simplicity, assume that $\|g\|_{\diamond_1} = 1$. Define $g' := g - c_g\rho$ with $c_g := \langle m_\phi, g\rangle$. By Lemma 5.8, both $|c_g|$ and $\|g'\|_{\diamond_1}$ are bounded by a constant. Since the norms $\|\cdot\|_{\diamond_1}$ and $\|\cdot\|_{\diamond_2}$ are equivalent, we deduce from Theorem 1.2 that $\|\lambda^{-n}\mathcal{L}^n(g')\|_{\diamond_1} \lesssim \beta^n$. As we have seen in the proof of Corollary 4.12, the LHS of the inequality in the theorem is bounded by

$$\big\|\lambda^{-n}\mathcal{L}^n(g) - c_g\rho\big\|_\infty = \big\|\lambda^{-n}\mathcal{L}^n(g - c_g\rho)\big\|_\infty = \big\|\lambda^{-n}\mathcal{L}^n(g')\big\|_\infty \lesssim \beta^n.$$

This implies the result. □

We also have the following useful result.

Proposition 6.2. *There exists a positive constant c such that, for every $g\colon \mathbb{P}^k \to \mathbb{R}$ with $\|g\|_{\diamond_1} \leq 1$, setting $c_{\rho g} := \langle m_\phi, \rho g\rangle = \langle \mu_\phi, g\rangle$ we have*

$$\Big\langle \mu_\phi, e^{\beta^{-n}|L^n(g) - c_{\rho g}|}\Big\rangle \leq c \qquad \textit{and} \qquad \|L^n g - c_{\rho g}\|_{\diamond_1} \leq c\beta^n.$$

Proof. Clearly, the first inequality follows from the second one by increasing the constant c if necessary. For the second inequality, define $\tilde{g} = \rho g$ and $\tilde{g}' := \tilde{g} - c_{\tilde{g}}\rho$. We have $\langle m_\phi, \tilde{g}'\rangle = 0$. By Lemma 5.8 and Theorem 1.2 we have that $\|\tilde{g}\|_{\diamond_1}$ is bounded by a constant because it is at most equal to $3\,\|\rho\|_{\diamond_1}\|g\|_{\diamond_1}$. We deduce that $c_{\tilde{g}}$ and hence $\|\tilde{g}'\|_{\diamond_1}$ are both bounded by a constant. Using again Lemma 5.8 and Theorem 1.2, we also obtain

$$\|L^n g - c_{\rho g}\|_{\diamond_1} = \big\|\lambda^{-n}\rho^{-1}\mathcal{L}^n\tilde{g} - c_{\tilde{g}}\big\|_{\diamond_1} = \big\|\rho^{-1}\lambda^{-n}\mathcal{L}^n\tilde{g}'\big\|_{\diamond_1} \leq 3\,\big\|\rho^{-1}\big\|_{\diamond_1}\big\|\lambda^{-n}\mathcal{L}^n\tilde{g}'\big\|_{\diamond_1} \lesssim \beta^n.$$

The result follows. □

6.2. **Mixing speed for the equilibrium state.** The speed of mixing in Theorem 1.1 is not controlled for $g_0, \ldots, g_r, g, l \in L^2(\mu_\phi)$, see Proposition 4.15. We establish here some uniform exponential bound for the speed of mixing of the system $\big(\mathbb{P}^k, f, \mu_\phi\big)$ for more regular observables. In the case of Hölder continuous weight ϕ and observable g, this was established in [Hay99] for $k = 1$ (see also [DPU96] for a uniform sub-exponential speed) and in [SUZ14] for $k > 1$, see also [DS10a] for the case when ϕ is constant.

Theorem 6.3. *Under the hypotheses of Theorem 1.2, for every integer $r \geq 0$, there is a positive constant $c = c(r)$ such that*

$$\Big|\,\langle \mu_\phi, g_0\,(g_1 \circ f^{n_1})\cdots(g_r \circ f^{n_r})\rangle - \prod_{j=0}^{r}\langle \mu_\phi, g_j\rangle\,\Big| \leq c\beta^n\Big(\prod_{j=0}^{r-1}\|g_j\|_{\diamond_1}\Big)\,\|g_r\|_{L^1(\mu_\phi)}$$

for $0 = n_0 \leq n_1 \leq \cdots \leq n_r$ and $n := \min_{0\leq j<r}(n_{j+1} - n_j)$. Here, the constant $0 < \beta < 1$ is the one from Theorem 1.2.

Proof. We prove the theorem by induction on r. Since when $r = 0$ the assertion is trivial, we can assume that the theorem holds for $r - 1$ and prove it for r.

The fact that the statement holds for $r - 1$ and the invariance of μ_ϕ imply the result for r when g_0 is constant. So, by subtracting from g_0 the constant $c_{\rho g_0} := \langle m_\phi, \rho g_0\rangle = \langle \mu_\phi, g_0\rangle$, we can assume that $\langle \mu_\phi, g_0\rangle = 0$. Setting $g' := g_1 (g_2 \circ f^{n_2-n_1}) \cdots (g_r \circ f^{n_r-n_1})$ and recalling that $L(\cdot) = (\lambda\rho)^{-1}\mathcal{L}(\rho\cdot)$, computing as at the end of the proof of Theorem 1.1 we get

$$\begin{aligned}\left|\left\langle \mu_\phi, g_0(g' \circ f^{n_1})\right\rangle\right| &= \left|\left\langle \mu_\phi, L^{n_1}(g_0(g' \circ f^{n_1}))\right\rangle\right| = \left|\left\langle \mu_\phi, L^{n_1}(g_0)g'\right\rangle\right| \\ &\leq \|L^{n_1}(g_0)\|_\infty \left\|g'\right\|_{L^1(\mu_\phi)} \\ &\leq \|L^{n_1}(g_0)\|_\infty \|g_1\|_\infty \cdots \|g_{r-1}\|_\infty \left\|g_r \circ f^{n_r-n_1}\right\|_{L^1(\mu_\phi)} \\ &= \|L^{n_1}(g_0)\|_\infty \|g_1\|_\infty \cdots \|g_{r-1}\|_\infty \|g_r\|_{L^1(\mu_\phi)}\,.\end{aligned}$$

The assertion follows from Proposition 6.2 applied to g_0 instead of g. □

6.3. **Central Limit Theorem (CLT).** We prove here the CLT for observables g such that $\|g\|_{\diamond_1} < \infty$ and which are not coboundaries, see Appendix A for the definitions. In the case of Hölder continuous weight ϕ and observable g, this was established in [DPU96] for $k = 1$ and in [SUZ14] for $k > 1$, see also [DS10a] for the case when ϕ is constant. We start with the following version that does not need the introduction of perturbed operators. More refined versions of this theorem will be given later in Sections 6.5 and 6.6.

Theorem 6.4. *Under the hypotheses of Theorem 1.2, consider a function $g\colon \mathbb{P}^k \to \mathbb{R}$ such that $\|g\|_{\diamond_1} < \infty$ and $\langle \mu_\phi, g\rangle = 0$. Assume that g is not a coboundary. Then g satisfies the CLT with variance $\sigma > 0$ given by*

$$\sigma^2 := \left\langle \mu_\phi, g^2\right\rangle + 2\sum_{n\geq 1} \langle \mu_\phi, g \cdot (g \circ f^n)\rangle\,. \tag{6.1}$$

We refer to Definition A.1 for the statement of the CLT. Recall that g is a coboundary if there exists $h \in L^2(\mu_\phi)$ such that $g = h \circ f - h$, and this is the case if and only if $\sigma = 0$. It is not difficult to check that a coboundary cannot satisfy the CLT as in Definition A.1.

Proof. Let $\mathcal{B}$ denote the Borel σ-algebra on $\mathbb{P}^k$. By Gordin Theorem A.2, we only need to show that

$$\sum_{n\geq 1} \left\|\mathbb{E}\left(g|f^{-n}\mathcal{B}\right)\right\|^2_{L^2(\mu_\phi)} < \infty.$$

We will transform the above series into a series of the norms $\|L^n g\|_{L^2(\mu_\phi)}$, that we will bound by means of Proposition 6.2. Recall that $\langle \mu_\phi, (L^n g)h\rangle = \langle \mu_\phi, g(h \circ f^n)\rangle$ for all $g, h\colon \mathbb{P}^k \to \mathbb{R}$. So

$$\langle \mu_\phi, g\,(h \circ f^n)\rangle = \langle \mu_\phi, (L^n g)h\rangle = \langle \mu_\phi, ((L^n g) \circ f^n)\,(h \circ f^n)\rangle\,,$$

where in the last passage we used the f_*-invariance of μ_ϕ. This precisely says that $\mathbb{E}(g, f^{-n}\mathcal{B}) = L^n(g) \circ f^n$. Note that the norm of f^* on $L^2(\mu_\phi)$ is 1. So we have

$$\sum_{n\geq 1} \left\|\mathbb{E}\left(g|f^{-n}\mathcal{B}\right)\right\|^2_{L^2(\mu_\phi)} \leq \sum_{n\geq 1} \|L^n(g)\|^2_{L^2(\mu_\phi)} \leq \sum_{n\geq 1} \|L^n(g)\|^2_\infty\,.$$

The last sum converges because of Proposition 6.2 and the hypothesis $\langle \mu_\phi, g\rangle = 0$. This completes the proof. □

We have the following characterization of coboundaries with bounded $\|\cdot\|_{\diamond_1}$ norm that will be used in Section 6.5.

Proposition 6.5. *Let g be a coboundary and assume that $\|g\|_{\diamond_1} < \infty$. Then there exists a function $\tilde{h}$ such that $\|\tilde{h}\|_{\diamond_1} < \infty$ and $g = \tilde{h} \circ f - \tilde{h}$ on the small Julia set of f.*

Proof. By the definition of coboundary, there exists a function $h \in L^2(\mu_\phi)$ such that $g = h\circ f - h$ in $L^2(\mu_\phi)$. This identity implies that $\langle \mu_\phi, g\rangle = 0$. By adding a constant to h we can assume that $\langle \mu_\phi, h\rangle = 0$. Since

$$L(h\circ f) = \lambda^{-1}\rho^{-1}\mathcal{L}(\rho(h\circ f)) = \lambda^{-1}\rho^{-1}\mathcal{L}(\rho)h = h$$

a direct computation gives

$$Lg + \cdots + L^n g = h - L^n h.$$

Since $\langle \mu_\phi, h\rangle = 0$, as in the proof of the K-mixing at the end of the proof of Theorem 1.1, we have $L^n h \to 0$ in $L^2(m_\phi)$ when n tends to infinity. Proposition 6.2 and the identity $\langle \mu_\phi, g\rangle = 0$ imply that the LHS of the last identity converges to a function $\tilde{h}$ with finite $\|\cdot\|_{\diamond_1}$ norm. We conclude that $\tilde{h} = h$ μ_ϕ-almost everywhere. Therefore, we have $g = \tilde{h}\circ f - \tilde{h}$ μ_ϕ-almost everywhere, and thus everywhere on $\mathrm{supp}(\mu_\phi)$ as both sides are continuous functions. The proposition follows because the support of μ_ϕ is equal to the small Julia set, see Theorem 1.1. □

6.4. **Properties of perturbed Perron-Frobenius operators.** The next statistical properties will be proved by means of spectral methods, and more precisely by the introduction of suitable (complex) perturbations of the operator $\mathcal{L} = \mathcal{L}_\phi$. This method was originally developed by Nagaev [Nag57] in the context of Markov chains. More details are given in Appendix A. We introduce here some notations and prove some preliminary results. We mainly follow the approach as presented in [Bro96, DNS07, RE83].

Definition 6.6. Given functions $\phi, g\colon \mathbb{P}^k \to \mathbb{R}$, $h\colon \mathbb{P}^k \to \mathbb{C}$, and a parameter $\theta \in \mathbb{C}$ we set

$$\mathcal{L}_{\phi+\theta g} h := \mathcal{L}_{\phi+\theta g}\Re h + i\mathcal{L}_{\phi+\theta g}\Im h,$$

where the operator in the RHS is the natural extension of (1.1) in the case with complex weight.

Since from now on we fix ϕ and g, we will just denote the above operator by $\mathcal{L}_{[\theta]}$ when no possible confusion arises. In particular, we have $\mathcal{L}_{[0]} = \mathcal{L}_\phi$. By means of Definition 6.6, we extended the operator $\mathcal{L}$ to complex weights and complex test functions. We naturally extend the norms $\|\cdot\|_{\diamond_1}$ and $\|\cdot\|_{\diamond_2}$ to these function spaces by setting

$$\|h\|_{\diamond_1} := \|\Re h\|_{\diamond_1} + \|\Im h\|_{\diamond_1} \quad \text{and} \quad \|h\|_{\diamond_2} := \|\Re h\|_{\diamond_2} + \|\Im h\|_{\diamond_2}.$$

We will be in particular interested in the case where θ is small or pure imaginary. We may develop our study given in the previous sections for a complex weight directly, but we prefer to use a shortcut which only requires to treat the real case. The next lemma collects the main properties of the family of operators $\mathcal{L}_{[\theta]}$ that we need.

Lemma 6.7. *Assume that $\|g\|_{\diamond_1}$ is finite. Then the following assertions hold for both of the norms $\|\cdot\|_{\diamond_1}$ and $\|\cdot\|_{\diamond_2}$.*

(i) $\mathcal{L}_{[0]} = \mathcal{L}_\phi$;
(ii) *For every $\theta \in \mathbb{C}$, $\mathcal{L}_{[\theta]}$ is a bounded operator;*
(iii) *The map $\theta \mapsto \mathcal{L}_{[\theta]}$ is analytic in θ;*
(iv) *For every $n \in \mathbb{N}$, $\theta \in \mathbb{C}$, and $h\colon \mathbb{P}^k \to \mathbb{C}$, we have*

$$\mathcal{L}^n_{[\theta]} h = \mathcal{L}^n_{[0]}(e^{\theta S_n(g)} h), \tag{6.2}$$

where

$$S_0(g) := 0 \quad \text{and} \quad S_n(g) := \sum_{j=0}^{n-1} g\circ f^j \text{ for } n \geq 1. \tag{6.3}$$

Proof. The first item is true by definition. For the second and the third items, it is enough to prove them for the norm $\|\cdot\|_{\diamond_1}$ since the norm $\|\cdot\|_{\diamond_2}$ is equivalent, see Lemma 5.8. By Lemma 5.8 we have, for all $h\colon \mathbb{P}^k \to \mathbb{C}$ and $n \in \mathbb{N}$,

$$\left\|\mathcal{L}_{[0]}(g^n h)\right\|_{\diamond_1} \leq \left\|\mathcal{L}_{[0]}\right\|_{\diamond_1} \|g^n h\|_{\diamond_1} \leq 3^n \left\|\mathcal{L}_{[0]}\right\|_{\diamond_1} \|g\|^n_{\diamond_1} \|h\|_{\diamond_1}.$$

So, the series $\sum_{n\geq 0}(\theta^n/n!)\mathcal{L}_{[0]}(g^n h)$ converges normally in norm $\|\cdot\|_{\diamond_1}$. The limit is equal to $\mathcal{L}_{[\theta]}(h)$ and we have

$$\left\|\mathcal{L}_{[\theta]}(h)\right\|_{\diamond_1} \leq e^{3|\theta|\|g\|_{\diamond_1}} \left\|\mathcal{L}_{[0]}\right\|_{\diamond_1} \|h\|_{\diamond_1}.$$

This proves the second and the third items.

The last item follows by a direct induction. The case $n = 0$ is trivial and we have

$$\begin{aligned}\mathcal{L}^n_{[0]}(e^{\theta S_n(g)}h) &= \mathcal{L}_{[0]}\Big(\mathcal{L}^{n-1}_{[0]}\big(e^{\theta\cdot g\circ f^{n-1}}e^{\theta S_{n-1}(g)}h\big)\Big)\\ &= \mathcal{L}_{[0]}\Big(e^{\theta g}\mathcal{L}^{n-1}_{[0]}(e^{\theta S_{n-1}(g)}h)\Big) = \mathcal{L}_{[\theta]}\mathcal{L}^{n-1}_{[0]}(e^{\theta S_{n-1}(g)}h) = \mathcal{L}_{[\theta]}\mathcal{L}^{n-1}_{[\theta]}h.\end{aligned}$$

The proof is complete. □

Recall that the operator $\lambda^{-1}\mathcal{L}_{[0]}$ has ρ as its unique (up to a multiplicative constant) eigenfunction of eigenvalue 1. It is a contraction with respect to the norm $\|\cdot\|_{\diamond_2}$ (which is equivalent to $\|\cdot\|_{\diamond_1}$) on the space of functions whose integrals with respect to m_ϕ are zero, see Theorem 1.2. The following is then a consequence of the Rellich perturbation method described in [DS58, Ch. VII], see also [Bro96, Proposition 5.2] and [Kat13, LP82]. Note that the last assertion of Theorem 1.3 is a direct consequence of the analyticity of α given by the fourth item.

Proposition 6.8. *Assume that $\|g\|_{\diamond_1}$ is finite and let $0 < \beta < 1$ be the constant in Theorem 1.2. Then, for all $\beta < \beta' < 1$, the following holds for θ sufficiently small and all $n \in \mathbb{N}$: there exists a decomposition*

$$\lambda^{-n}\mathcal{L}^n_{[\theta]} = \alpha(\theta)^n\Phi_\theta + \Psi^n_\theta$$

as operators on $\{h : \|h\|_{\diamond_1} < \infty\}$ such that

- (i) *$\alpha(\theta)$ is the (only) largest eigenvalue of $\mathcal{L}_{[\theta]}$, $\alpha(0) = 1$ and $|\alpha(\theta)| > \beta'$;*
- (ii) *Φ_θ is the projection on the (one dimensional) eigenspace associated to $\alpha(\theta)$ and we have $\Phi_0(h) = \langle m_\phi, h\rangle\,\rho$;*
- (iii) *Ψ_θ is a bounded operator on $\{h : \|h\|_{\diamond_1} < \infty\}$ whose spectral radius is $< \beta'$ and*

$$\Psi_\theta \circ \Phi_\theta = \Phi_\theta \circ \Psi_\theta = 0;$$

- (iv) *the maps $\theta \mapsto \Psi_\theta, \theta \mapsto \Phi_\theta$, and $\theta \mapsto \alpha(\theta)$ are analytic.*

The last property that we will need is the second order expansion of $\alpha(\theta)$ for θ near 0. It is a consequence of the above results.

Lemma 6.9. *Assume that $\|g\|_{\diamond_1}$ is finite and $\langle\mu_\phi, g\rangle = 0$. Let $\sigma > 0$ be as in (6.1) and $\alpha(\theta)$ be given by Proposition 6.8. Then we have*

$$\alpha(\theta) = e^{\frac{\sigma^2\theta^2}{2}+o(\theta^2)} = 1 + \frac{\theta^2\sigma^2}{2} + o(\theta^2).$$

Proof. We know from Lemma 6.7 and Proposition 6.8 that $\alpha(0) = 1$ and that $\theta \mapsto \alpha(\theta)$ is analytic near 0. So, we only need to prove that $\alpha'(0) = 0$ and $\alpha''(0) = \sigma^2$.

We start with the first equality. First of all notice that, for all $n \in \mathbb{N}$ and $h\colon \mathbb{P}^k \to \mathbb{C}$, the invariance of m_ϕ implies

$$\langle\mu_\phi, h\rangle = \langle m_\phi, \rho h\rangle = \left\langle m_\phi, \lambda^{-n}\mathcal{L}^n_{[0]}(\rho h)\right\rangle. \tag{6.4}$$

Applying these identities to $e^{S_n(g)/n}$ instead of h (with $S_n(g)$ as in (6.3)) and using (6.2), we get

$$\langle\mu_\phi, e^{S_n(g)/n}\rangle = \langle m_\phi, \lambda^{-n}\mathcal{L}^n_{[1/n]}(\rho)\rangle.$$

Then, the decomposition given in Proposition 6.8 (applied with $1/n$ instead of θ) gives

$$\langle\mu_\phi, e^{S_n(g)/n}\rangle = \alpha(1/n)^n\left\langle m_\phi, \Phi_{1/n}(\rho)\right\rangle + \left\langle m_\phi, \Psi^n_{1/n}(\rho)\right\rangle. \tag{6.5}$$

By Proposition 6.8, the second term in the RHS goes to zero exponentially fast as $n \to \infty$. Using that $\alpha(\theta) = 1 + \alpha'(0)\theta + o(\theta)$ and $\Phi_\theta(\rho) = \rho + o(1)$ as $\theta \to 0$ and the identity $\langle m_\phi, \rho\rangle = 1$, we get

$$\lim_{n\to\infty} \langle \mu_\phi, e^{S_n(g)/n}\rangle = e^{\alpha'(0)}.$$

Recall that μ_ϕ is mixing and hence ergodic. Moreover, g is continuous and $\langle \mu_\phi, g\rangle = 0$. Birkhoff's ergodic theorem then implies that $S_n(g)/n \to 0$ μ_ϕ-almost surely as n goes to infinity. We conclude that $\alpha'(0) = 0$ as desired.

Let us now prove that $\alpha''(0) = \sigma^2$. The f-invariance of μ_ϕ and a direct computation give

$$\sigma^2 = \lim_{n\to\infty} \Big\langle \mu_\phi, \Big(\frac{S_n g}{\sqrt{n}}\Big)^2 \Big\rangle \quad \text{and} \quad \Big\langle \mu_\phi, \Big(\frac{S_n g}{\sqrt{n}}\Big)^2 \Big\rangle = \frac{\partial^2}{\partial\theta^2} \Big\langle \mu_\phi, e^{(\theta/\sqrt{n})S_n g} \Big\rangle \Big|_{\theta=0}.$$

We use again (6.2) and Proposition 6.8 (applied with $\theta/\sqrt{n}$ instead of θ) to get

$$\Big\langle \mu_\phi, e^{(\theta/\sqrt{n})S_n g} \Big\rangle = \alpha(\theta/\sqrt{n})^n \Big\langle m_\phi, \Phi_{\theta/\sqrt{n}}(\rho) \Big\rangle + \Big\langle m_\phi, \Psi^n_{\theta/\sqrt{n}}(\rho) \Big\rangle .$$

A direct computation, together with the identities $\alpha(0) = 1$, $\alpha'(0) = 0$, $\langle m_\phi, \rho\rangle = 1$ and the properties in Proposition 6.8, gives

$$\frac{\partial^2}{\partial\theta^2} \Big\langle \mu_\phi, e^{(\theta/\sqrt{n})S_n g} \Big\rangle \Big|_{\theta=0} = \alpha''(0) + o(1).$$

It is now clear that $\alpha''(0) = \sigma^2$. This completes the proof of the lemma. □

6.5. **Local Central Limit Theorem (LCLT).** We establish here an improvement of the CLT, see Definition A.4, for observables satisfying a further cocycle condition. Our result is new for $k = 1$, ϕ non-constant, and for $k > 1$, even when $\phi = 0$; for $k = 1$ and $\phi = 0$, see [DNS07]. We need the following definition.

Definition 6.10. Let $g\colon \mathbb{P}^k \to \mathbb{R}$ be a measurable function. We say that g is a *multiplicative cocycle* if there exist $t > 0$, $s \in \mathbb{R}$, and a measurable function $\xi\colon \mathbb{P}^k \to \mathbb{C}$, not equal to zero μ_ϕ-almost everywhere, such that $e^{itg(z)}\xi(z) = e^{is}\xi(f(z))$. We say that g is a $(\mathcal{C}^0, \phi)$-*multiplicative cocycle* (resp. $(\|\cdot\|_{\diamond_1}, \phi)$-*multiplicative cocycle*) if there exist $t > 0$, $s \in \mathbb{R}$, and $\xi\colon \mathbb{P}^k \to \mathbb{C}$, not identically zero on the small Julia set of f, which is continuous (resp. with finite $\|\cdot\|_{\diamond_1}$ norm), such that $e^{itg(z)}\xi(z) = e^{is}\xi(f(z))$ on the small Julia set of f.

Recall that the supports of μ_ϕ and m_ϕ are both equal to the small Julia set of f, see Theorem 1.1.

Theorem 6.11. *Under the hypotheses of Theorem 1.2, let $g\colon \mathbb{P}^k \to \mathbb{R}$ be such that $\|g\|_{\diamond_1}$ is finite, $\langle \mu_\phi, g\rangle = 0$, and g is not a $(\|\cdot\|_{\diamond_1}, \phi)$-multiplicative cocycle. Then g satisfies the LCLT with variance σ given by* (6.1)*.*

We first need some properties of multiplicative cocycles. Note that the following lemma still holds if we only assume that ϕ and g have bounded $\|\cdot\|_{\log^q}$ norms for some $q > 2$.

Lemma 6.12. *There is a positive constant c independent of g, t, and n such that $\|\lambda^{-n}\mathcal{L}^n_{[it]}\|_\infty \leq c$. Let K be a compact subset of $\mathbb{R} \setminus \{0\}$ (e.g., a singleton in $\mathbb{R} \setminus \{0\}$). Let $\mathcal{F}$ be a uniformly bounded and equicontinuous family of functions on $\mathbb{P}^k$. Then the family*

$$\mathcal{F}^K_{\mathbb{N}} := \big\{\lambda^{-n}\mathcal{L}^n_{[it]}h \ : \ t \in K,\ h \in \mathcal{F},\ n \in \mathbb{N}\big\}$$

is also uniformly bounded and equicontinuous. Furthermore, if $K \subset (0, \infty)$ and g is not a $(\mathcal{C}^0, \phi)$-multiplicative cocycle, then $\|\lambda^{-n}\mathcal{L}^n_{[it]}h\|_\infty$ tends to 0 when n goes to infinity, uniformly in $t \in K$ and $h \in \mathcal{F}$.

Proof. Define $\phi_t := \phi + itg$. Observe that $|e^{\phi_t}| = e^\phi$. It follows that $\|\mathcal{L}^n_{[it]}\|_\infty \leq \|\mathcal{L}^n\|_\infty \leq c\lambda^n$ for some positive constant c, according to Lemma 4.9. Using this, we can follow the proof of Lemma 4.10, with ϕ_t instead of ϕ, and obtain that $\mathcal{F}^K_{\mathbb{N}}$ is uniformly bounded and equicontinuous. It remains to prove the last assertion in the lemma.

We will use the ideas in the proof of Theorem 4.1. Let $\mathcal{F}_\infty^K$ denote the family of the limit functions of all sequences $\lambda^{-n_j}\mathcal{L}_{[it_j]}^{n_j}(h_j)$ with $t_j \in K$, $h_j \in \mathcal{F}$, and n_j going to infinity. By Arzelà-Ascoli theorem, this is a uniformly bounded and equicontinuous family of functions which is compact for the uniform topology. Define

$$M := \max\big\{|l(a)/\rho(a)| \ : \ l \in \mathcal{F}_\infty^K, \ a \text{ in the small Julia set}\big\}.$$

Claim 1. If $M = 0$, then $\mathcal{F}_\infty^K$ only contains the zero function.

Proof of Claim 1. Assume that $M = 0$. Consider a function l in $\mathcal{F}_\infty^K$. We show that $l = 0$. As in the proof of Theorem 4.1, we can find functions $l_{-n} \in \mathcal{F}_\infty^K$ such that $l = \lambda^{-n}\mathcal{L}_{[it]}^n l_{-n}$ for some $t \in K$ (take t as a limit of t_j above). Since $M = 0$, the function l_{-n} vanishes on the support of m_ϕ (which is the small Julia set) and we have $c_{|l_{-n}|} = \langle m_\phi, |l_{-n}|\rangle = 0$. Moreover, we also have

$$|l| = |\lambda^{-n}\mathcal{L}_{[it]}^n l_{-n}| \leq \lambda^{-n}\mathcal{L}^n|l_{-n}|.$$

By the last assertion of Proposition 4.11, applied to the family $\{|h| : \ h \in \mathcal{F}_\infty^K\}$, the last function converges to 0 when n tends to infinity. The claim follows. □

Claim 2. There is $l \in \mathcal{F}_\infty^K$ such that $|l/\rho| = M$ on the small Julia set.

Proof of Claim 2. Choose a function l in $\mathcal{F}_\infty^K$ such that $\max|l/\rho| = M$. Consider l_{-n} and t as in the proof of Claim 1. Let a in the small Julia set be such that $|l(a)/\rho(a)| = M$. Then we have

$$\begin{aligned}|l(a)| = |\lambda^{-n}\mathcal{L}_{[it]}^n l_{-n}(a)| &= \Big|\lambda^{-n}\sum_{b\in f^{-n}(a)} e^{\phi_t(b)+\cdots+\phi_t(f^{n-1}(b))}l_{-n}(b)\Big| \\ &\leq \lambda^{-n}\sum_{b\in f^{-n}(a)} e^{\phi(b)+\cdots+\phi(f^{n-1}(b))}M\rho(b) = M\rho(a).\end{aligned}$$

So the last inequality is actually an equality. In particular, we have $|l_{-n}/\rho| = M$ on $f^{-n}(a)$. Recall that when n goes to infinity, $f^{-n}(a)$ tends to the small Julia set. Therefore, if $l_{-\infty}$ is a limit of l_{-n}, we have $|l_{-\infty}/\rho| = M$ on the small Julia set. Replacing l by $l_{-\infty}$ gives the claim. □

Claim 3. There are $t \in K$ and $l \in \mathcal{F}_\infty^K$ such that $|\lambda^{-N}\mathcal{L}_{[it]}^N l| = M\rho$ on the small Julia set for every $N \geq 0$.

Proof of Claim 3. Choose l satisfying Claim 2 and t as in its proof. Define l_{-n} and $l_{-\infty}$ as in the proof of that claim. Fix an $N \geq 0$. For every $n \geq N$, on the small Julia set we have

$$\begin{aligned}M\rho = |l| = |\lambda^{-n}\mathcal{L}_{[it]}^n l_{-n}| &= |\lambda^{-n+N}\mathcal{L}_{[it]}^{n-N}(\lambda^{-N}\mathcal{L}_{[it]}^N l_{-n})| \\ &\leq \lambda^{-n+N}\mathcal{L}^{n-N}|\lambda^{-N}\mathcal{L}_{[it]}^N l_{-n}| \leq \lambda^{-n+N}\mathcal{L}^{n-N}\lambda^{-N}\mathcal{L}^N(M\rho) = M\rho.\end{aligned}$$

So the two last inequalities are in fact equalities and we deduce that $|\lambda^{-N}\mathcal{L}_{[it]}^N l_{-n}| = M\rho$. It follows that $|\lambda^{-N}\mathcal{L}_{[it]}^N l_{-\infty}| = M\rho$ for every $N \geq 0$. Replacing l by $l_{-\infty}$ gives the claim. □

Assume now that g is not a $(\mathcal{C}^0,\phi)$-multiplicative cocycle. By Claim 1, we only need to show that $M = 0$. Assume by contradiction that $M \neq 0$. Consider t and l as in Claim 3. Define $\xi(a) := l(a)/\rho(a)$ for $a \in \mathbb{P}^k$ and $\vartheta(a) := e^{itg(a)}\xi(a)/\xi(f(a))$ for a in the small Julia set. These functions are continuous and we have $|\xi(a)| = M$ and $|\vartheta(a)| = 1$ on the small Julia set. We have for a in the small Julia set

$$\begin{aligned}M\rho(a) = |\lambda^{-n}\mathcal{L}_{[it]}^n l(a)| &= \Big|\lambda^{-n}\sum_{b\in f^{-n}(a)} e^{\phi_t(b)+\cdots+\phi_t(f^{n-1}(b))}l(b)\Big| \\ &\leq \lambda^{-n}\sum_{b\in f^{-n}(a)} e^{\phi(b)+\cdots+\phi(f^{n-1}(b))}M\rho(b) = M\rho(a).\end{aligned}$$

So, the last inequality is an equality. Using the function ξ, we can rewrite this equality as (we remove the factors λ^{-n} and also the factors $|\xi(f^n(b))|$ and M as they are both equal to $|\xi(a)|$ and independent of $b \in f^{-n}(a)$)

$$\Big|\sum_{b\in f^{-n}(a)} \vartheta(b)\dots\vartheta(f^{n-1}(b))e^{\phi(b)+\dots+\phi(f^{n-1}(b))}\rho(b)\Big| = \sum_{b\in f^{-n}(a)} e^{\phi(b)+\dots+\phi(f^{n-1}(b))}\rho(b).$$

As $|\vartheta| = 1$, we deduce that if b and b' are two points in $f^{-n}(a)$ then

$$\vartheta(b)\dots\vartheta(f^{n-1}(b)) = \vartheta(b')\dots\vartheta(f^{n-1}(b')).$$

This and a similar equality for $f(b), f(b'), n-1$ instead of b, b', n imply that $\vartheta(b) = \vartheta(b')$. We conclude that ϑ is constant on $f^{-n}(a)$ for every n. As $f^{-n}(a)$ tends to the small Julia set when n going to infinity, it follows that ϑ is constant. From the definition of ϑ, and since ξ is continuous, we obtain that g is a $(\mathcal{C}^0, \phi)$-multiplicative cocycle for a suitable real number s such that $\vartheta = e^{is}$. This is a contradiction. So we have $M = 0$, as desired. □

Recall that ϕ and g have bounded $\|\cdot\|_{\diamond_1}$ norms.

Lemma 6.13. *Let K be a compact subset of $\mathbb{R}$. There is a positive constant c such that $\|\lambda^{-n}\mathcal{L}^n_{[it]}\|_{\diamond_1} \leq c$ for every $n \geq 0$ and $t \in K$. If $K \subset \mathbb{R}\setminus\{0\}$ and g is not a $(\|\cdot\|_{\diamond_1}, \phi)$-multiplicative cocycle, then there are constants $c > 0$ and $0 < r < 1$ such that $\|\lambda^{-n}\mathcal{L}^n_{[it]}\|_{\diamond_1} \leq cr^n$ for every $t \in K$ and $n \geq 0$.*

Proof. Consider the functional ball $\mathcal{F} := \{h \,:\, \|h\|_{\diamond_1} \leq 1\}$. By Arzelà-Ascoli theorem this ball is compact for the uniform topology. Define $\mathcal{F}^K_\infty$ as in the proof of Lemma 6.12. Using that lemma, we can follow the proof of Proposition 5.7 and obtain that $\|\lambda^{-n}\mathcal{L}^n_{[it]}\|_{\langle p,\alpha\rangle,\gamma}$ is bounded uniformly on n and $t \in K$. It follows that a similar property holds for the norm $\|\cdot\|_{\diamond_1}$. This gives the first assertion in the lemma. We also obtain that the family $\mathcal{F}^K_\infty$ is bounded in the $\|\cdot\|_{\diamond_1}$ norm and, using again Arzelà-Ascoli theorem, we obtain that it is compact in the uniform topology.

Consider now the second assertion and assume that g is not a $(\|\cdot\|_{\diamond_1}, \phi)$-multiplicative cocycle. We first show that $\mathcal{F}^K_\infty$ reduces to $\{0\}$. Assume by contradiction that this is not true. Consider t and l as in Claim 3 in the proof of Lemma 6.12. Recall that both $\|l\|_{\diamond_1}$ and $\|1/\rho\|_{\diamond_1}$ are finite, see Theorem 1.2. By Lemma 3.17, the function $\xi := l/\rho$ satisfies the same property and we conclude, as at the end of the proof of Lemma 6.12, that g is a $(\|\cdot\|_{\diamond_1}, \phi)$-multiplicative cocycle. This contradicts the hypothesis.

So $\mathcal{F}^K_\infty$ is reduced to $\{0\}$. By definition of $\mathcal{F}^K_\infty$, we obtain that $\lambda^{-n}\mathcal{L}^n_{[it]}h$ converges to 0 uniformly on $h \in \mathcal{F}$ and $t \in K$. Using this property, we can follow the proof of Proposition 5.4 (take $\beta = 1/4$) to obtain that $\|\lambda^{-N}\mathcal{L}^N_{[it]}h\|_{\langle p,\alpha\rangle,\gamma} \leq 1/4$ for N large enough and for all $h \in \mathcal{F}$ and $t \in K$. When N is large enough, we also have $\|\lambda^{-N}\mathcal{L}^N_{[it]}h\|_\infty \leq 1/4$. Therefore, we have $\|\lambda^{-N}\mathcal{L}^N_{[it]}\|_{\diamond_1} \leq 1/2$ which implies the desired property with $r = 2^{-1/N}$. □

We have the following characterizations of multiplicative cocycles.

Proposition 6.14. *Let $g\colon \mathbb{P}^k \to \mathbb{R}$ be such that $\|g\|_{\diamond_1}$ is finite. Then the following properties are equivalent:*

- (ia) *g is a multiplicative cocycle;*
- (ib) *g is a $(\mathcal{C}^0, \phi)$-multiplicative cocycle;*
- (ic) *g is a $(\|\cdot\|_{\diamond_1}, \phi)$-multiplicative cocycle;*
- (iia) *there exists a number $t > 0$ such that the spectral radius with respect to the norm $\|\cdot\|_{\diamond_1}$ of $\lambda^{-1}\mathcal{L}_{[it]}$ is ≥ 1.*
- (iib) *there exists a number $t > 0$ such that the spectral radius with respect to the norm $\|\cdot\|_{\diamond_1}$ of $\lambda^{-1}\mathcal{L}_{[it]}$ is equal to 1.*

Moreover, every coboundary with finite $\|\cdot\|_{\diamond_1}$ norm is a $(\|\cdot\|_{\diamond_1}, \phi)$-multiplicative cocycle.

Proof. The last assertion is a direct consequence of Proposition 6.5. Indeed, if g is such a coboundary, we have $g = \tilde{h} \circ f - \tilde{h}$ on the small Julia set for some $\tilde{h}$ with $\|\tilde{h}\|_{\diamond_1} < \infty$. So g is a $(\|\cdot\|_{\diamond_1}, \phi)$-multiplicative cocycle as in Definition 6.10 for $t := 1$, $s := 0$ and $\xi := e^{i\tilde{h}}$. We prove now the equivalence of the above five properties. It is clear that (ic) ⇒ (ib) ⇒ (ia) and (iib) ⇒ (iia). Lemma 6.13 implies that (iia) ⇒ (ic). The following implications complete the proof.

(iia) ⇒ (iib). By Lemma 6.13, $\|\lambda^{-n}\mathcal{L}^n_{[it]}\|_{\diamond_1}$ is bounded uniformly on n. Thus, the spectral radius of $\lambda^{-1}\mathcal{L}_{[it]}$ with respect to the norm $\|\cdot\|_{\diamond_1}$ is at most equal to 1. We easily deduce that (iia) ⇒ (iib).

(ia) ⇒ (iia). Assume by contradiction that (ia) is true and (iia) is not true. Consider t, s, and ξ as in Definition 6.10. If $\xi(z) \neq 0$ we divide it by $|\xi(z)|$. This allows us to assume that ξ is bounded. Define $h := \xi\rho$. Using the cocycle property of g and the $\lambda^{-1}\mathcal{L}$-invariance of ρ, we have

$$\lambda^{-1}\mathcal{L}_{[it]}h = \lambda^{-1}\mathcal{L}(e^{itg}\xi\rho) = e^{is}\lambda^{-1}\mathcal{L}((\xi \circ f)\rho) = e^{is}\xi\lambda^{-1}\mathcal{L}(\rho) = e^{is}h.$$

It follows that $\lambda^{-n}\mathcal{L}^n_{[it]}h = e^{ins}h$ for $n \geq 1$. Fix an arbitrary positive constant ϵ and choose a smooth function $\tilde{h}$ such that $\|h - \tilde{h}\|_{L^1(m_\phi)} \leq \epsilon$. Since (iia) is not true, $\lambda^{-n}\mathcal{L}^n_{[it]}\tilde{h}$ converges uniformly to 0. Moreover, we have

$$|h| = |\lambda^{-n}\mathcal{L}^n_{[it]}h| \leq |\lambda^{-n}\mathcal{L}^n_{[it]}\tilde{h}| + |\lambda^{-n}\mathcal{L}^n_{[it]}(h - \tilde{h})| \leq |\lambda^{-n}\mathcal{L}^n_{[it]}\tilde{h}| + \lambda^{-n}\mathcal{L}^n|h - \tilde{h}|.$$

This and the $\lambda^{-1}\mathcal{L}^*$-invariance of m_ϕ imply that

$$\|h\|_{L^1(m_\phi)} \leq \|\lambda^{-n}\mathcal{L}^n_{[it]}\tilde{h}\|_{L^1(m_\phi)} + \|h - \tilde{h}\|_{L^1(m_\phi)} \leq \|\lambda^{-n}\mathcal{L}^n_{[it]}\tilde{h}\|_{L^1(m_\phi)} + \epsilon.$$

By taking n going to infinity, we obtain that $\|h\|_{L^1(m_\phi)} \leq \epsilon$, which implies that $h = 0$ and hence $\xi = 0$, both m_ϕ-almost everywhere. This contradicts the requirement on ξ in Definition 6.10 and ends the proof of the proposition. □

Proof of Theorem 6.11. We follow here the proof of [DNS07, Th. C], which is based on [Bre92, Theorem 10.17]. Let ψ be any real-valued function in $L^1(\mathbb{R})$ whose Fourier transform

$$\hat{\psi}(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \psi(t)e^{-itx}dt$$

is a continuous function with support contained in some interval $[-\delta, \delta]$. Define

$$A_n(x) := \sigma\sqrt{2\pi n}\,\langle \mu_\phi, \psi(x + S_n(g))\rangle - e^{-x^2/(2\sigma^2 n)} \int_{-\infty}^{\infty} \psi(t)dt. \tag{6.6}$$

By [Bre92, Th. 10.7], it suffices to show that $A_n(x)$ converges to 0 as n goes to infinity, uniformly on x.

Claim. We have

$$A_n(x) = \int_{-\delta\sigma\sqrt{n}}^{\delta\sigma\sqrt{n}} \hat{\psi}\Big(\frac{t}{\sigma\sqrt{n}}\Big)e^{itx/(\sigma\sqrt{n})}\big\langle m_\phi, \lambda^{-n}\mathcal{L}^n_{[it/(\sigma\sqrt{n})]}\rho\big\rangle dt - \hat{\psi}(0)\int_{-\infty}^{\infty} e^{itx/(\sigma\sqrt{n})}e^{-t^2/2}dt. \tag{6.7}$$

Proof of Claim. The second term in the RHS of (6.6) is equal to the second term in the RHS of (6.7) because

$$e^{-x^2/(2\sigma^2 n)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{itx/(\sigma\sqrt{n})}e^{-t^2/2}dt \qquad \text{and} \qquad \hat{\psi}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(t)dt.$$

It remains to compare the first terms. Using the identities

$$\big\langle m_\phi, e^{itS_n(g)}h\big\rangle = \big\langle m_\phi, \lambda^{-n}\mathcal{L}^n(e^{itS_n(g)}h)\big\rangle = \big\langle m_\phi, \lambda^{-n}\mathcal{L}^n_{[it]}h\big\rangle \quad \text{and} \quad \psi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\delta}^{\delta} \hat{\psi}(t)e^{itz}dt$$

and the fact that $\mu_\phi = \rho m_\phi$, we obtain

$$\begin{aligned} \sigma\sqrt{2\pi n}\,\langle \mu_\phi, \psi(x+S_n(g))\rangle &= \sigma\sqrt{n}\int_{-\delta}^{\delta}\hat\psi(t)e^{itx}\big\langle \mu_\phi, e^{itS_n(g)}\big\rangle dt \\ &= \sigma\sqrt{n}\int_{-\delta}^{\delta}\hat\psi(t)e^{itx}\big\langle m_\phi, \lambda^{-n}\mathcal{L}^n_{[it]}\rho\big\rangle dt. \end{aligned}$$

The last expression is equal to the first term in the RHS of (6.7) by the change of variable $t\mapsto t/(\sigma\sqrt{n})$. This ends the proof of the claim. □

Notice that for any constant $\delta_0>0$ we have the following partial estimate for the second term in the RHS of (6.7):

$$\lim_{n\to\infty}\Big|\int_{|t|>\delta_0\sigma\sqrt{n}} e^{itx/(\sigma\sqrt{n})}e^{-t^2/2}dt\Big| \le \lim_{n\to\infty}\int_{|t|>\delta_0\sigma\sqrt{n}} e^{-t^2/2}dt = 0. \tag{6.8}$$

Moreover, it follows from Proposition 6.8, Lemma 6.9, and the identity $\langle m_\phi,\rho\rangle = 1$, that

$$\lim_{n\to\infty}\hat\psi\Big(\frac{t}{\sigma\sqrt{n}}\Big)\big\langle m_\phi, \lambda^{-n}\mathcal{L}^n_{[it/(\sigma\sqrt{n})]}\rho\big\rangle - \hat\psi(0)e^{-t^2/2} = 0 \quad \text{for all } t\in\mathbb{R}, \tag{6.9}$$

and, for $\delta_0<\delta$ sufficiently small,

$$\Big|\hat\psi\Big(\frac{t}{\sigma\sqrt{n}}\Big)\big\langle m_\phi, \lambda^{-n}\mathcal{L}^n_{[it/(\sigma\sqrt{n})]}\rho\big\rangle - \hat\psi(0)e^{-t^2/2}\Big| \lesssim e^{-t^2/4}\|\hat\psi\|_\infty \quad \text{when } |t|\le\delta_0\sigma\sqrt{n}. \tag{6.10}$$

Since the RHS in (6.10) defines an integrable function of $t\in\mathbb{R}$, (6.9) and (6.10) imply that

$$\lim_{n\to\infty}\int_{-\delta_0\sigma\sqrt{n}}^{\delta_0\sigma\sqrt{n}}\hat\psi\Big(\frac{t}{\sigma\sqrt{n}}\Big)e^{itx/(\sigma\sqrt{n})}\big\langle m_\phi, \lambda^{-n}\mathcal{L}^n_{[it/(\sigma\sqrt{n})]}\rho\big\rangle dt - \hat\psi(0)\int_{-\delta_0\sigma\sqrt{n}}^{\delta_0\sigma\sqrt{n}} e^{itx/(\sigma\sqrt{n})}e^{-t^2/2}dt = 0.$$

Using this, (6.8), and the above claim, it is enough to prove that (compare with the first term in the RHS of (6.7))

$$A'_n(x) := \int_{\delta_0\le|t/(\sigma\sqrt{n})|<\delta}\hat\psi\Big(\frac{t}{\sigma\sqrt{n}}\Big)e^{itx/(\sigma\sqrt{n})}\big\langle m_\phi, \lambda^{-n}\mathcal{L}^n_{[it/(\sigma\sqrt{n})]}\rho\big\rangle dt$$

converges to 0. Here is where we use the assumption that g is not a multiplicative cocycle. By Lemma 6.13 and Proposition 6.14, we can find two constants $c>0$ and $0<r<1$ such that $\big\|\lambda^{-n}\mathcal{L}^n_{[it/(\sigma\sqrt{n})]}\big\|_{\diamond_1}\le cr^n$ for $|t/(\sigma\sqrt{n})|\in[\delta_0,\delta]$. This shows that the term $\langle\cdot,\cdot\rangle$ in the definition of $A'_n(x)$ goes to zero exponentially fast, and we deduce that $A'_n(x) = O(\sqrt{n}r^n)$ as n goes to infinity. This completes the proof of the theorem. □

6.6. **Almost Sure Invariant Principle (ASIP) and consequences.** We can now prove the ASIP for observables which are not coboundaries, see Definition A.5. The ASIP was proved by Dupont [Dup10] in the case where $\phi=0$ for observables which are Hölder continuous, or admit analytic singularities, by using [PS75], see also Przytycki-Urbański-Zdunik [PUZ89] for $k=1$ and $\phi=0$.

Theorem 6.15. *Under the hypotheses of Theorem 1.2, let $g\colon\mathbb{P}^k\to\mathbb{R}$ be such that $\|g\|_{\diamond_1}$ is finite and $\langle\mu_\phi,g\rangle=0$. Assume that g is not a coboundary. Then g satisfies the ASIP with error rate $o(n^q)$ for all $q>1/4$.*

Proof. We will prove that the assumptions of Theorem A.6 are satisfied by the system $f\colon\mathbb{P}^k\to\mathbb{P}^k$ and the operators $\mathcal{T}_t := \lambda^{-1}\mathcal{L}_{[it]}$ for $t\in\mathbb{R}$. These operators act on the functional space $\mathcal{H} := \{h\ :\ \|h\|_{\diamond_1}<\infty\}$ which is endowed with the norm $\|\cdot\|_{\diamond_1}$. We consider here the probability measures $\nu:=\mu_\phi$, $\nu^*:=m_\phi$, the function $\xi:=\rho$, and the constant p (of Theorem A.6) large enough. We only need to check the strong coding condition in Theorem A.6. Indeed, the spectral description is a consequence of Theorem 1.2, the weak regularity follows from Lemma 6.7 or Lemma 6.13, and the last condition is satisfied since g is continuous.

Let us prove that the strong coding condition holds for all $t_0, \ldots, t_{n-1} \in \mathbb{R}$. As in the proof of (6.2), we obtain

$$\begin{aligned}
\mathcal{L}^n_{[0]}(e^{i\sum_{l=0}^{n-1} t_l g\circ f^l}\rho) &= \mathcal{L}_{[0]}\Big(\mathcal{L}^{n-1}_{[0]}\big(e^{it_{n-1}g\circ f^{n-1}}e^{i\sum_{l=0}^{n-2} t_l g\circ f^l}\rho\big)\Big)\\
&= \mathcal{L}_{[0]}\Big(e^{it_{n-1}g}\mathcal{L}^{n-1}_{[0]}\big(e^{i\sum_{l=0}^{n-2} t_l g\circ f^l}\rho\big)\Big)\\
&= \mathcal{L}_{[it_{n-1}]}\mathcal{L}^{n-1}_{[0]}\big(e^{i\sum_{l=0}^{n-2} t_l g\circ f^l}\rho\big).
\end{aligned}$$

Hence, by induction, we get

$$\mathcal{L}^n_{[0]}(e^{i\sum_{l=0}^{n-1} t_l g\circ f^l}\rho) = \mathcal{L}_{[it_{n-1}]}\circ\cdots\circ\mathcal{L}_{[it_0]}\rho.$$

The identity (6.4) and the last one imply

$$\langle\mu_\phi, e^{i\sum_{l=0}^{n-1} t_l g\circ f^l}\rangle = \langle m_\phi, \lambda^{-n}\mathcal{L}^n_{[0]}(e^{i\sum_{l=0}^{n-1} t_l g\circ f^l}\rho)\rangle = \big\langle m_\phi, \lambda^{-n}\mathcal{L}_{[it_{n-1}]}\circ\cdots\circ\mathcal{L}_{[it_0]}\rho\big\rangle\,.$$

This is the desired strong coding condition. Applying Theorem A.6 gives the result. □

We also have the following direct consequence of the ASIP and Theorem A.3, see Definitions A.7, A.8, and Theorem A.9. The LIL was established in [SUZ14] in the case where both the weight ϕ and the observable g are Hölder continuous.

Corollary 6.16. *Under the hypotheses of Theorem 1.2, let g be such that $\|g\|_{\diamond_1}$ is finite and $\langle\mu_\phi, g\rangle = 0$. Assume that g is not a coboundary. Then g satisfies the ASCLT, the LIL, and the CLT with error rate $O(n^{-1/2})$.*

Remark 6.17. The LCLT is stronger that the CLT, even in its quantitative version given in Corollary 6.16. On the other hand, notice that it requires a stronger assumption on the observable g.

6.7. **Large Deviation Principle (LDP).** We conclude the statistical study of $(\mathbb{P}^k, f, \mu_\phi)$ with the following property, see Definition A.10. This is new in this generality for all $k \geq 1$, even for $\phi = 0$ (see [CRL11] for the case when $k = 1$ and some kind of weak hyperbolicity is assumed). The LDP in particular implies the Large Deviation Theorem, which is proved in [DNS10] in the case $\phi = 0$, see also [PS96, DS10a].

Theorem 6.18. *Under the hypotheses of Theorem 1.2, let $g\colon \mathbb{P}^k \to \mathbb{R}$ be such that $\|g\|_{\diamond_1}$ is finite and $\langle\mu_\phi, g\rangle = 0$. Assume that g is not a coboundary. Then g satisfies the LDP.*

Proof. We apply Theorem A.11 for the random variables $\mathcal{S}_n := S_n(g)$ (see (6.3)) with respect to the probability μ_ϕ, and for $\Lambda(\theta) := \log\alpha(\theta)$, where $\alpha(\theta)$ is defined in Proposition 6.8. So, $\Lambda(\theta)$ is analytic and well defined for θ small enough. Since g is not a coboundary, we have $\sigma > 0$ and the convexity of Λ follows from the expansion of $\alpha(\theta)$ given in Lemma 6.9. Finally, as in (6.5), using θ instead of $1/n$, we get

$$\langle\mu_\phi, e^{\theta S_n(g)}\rangle = \alpha(\theta)^n\,\langle m_\phi, \Phi_\theta(\rho)\rangle + \big\langle m_\phi, \Psi^n_\theta(\rho)\big\rangle$$

which implies (note that $\langle m_\phi, \Phi_\theta(\rho)\rangle$ does not vanish for θ small as it is equal to 1 when $\theta = 0$)

$$\lim_{n\to\infty}\frac{1}{n}\log\langle\mu_\phi, e^{\theta S_n(g)}\rangle = \log\alpha(\theta) = \Lambda(\theta).$$

Thus, Condition (A.5) in Theorem A.11 is satisfied. Applying that theorem gives the result. □

Appendix A. Abstract statistical theorems

We provide here the precise definitions for the statistical properties studied in Section 6, as well as the criteria, used in the previous section, ensuring their validity in an abstract setting. We will consider in what follows a dynamical system $T\colon (X, \mathcal{F}) \to (X, \mathcal{F})$, where $\mathcal{F}$ is a given σ-algebra, and ν a probability measure which is invariant with respect to T. Given any observable (real-valued measurable function) $g \in L^1(\nu)$ and $n \geq 1$ we denote by $S_n(g)$ the Birkhoff sum

$S_n(g) := \sum_{j=0}^{n-1} g \circ T^j$. We are interested in comparing the behaviour of the sequence $S_n(g)$ with that of a sum of independent identically distributed random variables Z_n of mean $\langle \nu, g \rangle$. Notice that the invariance of ν precisely implies that the functions $g \circ T^j$, considered as random variables, are identically distributed. The goal is to prove that, under suitable assumptions, the sequence of *weakly dependent* random variables $g \circ T^j$ and their sums $S_n(g)$ enjoy many of the statistical properties of the sequences Z_n and $\sum_{j=0}^{n-1} Z_j$. For simplicity, we only consider observables g of zero mean, i.e., such that $\langle \nu, g \rangle = 0$. It is easy to extend the discussion to the case of non-zero mean.

In what follows, we will denote by $\mathbb{E}(h|\mathcal{G})$ the conditional expectation of an $\mathcal{F}$-measurable function h with respect to a σ-algebra $\mathcal{G} \subseteq \mathcal{F}$. The σ-algebra $T^{-n}\mathcal{F}$ is generated by the sets of the form $T^{-n}(B)$ with B in $\mathcal{F}$. We will say that h is a *coboundary* if there exists $\tilde{h} \in L^2(\nu)$ such that $h = \tilde{h} \circ T - \tilde{h}$. A coboundary belongs to $L^2(\nu)$ and has zero mean.

Definition A.1 (CLT - Central Limit Theorem)**.** Let $T\colon (X, \mathcal{F}) \to (X, \mathcal{F})$ be a dynamical system and ν an invariant probability measure. Let g be a real-valued integrable function with $\langle \nu, g \rangle = 0$. We say that g satisfies the Central Limit Theorem (CLT) with variance $\sigma > 0$ if for any interval $I \subset \mathbb{R}$ we have

$$\lim_{n\to\infty} \nu\Big\{ \frac{1}{\sqrt{n}} S_n(g) \in I \Big\} = \frac{1}{\sqrt{2\pi}\sigma} \int_I e^{-\frac{t^2}{2\sigma^2}} dt. \tag{A.1}$$

We give two criteria to ensure that an observable g satisfies the CLT. The first one, due to Gordin (see also [Liv96]), is more classical and easier to use.

Theorem A.2 (Gordin [Gor69])**.** *Let $T\colon (X, \mathcal{F}) \to (X, \mathcal{F})$ be a dynamical system and ν an invariant probability measure. Let g be a real-valued function in $L^2(\nu)$ which is not a coboundary. Assume that*

$$\sum_{n\geq 1} \big\| \mathbb{E}\left(g|T^{-n}\mathcal{F}\right) \big\|^2_{L^2(\nu)} < \infty. \tag{A.2}$$

Then g satisfies the CLT with variance $\sigma > 0$ given by

$$\sigma^2 := \langle \nu, g^2 \rangle + 2 \sum_{n\geq 1} \langle \nu, g \cdot (g \circ T^n) \rangle . \tag{A.3}$$

Note that under the hypothesis (A.2), the observable g is not a coboundary if and only if the constant σ defined in (A.3) is non-zero. Note also that this hypothesis implies that g has zero mean because each term in the infinite sum in (A.2) is larger than or equal to $|\langle \nu, g \rangle|^2$ thanks to Cauchy-Schwarz's inequality and the invariance of ν.

The second criterion is based on the theory of perturbed operators, which also allows to study the speed of convergence in (A.1), i.e., to get a version of Berry-Esseen theorem. Recall that the spectrum of a linear operator $\mathcal{T}$, from a complex Banach space to itself, is the closure of the set of $z \in \mathbb{C}$ such that the operator $zI - \mathcal{T}$ is not invertible. The essential spectrum is obtained from the spectrum by removing its isolated points corresponding to eigenvalues of finite multiplicity. The spectral radius $r(\mathcal{T})$ and the essential spectral radius $r_{\text{ess}}(\mathcal{T})$ are the radii of the smallest disks centered at the origin which contain the spectrum and the essential spectrum, respectively. The following criterion, based on ideas on Nagaev and Guivarc'h [Nag57, GH88], is a translation in our setting of [Gou15, Th. 3.7].

Theorem A.3 (Nagaev-Guivarc'h-Gouëzel)**.** *Let $T\colon (X, \mathcal{F}) \to (X, \mathcal{F})$ be a dynamical system and ν an invariant probability measure. Let g be a real-valued integrable function. Assume that there exist a complex Banach space $\mathcal{H}$, a family of operators $\mathcal{T}_t$ acting on $\mathcal{H}$, for $t \in [-\delta, \delta]$ with some constant $\delta > 0$, and elements $\xi \in \mathcal{H}$ and $\nu^* \in \mathcal{H}^*$ such that*

(i) *coding: for all $n \in \mathbb{N}$ and all $t \in [-\delta, \delta]$, we have $\langle \nu, e^{itS_n(g)} \rangle = \langle \nu^*, \mathcal{T}_t^n \xi \rangle$;*
(ii) *spectral description: we have $r_{\text{ess}}(\mathcal{T}_0) < 1$, and $\mathcal{T}_0$ has a unique eigenvalue of modulus ≥ 1; moreover, this eigenvalue has multiplicity 1 and is located at 1;*
(iii) *regularity: the family $t \mapsto \mathcal{T}_t$ is of class C^3.*

Assume also that the largest eigenvalue $\alpha(t)$ *of* $\mathcal{T}_t$ *for* t *near* 0 *admits the order* 2 *expansion* $\alpha(t) = e^{iAt-\sigma^2 t^2/2+o(t^2)}$ *for some* $A, \sigma \in \mathbb{R}$ *with* $\sigma > 0$. *Then* $g - A$ *satisfies the CLT with variance* σ *and the speed of convergence in* (A.1) *is of order* $O(n^{-1/2})$.

The following is an improvement of the CLT, see also Remark 6.17.

Definition A.4 (LCLT - Local Central Limit Theorem)**.** Let $T\colon (X,\mathcal{F}) \to (X,\mathcal{F})$ be a dynamical system, and ν an invariant probability measure. Let g be a real-valued integrable function with $\langle \nu, g\rangle = 0$. We say that g satisfies the Local Central Limit Theorem (LCLT) with variance $\sigma > 0$ if for every bounded interval $I \subset \mathbb{R}$ the convergence

$$\lim_{n\to\infty} \left| \sigma\sqrt{n}\nu\{x + S_n(g) \in I\} - \frac{1}{\sqrt{2\pi}} e^{-x^2/(2\sigma^2 n)} |I| \right| = 0$$

holds uniformly in $x \in \mathbb{R}$. Here $|I|$ denotes the length of I.

Partial sums of independent and identically distributed random variables can be compared with Brownian motions of suitable variance, and one can get precise estimates for the fluctuations around the mean value. In the context of dynamical systems, we have the following analogous definition.

Definition A.5 (ASIP - Almost Sure Invariant Principle)**.** Let $T\colon (X,\mathcal{F}) \to (X,\mathcal{F})$ be a dynamical system and ν an invariant probability measure. Let g be a real-valued integrable function with $\langle \nu, g\rangle = 0$. We say that g satisfies the Almost Sure Invariance Principle (ASIP) of variance $\sigma > 0$ and error rate n^γ if there exist, on some probability space $\mathcal{X}$, a sequence of random variables $(\mathcal{S}_n)_{n\geq 0}$ and a Brownian motion $\mathcal{W}$ of variance σ such that

(i) $\mathcal{S}_n = \mathcal{W}(n) + o(n^\gamma)$ almost everywhere on $\mathcal{X}$;
(ii) $S_n(g)$ and $\mathcal{S}_n$ have the same distribution for any $n \geq 0$.

General criteria that allow one to establish the ASIP in various contexts are given in [PS75]. We state here the following criterion by Gouëzel, see [Gou10, Th. 1.2] and [Gou15, Th. 5.2]. Notice that Gouëzel's result actually holds in the more general case of random variables with values in $\mathbb{R}^d$.

Theorem A.6 (Gouëzel)**.** *Let* $T\colon (X,\mathcal{F}) \to (X,\mathcal{F})$ *be a dynamical system and* ν *an invariant probability measure. Let* g *be a real-valued integrable function. Assume that there exist a complex Banach space* $\mathcal{H}$, *a family of operators* $\mathcal{T}_t$ *acting on* $\mathcal{H}$, *for* $t \in [-\delta, \delta]$ *with some constant* $\delta > 0$, *and elements* $\xi \in \mathcal{H}$ *and* $\nu^* \in \mathcal{H}^*$ *such that*

(i) *strong coding: for all* $n \in \mathbb{N}$ *and all* $t_0, \ldots, t_{n-1} \in [-\delta, \delta]$, *we have*
$$\left\langle \nu, e^{i\sum_{j=0}^{n-1} t_j g\circ T^j} \right\rangle = \left\langle \nu^*, \mathcal{T}_{t_{n-1}} \circ \cdots \circ \mathcal{T}_{t_0}\xi \right\rangle;$$
(ii) *spectral description: we have* $r_{\mathrm{ess}}(\mathcal{T}_0) < 1$, *and* $\mathcal{T}_0$ *has a unique eigenvalue of modulus* ≥ 1; *moreover, this eigenvalue has multiplicity* 1 *and is located at* 1;
(iii) *weak regularity: either* $t \mapsto \mathcal{T}_t$ *is continuous, or* $\|\mathcal{T}_t^n\|_{\mathcal{H}\to\mathcal{H}} \leq c$ *for some constant* $c > 0$ *and for all* $t \in [-\delta, \delta]$ *and* $n \geq 0$;
(iv) *there exist* $p > 2$ *and* $c > 0$ *such that* $\|g \circ T^n\|_{L^p(\nu)} \leq c$ *for all* $n \geq 0$.

Assume also that the largest eigenvalue $\alpha(t)$ *of* $\mathcal{T}_t$ *for* t *near* 0 *admits the order* 2 *expansion* $\alpha(t) = e^{iAt-\sigma^2 t^2/2+o(t^2)}$ *for some* $A, \sigma \in \mathbb{R}$ *with* $\sigma > 0$. *Then* $g - A$ *satisfies the ASIP of variance* σ *and error rate* $o(n^q)$ *for all* q *such that*

$$q > \frac{p}{4p-4} = \frac{1}{4} + \frac{1}{4p-4}.$$

The following properties are general consequences of the ASIP, see for instance [PS75, LP89, CG07].

Definition A.7 (LIL - Law of iterated logarithms)**.** Let $T\colon (X,\mathcal{F}) \to (X,\mathcal{F})$ be a dynamical system and ν an invariant probability measure. Let g be a real-valued integrable function such that $\langle \nu, g\rangle = 0$. We say that g satisfies the Law of Iterated Logarithms (LIL) with variance $\sigma > 0$ if

$$\limsup_{n\to\infty} \frac{S_n(g)}{\sigma\sqrt{2n\log\log n}} = 1 \quad \nu\text{-almost everywhere.}$$

Definition A.8 (ASCLT - Almost sure Central Limit Theorem)**.** Let $T\colon (X,\mathcal{F}) \to (X,\mathcal{F})$ be a dynamical system and ν an invariant probability measure. Let g be a real-valued integrable function such that $\langle \nu, g\rangle = 0$. We say that g satisfies the Almost Sure Central Limit Theorem (ASCLT) with variance $\sigma > 0$ if, for ν-almost every point $x \in X$,

$$\frac{1}{\log n}\sum_{j=1}^{n} \frac{1}{j}\delta_{j^{-1/2}S_j(g)(x)} \to \mathcal{N}(0,\sigma).$$

In particular, ν-almost surely

$$\frac{1}{\log n}\sum_{j=1}^{n} \frac{1}{j}\mathbb{1}_{\{j^{-1/2}S_j(g)\le t_0\}} \to \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{t_0} e^{-\frac{t^2}{2\sigma^2}}\,dt$$

for all $t_0 \in \mathbb{R}$.

Theorem A.9. *Let $T\colon (X,\mathcal{F}) \to (X,\mathcal{F})$ be a dynamical system and ν an invariant probability measure. Let g be a real-valued integrable function with $\langle \nu, g\rangle = 0$. Assume that g satisfies the ASIP with variance $\sigma > 0$ and error rate n^γ for some $\gamma < 1/2$. Then g satisfies the LIL and the ASCLT, both with the same variance σ.*

The last property that we recall is the Large Deviation Principle. It gives very precise estimates on the measure of the set where the partial sums are far from the mean value. It implies in particular the Large Deviation Theorem, which only requires an upper bound for the measure in (A.4) below.

Definition A.10 (LDP - Large Deviation Principle)**.** Let $T\colon (X,\mathcal{F}) \to (X,\mathcal{F})$ be a dynamical system and ν an invariant probability measure. Let g be a real-valued integrable function such that $\langle \nu, g\rangle = 0$. We say that g satisfies the Large Deviation Principle (LDP) if there exists a non-negative, strictly convex function c which is defined on a neighbourhood of $0 \in \mathbb{R}$, vanishes only at 0, and such that, for all $\epsilon > 0$ sufficiently small,

$$\lim_{n\to\infty} \frac{1}{n}\log \nu\Big\{x \in X\colon \frac{S_n(g)(x)}{n} > \epsilon\Big\} = -c(\epsilon). \tag{A.4}$$

The following result, established in [BL85], is a local version of Gärtner-Ellis Theorem [Gär77, Ell84, DZ98] which can be used to prove the LDP, see [HH01, Lem. XIII.2].

Theorem A.11 (Bougerol-Lacroix)**.** *For all $n \ge 1$ denote by $\mathbb{P}_n$ a probability distribution, by $\mathbb{E}_n$ the corresponding expectation operator and by $\mathcal{S}_n$ a real-valued random variable. Assume that on some interval $[-\theta_0, \theta_0]$, with $\theta_0 > 0$, we have*

$$\lim_{n\to\infty} \frac{1}{n}\log \mathbb{E}_n(e^{\theta\mathcal{S}_n}) = \Lambda(\theta), \tag{A.5}$$

where Λ is a $\mathcal{C}^1$, strictly convex function satisfying $\Lambda'(0) = 0$. Then, for all $0 < \epsilon < \Lambda(\theta_0)/\theta_0$, we have

$$\lim_{n\to\infty} \frac{1}{n}\log \mathbb{P}_n(\mathcal{S}_n > n\epsilon) = -c(\epsilon) < 0,$$

where $c(\epsilon) := \sup\{\theta\epsilon - \Lambda(\epsilon)\colon |\theta| \le \theta_0\}$ is the Legendre transform of Λ, which is non-negative, strictly convex, and vanishes only at 0.

Notice that the symmetric statement for the measure of the set where $n^{-1}S_n(g) < -\epsilon$ in (A.4) can be obtained by applying the result above to the sequence of random variables $-\mathcal{S}_n$.

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