ON THE SUPPORT OF MEASURES OF LARGE ENTROPY FOR POLYNOMIAL-LIKE MAPS

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Abstract. Let f be a polynomial-like map with dominant topological degree $d_t \geq 2$ and let $d_{k-1} < d_t$ be its dynamical degree of order $k-1$. We show that the support of every ergodic measure whose measure-theoretic entropy is strictly larger than $\log \sqrt{d_{k-1} d_t}$ is supported on the Julia set, i.e., the support of the unique measure of maximal entropy μ. The proof is based on the exponential speed of convergence of the measures $d_t^{-n}(f^n)^*\delta_a$ towards μ, which is valid for a generic point a and with a controlled error bound depending on a. Our proof also gives a new proof of the same statement in the setting of endomorphisms of $\mathbb{P}^k(\mathbb{C})$ – a result due to de Thélin and Dinh – which does not rely on the existence of a Green current.

1. Introduction

The study of the dynamics of holomorphic endomorphisms of complex projective spaces $\mathbb{P}^k := \mathbb{P}^k(\mathbb{C})$ is a central topic in complex dynamics, see for instance [15, 24] for an overview of the subject. Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be an endomorphism of algebraic degree $d \geq 2$. There exists a canonical positive closed f^*-invariant $(1,1)$-current T, called the *Green current of* f, with the property that the sequence $d^{-n}(f^n)^*\omega_0$ converges to T for every smooth positive closed $(1,1)$-form ω_0 of mass 1. The current T has strong geometric properties, in particular, it has Hölder continuous potentials. As a consequence, the measure $\mu := T^{\wedge k}$ is well-defined, and it is the unique measure of maximal entropy $k \log d$ of f [8, 20]. Its support is called the *Julia set* of f. By a result of de Thélin and Dinh [9, 12], every ergodic measure whose measure-theoretic entropy is strictly larger than $(k-1)\log d$ is also supported on the Julia set of f. Large classes of examples of such measures are constructed and studied in [4, 5, 19, 26, 27].

The proof given in [9, 12] of the above property crucially relies on the existence of the Green current. In particular, it follows from a delicate induction which makes use of the successive self-intersections $T^{\wedge j}$ of the Green current T. It is then unclear how to generalize this result to more general non-algebraic settings, where a dynamical Green current does not exist. In this paper, we address this problem in the case of polynomial-like maps with dominant topological degree. Our proof will in particular also give a new proof of the result by de Thélin and Dinh, which makes no use of the Green current.

Recall that polynomial-like maps are proper holomorphic maps $f : U \to V$, where $U \Subset V$ are open subsets of $\mathbb{C}^k$ and V is convex. By definition, every polynomial-like map defines a ramified covering $U \to V$ and the *topological degree* d_t of f is well-defined. For every

$0 \leq p \leq k$, one can define the *dynamical degrees*[1]

$$d_p = d_p(f) := \limsup_{n\to\infty} \sup_S \|(f^n)_*(S)\|_U^{1/n},$$

where the supremum is taken over all positive closed $(k-p, k-p)$-currents on U whose mass is less than or equal to 1 see [13, 15] and Definition 2.1 below. Note that we always have $d_0 = 1$ and $d_k = d_t$. By [6], the sequence $\{d_p\}_{0\leq p\leq k}$ is non-decreasing. Hence, in particular, we have $\max_{0\leq p\leq k-1} d_p = d_{k-1}$. We say that f has *dominant topological degree*[2] if $d_{k-1} < d_t$. Observe that in this case we always have $d_t \geq 2$.

Polynomial-like maps with dominant topological degree enjoy many of the dynamical properties of endomorphisms (however, their study is usually technically more involved, because of the lack of a naturally defined Green function). In particular, for every such f there exists a unique measure μ of maximal entropy $\log d_t$ and a proper analytic set $\mathcal{E} \subset V$ such that

$$d_t^{-n}(f^n)^*\delta_a \to \mu \quad \text{for all} \quad a \in V \setminus \bigcup_{j=0}^{\infty} f^j(\mathcal{E}), \tag{1.1}$$

see [13, 15]. Note that, unlike the case of endomorphisms of $\mathbb{P}^k$, here the set $\mathcal{E}$ may not be f-invariant and hence $\cup_{j=0}^{\infty} f^j(\mathcal{E})$ is not, a priori, an analytic set.

The following is our main result.

Theorem 1.1. *Let $f : U \to V$ be a polynomial-like map with dominant topological degree. Every ergodic measure ν whose measure-theoretic entropy satisfies $h_\nu(f) > \log\sqrt{d_{k-1}d_t}$ is supported on the Julia set J.*

The proof of Theorem 1.1 consists of a quantified version of the classical estimate by Gromov [20] for the topological entropy in terms of the volume growth of suitable analytic sets in the space of orbits. It is given in Section 3 and exploits in a crucial way an explicit exponential rate of the convergence (1.1), which we discuss in Section 2. In the same section, we give a more precise bound $\log\beta(f)$ for the entropy in Theorem 1.1, see also Theorem 3.5.

In the case of endomorphisms of $\mathbb{P}^k$, we have $\beta(f) = d^{k-1}$, hence the bound $\log\sqrt{d_{k-1}d_t}$ can be improved to $\log d_{k-1} = (k-1)\log d$. In particular, Theorem 1.1 gives an alternative proof of the result by de Thélin and Dinh mentioned above [9, 12].

Corollary 1.2. *Let f be an endomorphism of $\mathbb{P}^k$ of algebraic degree $d \geq 2$. Every ergodic measure whose measure-theoretic entropy is strictly larger than $(k-1)\log d$ is supported on the Julia set.*

We refer to Section 4 for further applications and corollaries of Theorem 1.1.

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[1] Sometimes, see for instance [15], these degrees are denoted by d_p^* to distinguish them from a different type of dynamical degree that can also be considered. Since here we will only use one type of dynamical degree, we will use the simpler notation d_p.

[2] In some references, maps with $d_{k-1} < d_t$ are said to have *large* topological degree, and the name *dominant* is reserved to maps for which $\max_{0\leq p\leq k-1} d_p < d_t$. We use here the name *dominant* as, by [6], these notions are equivalent.

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2. Preliminaries

2.1. Polynomial-like maps.

A *polynomial-like map* is a proper holomorphic map $f : U \to V$, where $U \Subset V$ are open subsets of $\mathbb{C}^k$ and V is convex. Homogeneous lifts to $\mathbb{C}^{k+1}$ of endomorphisms of $\mathbb{P}^k$ give examples of polynomial-like maps. In dimension $k = 1$, any polynomial-like map is conjugate to an actual polynomial on the Julia set [17]. However, in higher dimensions, the class of polynomial-like maps is significantly larger than that of regular polynomial endomorphisms of $\mathbb{C}^k$ (i.e., those extending holomorphically to $\mathbb{P}^k$), see for instance [15, Example 2.25].

Every polynomial-like map f gives a ramified covering from U to V and the *topological degree* d_t of f is well-defined. We will always assume that we have $d_t \geq 2$. The (compact) set $K := \bigcap_{n=1}^{\infty} f^{-n}(U)$ is called the *filled-in Julia set* of f. It consists of the points whose orbit is well-defined. The system (K, f) is a true dynamical system.

Definition 2.1. Let $f : U \to V$ be a polynomial-like map. For every $0 \leq p \leq k$ define

$$d_p = d_p(f) := \limsup_{n\to\infty} \sup_S \|(f^n)_*(S)\|_W^{1/n}, \tag{2.1}$$

where $W \Subset V$ is an open neighbourhood of K and the supremum in (2.1) is taken over all positive closed $(k-p, k-p)$-currents whose mass is less than or equal to 1 on a fixed neighbourhood $W' \Subset V$ of K. We say that d_p is the *dynamical degree of order* p of f.

Recall that the mass of a positive (p,p)-current S on the open set W is given by $||S||_W := \int_W S \wedge \omega^{k-p}$, where ω is the standard Kähler form of $\mathbb{C}^k$. The definition above is independent of W, W' [13, 15]. Moreover, we have $d_0 = 1$ and $d_k = d_t$. In the case of endomorphisms of $\mathbb{P}^k$ of algebraic degree d, the above definitions reduce to $d_p = d^p$. We say that f has *dominant topological degree* if $d_{k-1} < d_t$. By [6, Theorem 1.3], the sequence $\{d_p\}_{0\leq p\leq k}$ is non-decreasing, hence we have $\max_{0\leq p\leq k-1} d_p = d_{k-1} < d_t$ (and therefore also $d_t \geq 2$) for every polynomial-like map with dominant topological degree.

Polynomial-like maps with dominant topological degree enjoy many of the dynamical properties of endomorphisms. For instance, they admit a unique measure of maximal entropy $\log d_t$ [13, 15]. We will denote this measure by μ and define the *Julia set* J as the support of μ. Observe that J is a subset of the boundary of K.

2.2. Speed of convergence.

Let us fix a polynomial-like map $f : U \to V$ with topological degree $d_t \geq 2$. By [13, Proposition 3.2.5] see also [15, Proposition 2.15], there exists a constant $0 < \gamma < 1$ such that

$$|d_t^{-n}(f^n)_*\psi - \langle \mu, \psi\rangle| \lesssim \gamma^n \tag{2.2}$$

for every function ψ in a given compact family of pluriharmonic functions on V, where the implicit constant depends only on the family. We will denote by $\gamma_0 = \gamma_0(f)$ the infimum of the constants γ for which (2.2) holds. Assuming now that f has dominant topological degree, we set

$$\beta = \beta(f) := \max\{d_{k-1}, \gamma_0 d_t\}. \tag{2.3}$$

In the case of endomorphisms of $\mathbb{P}^k$, every pluriharmonic function is constant. Hence, (2.2) is trivial and we can take $\beta = d_{k-1} = d^{k-1}$, where d is the algebraic degree of the endomorphism. More generally, in a compact setting, one can take $\beta = d_{k-1}$. In our setting, we have the following weaker bound for β.

Lemma 2.2. *Let $f : U \to V$ be a polynomial-like map with dominant topological degree d_t. Then we have $\beta(f) \leq \sqrt{d_{k-1}d_t}$.*

Proof. It is enough to show that γ_0 satisfies $\gamma_0 \leq \sqrt{d_{k-1}/d_t}$. Hence, it suffices to show that (2.2) holds for every $\gamma > \sqrt{d_{k-1}/d_t}$.

Observe that the map $\psi \mapsto \sqrt{\|i\partial\psi \wedge \bar{\partial}\psi\|_U}$ defines a norm on the space of pluriharmonic functions on V with $\langle \mu, \psi\rangle = 0$. Moreover, by the Cauchy-Schwarz inequality, for any such function we have

$$0 \leq i\partial(d_t^{-n}(f^n)_*\psi) \wedge \bar{\partial}(d_t^{-n}(f^n)_*\psi) \leq d_t^{-n}(f^n)_*(i\partial\psi \wedge \bar{\partial}\psi).$$

As $i\partial\psi \wedge \bar{\partial}\psi$ is a positive closed $(1,1)$-current, the mass of the last term in the above expression satisfies

$$\|d_t^{-n} f^n_*(i\partial\psi \wedge \bar{\partial}\psi)\|_U \lesssim d_t^{-n}(d_*)^n \|i\partial\psi \wedge \bar{\partial}\psi\|_U \quad \text{as } n \to \infty$$

for every $d_* > d_{k-1}$. The assertion follows. □

The following result, whose proof in a compact setting is essentially given in [14, Lemme 4.2], gives the estimates for the rate in the convergence (1.1) that we will need. We give the details of the proof for the reader's convenience. A more precise version in the setting of endomorphisms of $\mathbb{P}^k$ is given in [16].

Theorem 2.3. *Let $f : U \to V$ be a polynomial-like map with dominant topological degree d_t and μ its equilibrium measure. Let β be as in (2.3) and take λ such that $\beta < \lambda < d_t$. Then there exists a psh function u_λ with $u_\lambda < -1$ such that for every $a \in U$, $\psi \in \mathcal{C}^2(U)$, and $n \in \mathbb{N}$ we have*

$$|\langle d_t^{-n}(f^n)^*\delta_a - \mu, \psi\rangle| \leq A\|\psi\|_{\mathcal{C}^2(U)} \left(\frac{\lambda}{d_t}\right)^n |u_\lambda(a)|, \tag{2.4}$$

where A is a constant depending on λ but independent of a, ψ, and n.

Proof. By linearity, it is enough to show the assertion in the assumption that ψ is psh and satisfies $\langle \mu, \psi\rangle = 0$ and $0 \leq dd^c\psi \leq dd^c\|z\|^2 = \omega$. Define also the function $v_0 := \|z^2\| - \langle \mu, \|z^2\|\rangle$, and observe that it satisfies $\langle \mu, v_0\rangle = 0$. Set also

$$v_n := d_t^{-n}(f^n)_* v_0 \quad \text{and} \quad u_\lambda := \sum_{n=0}^{\infty} \left(\frac{d_t}{\lambda}\right)^n v_n - C_\lambda,$$

where the constant C_λ is chosen so that

$$u_\lambda < -1 \qquad \text{and} \qquad \sum_{n\neq n_0} \left(\frac{d_t}{\lambda}\right)^n v_n \leq C_\lambda \quad \text{for every } n_0 \in \mathbb{N}.$$

Fix λ' with $\beta < \lambda' < \lambda$. As $v_n \lesssim (\lambda'/d_t)^n$ and $\|v_n\|_{L^p} \lesssim (\lambda'/d_t)^n$ for any $p \geq 1$ (see for instance [15, Theorem 2.33] and [15, Corollary 2.35]), we have that u_λ is a well-defined (i.e., not identically equal to $-\infty$) psh function on V. Setting $A_n(\psi) := \langle d_t^{-n}(f^n)^*\delta_a, \psi\rangle$, we will show the inequality

$$|A_n(\psi)| \lesssim \left(\frac{\lambda}{d_t}\right)^n |u_\lambda(a) - 1|.$$

This will show the assertion, up to replacing u_λ by $u_\lambda - 1$.

By [15, Theorem 2.34], the definition of β, and the choice of λ, for every $\phi \in \mathcal{C}^2(U)$ with $\langle \mu, \phi\rangle = 0$ and $n \in \mathbb{N}$ we have

$$|\langle d_t^{-n}(f^n)^*\nu, \phi\rangle| \lesssim \|\phi\|_{\mathcal{C}^2(U)} \left(\frac{\lambda}{d_t}\right)^n$$

for every *smooth* probability measure ν compactly supported on V (here the implicit constant can depend on ν, but is independent of ν if this is taken in a compact family of probability measures). The mean inequality for psh functions implies that we have $A_n(\psi) \lesssim (\lambda/d_t)^n$ (where the implicit constant is now independent of a). In order to conclude, we need to prove a similar bound for $-A_n(\psi)$.

It follows from the properties at the beginning of the proof that $v_0 - \psi$ is psh. Hence, we can write $\psi = v_0 - (v_0 - \psi)$ as a difference of psh functions, and we have

$$-A_n(\psi) = -A_n(v_0) + A_n(v_0 - \psi).$$

As an upper bound for $A_n(v_0 - \psi)$ can be found with the same arguments as above, we only need to prove an upper bound for $-A_n(v_0) = -v_n(a)$. By the definition of u_λ, we have $(d_t/\lambda)^n v_n \geq u_\lambda$ for every $n \in \mathbb{N}$. It follows that we have

$$-A_n(v_0) = -v_n(a) \leq \left(\frac{\lambda}{d_t}\right)^n |u_\lambda|.$$

The assertion follows. □

The following immediate consequence of Theorem 2.3 will be used to prove Theorem 1.1.

Corollary 2.4. *Let $f : U \to V$ be a polynomial-like map with dominant topological degree d_t and μ its equilibrium measure. Let β be as in (2.3). Fix an open set U' with $K \subseteq U' \subseteq U$ and let ω_0 be a smooth probability measure on U'. For any λ with $\beta < \lambda < d_t$ there exists a psh function u_λ such that for every $\psi \in \mathcal{C}^2(U)$ and $n \in \mathbb{N}$ we have*

$$|\langle d_t^{-n}(f^n)^*\omega_0 - \mu, \psi\rangle| \leq A\|\psi\|_{\mathcal{C}^2(U)} \left(\frac{\lambda}{d_t}\right)^n \int_{U'} |u_\lambda|\omega_0, \tag{2.5}$$

where A is a constant independent of ω_0, ψ, U' and n.

3. Proof of Theorem 1.1

3.1. **Preparatory lemmas.** In this section we prove a couple of technical lemmas that we will need in the proof of Theorem 1.1. Theorem 2.3 and Corollary 2.4 are not used here.

We fix a polynomial-like map $f : U \to V$ and the constant

$$M = M(f) := \max_{1\leq l\leq k} \max_{z\in\overline{f^{-1}(U)}} \|\nabla f_l(z)\|, \tag{3.1}$$

where the f_l's denote the components of f. Recall that we denote by ω the standard Kahler form on $\mathbb{C}^k$. For every $m \in \mathbb{N}$ and $1 \leq i, j \leq k$, we also set

$$\alpha_{i,j}^{(m)} := \sum_{l=1}^{k} \frac{\partial f_l^m}{\partial z_i}\frac{\partial \bar{f}_l^m}{\partial \bar{z}_j},$$

where the f_l^m's denote the components of f^m. Observe that the $\alpha_{i,j}^{(m)}$'s are smooth functions on $f^{-m}(V)$ and for every $N \geq m$ we have

$$\left.(f^m)^* \omega\right|_{f^{-N}(U)} = \sum_{1\leq i,j\leq k} \alpha_{i,j}^{(m)} dz_i \wedge d\bar{z}_j.$$

Lemma 3.1. *For every* $1 \leq m \leq N-1$*, we have*

$$|\alpha_{i,j}^{(m)}|_{f^{-N}(U)} \leq k^{2m-1} M^{2m}.$$

Proof. We claim that for every $z \in \overline{f^{-N}(U)}$, every $1 \leq m \leq N-1$, and every $1 \leq i, j \leq k$ we have

$$\left|\frac{\partial f_j^m(z)}{\partial z_i}\right| \leq k^{m-1} M^m. \tag{3.2}$$

For $m = 1$, (3.2) follows from the definition of M. Assume now that (3.2) holds for $m-1 \geq 0$ instead of m. As $f^{m-N}(U) \Subset f^{-1}(U)$, we have $\left|\frac{\partial f_j}{\partial z_l}\left(f^{m-1}(z)\right)\right| \leq M$ for every $1 \leq j, l \leq k$. So, for every $1 \leq i, j \leq k$, we have

$$\begin{aligned}\left|\frac{\partial f_j^m(z)}{\partial z_i}\right| &= \left|\sum_{l=1}^k \frac{\partial f_j}{\partial z_l}\left(f^{m-1}(z)\right)\frac{\partial f_l^{m-1}(z)}{\partial z_i}\right| \leq \sum_{l=1}^k \left|\frac{\partial f_j}{\partial z_l}\left(f^{m-1}(z)\right)\right|\left|\frac{\partial f_l^{m-1}(z)}{\partial z_i}\right| \\ &\leq \sum_{l=1}^k M\cdot k^{m-2}M^{m-1} = k^{m-1}M^m\end{aligned}$$

on $\overline{f^{-N}(U)}$. Hence, (3.2) holds for all $1 \leq m \leq N-1$. It follows that we have

$$|\alpha_{i,j}^{(m)}(z)| = \left|\sum_{l=1}^k \frac{\partial f_l^m}{\partial z_i}\frac{\partial \bar{f}_l^m}{\partial \bar{z}_j}\right| \leq \sum_{l=1}^k \left|\frac{\partial f_l^m}{\partial z_i}\right|\left|\frac{\partial \bar{f}_l^m}{\partial \bar{z}_j}\right| \leq k^{2m-1}M^{2m},$$

as desired. □

From now on, for simplicity, given $m_1 \leq \ldots \leq m_k \in \mathbb{N}$, we will use the notation

$$\Omega_{m_1,\ldots,m_k} := (f^{m_1})^*\omega \wedge \ldots \wedge (f^{m_k})^*\omega. \tag{3.3}$$

Observe that $\Omega_{m_1,\ldots,m_k}$ is a smooth volume form on $f^{-m_k}(V)$. We will also denote by $\varphi_{m_1,\ldots,m_k}$ the Radon-Nikodym density of $\Omega_{m_1,\ldots,m_k}$ with respect to ω^k. This is a positive smooth function on $f^{-m_k}(V)$. The following bound for $\varphi_{m_1,\dots,m_k}$ immediately follows from Lemma 3.1.

Corollary 3.2. *There exists a constant C such that, for every $0 \le m_1 \le \dots \le m_k \le n \le N-1$ we have*

$$\varphi_{m_1,\dots,m_k} \le C(kM)^{2kn} \quad \text{on} \quad f^{-N}(U).$$

Observe that $\Omega_{m_1,\dots,m_k}$ as above satisfies

$$\int_{f^{-m_k}(V)} \Omega_{m_1,\dots,m_k} \lesssim d_t^{m_k},$$

where the implicit constant is independent of $m_1,\dots,m_k$, see for instance [13, 15]. In the following, we will need the following better bound for the integral above in the case where $m_1 = 1$.

Lemma 3.3. *There exists a function $\eta : \mathbb{N} \to \mathbb{R}$ with $\limsup_{n\to\infty} \eta(n)^{1/n} = 1$ such that*

$$\int_{f^{-N}(V)} \Omega_{0,m'_1,\dots,m'_{k-1}} \le \eta(N)(d_{k-1})^N \tag{3.4}$$

for every $1 \le m'_1 \le \dots \le m'_{k-1} \le N-1$.

Proof. Let X be an analytic subset of V of pure dimension $k-1$. There exists a function $\eta : \mathbb{N} \to \mathbb{R}$ with $\limsup_{n\to\infty} \eta(n)^{1/n} = 1$ and depending only from the mass of $[X]$ such that

$$\int_{f^{-N}(V)} [X] \wedge (f^{m'_1})^*\omega \wedge \dots \wedge (f^{m'_{k-1}})^*\omega \le \eta(N)\,(d_{k-1})^N \tag{3.5}$$

for every $1 \le m'_1 \le \dots \le m'_{k-1} \le N-1$. The proof of (3.5) follows the strategy used by Gromov to estimate the topological entropy of endomorphisms of $\mathbb{P}^k$, see for instance [20], and adapted by Dinh and Sibony [13, 15] to the setting of polynomial-like maps. Since only minor modifications are needed, we refer to [2, Lemma A.2.6] for a complete proof. The inequality (3.4) is deduced from (3.5) (by possibly multiplying the function $\eta(n)$ by a bounded factor) as ω can be written as an average of currents of integration on $(k-1)$-dimensional analytic sets. □

Finally, we will need the following lemma about the integrals of psh functions.

Lemma 3.4. *Let u be a psh function on V. Then, there exists a positive constant A_1 depending on u such that for every $0 \le m_1 \le \dots \le m_k \le N \in \mathbb{N}$ we have*

$$\int_{f^{-N}(U)} |u|\Omega_{m_1,\dots,m_k} \le A_1 + m_k^2 \int_{f^{-N}(U)} \Omega_{m_1,\dots,m_k}.$$

Proof. For every $n \le N \in \mathbb{N}$, set

$$W_{n,N} := f^{-N}(U) \cap \{u < -n^2\}.$$

Then, for all $m_1,\dots,m_k, N$ as in the statement and $m_k \le n \le N$, we have

$$\begin{aligned}\int_{f^{-N}(U)} |u|\Omega_{m_1,\dots,m_k} &\le \int_{f^{-N}(U)\setminus W_{n,N}} |u|\Omega_{m_1,\dots,m_k} + \int_{W_{n,N}} |u|\Omega_{m_1,\dots,m_k} \\ &\le n^2 \int_{f^{-N}(U)} \Omega_{m_1,\dots,m_k} + \int_{W_{n,N}} |u|\varphi_{m_1,\dots,m_k}\omega^k.\end{aligned}$$

Hence, it is enough to show that the last integral in the above expression is bounded uniformly in $m_1, \dots, m_k, n$, and N (we will actually show that it tends to 0 if $n \to \infty$).

As we have $W_{n,N} \subset \{u < -n^2\}$, by the Skoda estimates for psh functions [25] there exists $\alpha > 0$ and $n_0 \in \mathbb{N}$ such that for any $n > n_0$ we have

$$\int_{W_{n,N}} |u|\omega^k \le C_1 e^{-\alpha n^2}, \tag{3.6}$$

for some positive constant C_1 independent from n. By Corollary 3.2, there exist two constants C_2 and M, independent from $m_1, \dots, m_k$, n, and N such that $\varphi_{m_1,\dots,m_k} \le C_2(kM)^{2kn}$ on $f^{-n}(U) \supseteq f^{-N}(U)$. We deduce from this estimate and (3.6) that we have

$$\int_{W_{n,N}} |u|\varphi_{m_1,\dots,m_k}\omega^k \le C_2(kM)^{2kn} \int_{W_{n,N}} |u|\omega^k \le C_1C_2(kM)^{2kn}e^{-\alpha n^2}$$

for every $n > n_0$. Choosing $A_1 := \max_{n\in\mathbb{N}} C_1C_2(kM)^{2kn}e^{-\alpha n^2}$ completes the proof. □

3.2. **Proof of Theorem 1.1.** We can now prove the following statement, which, by Lemma 2.2, gives a more precise version of Theorem 1.1.

Theorem 3.5. *Let $f : U \to V$ be a polynomial-like map with dominant topological degree and β be as in* (2.3)*. Then, every ergodic measure ν whose measure-theoretic entropy satisfies $h_\nu(f) > \log\beta$ is supported on the Julia set J.*

Proof. Fix ν as in the statement and λ with $\beta < \lambda < e^{h_\nu(f)} \le d_t$. Let $F \subset U \setminus J$ be a closed set. We are going to show that we have $h_t(f,F) \le \log\lambda$, where $h_t(f,F)$ denotes the topological entropy of f on F. By the relative variational principle, this implies $\nu(F) = 0$, and hence that the support of ν is contained in J, as desired. Observe that we can assume, with no loss of generality, that we have $F \subset K$.

Let W be an open neighbourhood of F with $\overline{W} \cap J = \emptyset$. Using Gromov's contruction [20] and the same arguments as in the proof of [15, Theorem 1.108] (see also the proof of [2, Lemma A.2.6]), we have

$$h_t(f,F) = h_t(f, F\cap K) \le \operatorname{lov}(f,W) := \limsup_{N\to\infty} \frac{1}{N}\log \operatorname{vol}(\Gamma_N^W),$$

where, for every $N \in \mathbb{N}$, Γ_N^W denotes the subset of U^{N+1} given by

$$\Gamma_N^W := \{(z, f(z), \dots, f^{N-1}(z)), z \in W \cap f^{-N}(U)\}.$$

Note that, for every $N \in \mathbb{N}$, we have

$$\operatorname{vol}(\Gamma_N^W) = \sum_{\substack{0\le n_i\le N-1,\\ 1\le i\le k}} \int_{W\cap f^{-N}(U)} (f^{n_1})^*\omega \wedge \dots \wedge (f^{n_k})^*\omega.$$

As the number of terms in the sum is polynomial in N, it is enough to consider separately each term of the sum on the right-hand side of the above expression. Hence, without loss of generality, we can assume that we have $0 \le n_1 \le \dots \le n_k \le N-1$ and we need to show the inequality

$$\limsup_{N\to\infty} \frac{1}{N}\log\int_{W\cap f^{-N}(U)} (f^{n_1})^*\omega \wedge \dots \wedge (f^{n_k})^*\omega \le \log\lambda.$$

Recall the notation (3.3). We will now consider the smooth form

$$\Omega'_{n_1,\ldots,n_k} := \Omega_{0,n_2-n_1,\ldots n_k-n_1} = \omega \wedge (f^{n_2-n_1})^*\omega \wedge \ldots \wedge (f^{n_k-n_1})^*\omega.$$

Observe that we have

$$\Omega_{n_1,\ldots,n_k} = (f^{n_1})^*(\Omega'_{n_1,\ldots,n_k}).$$

Fix also an open set $\tilde{W}$ with $\tilde{W} \Supset W$ and $\tilde{W} \cap J = \emptyset$ and a smooth function $0 \le \psi \le 1$ with compact support in $\tilde{W}$ and such that $\psi|_W = 1$. Since $\mu|_{\tilde{W}} = 0$, by Corollary 2.4 applied with

$$U' = f^{-(N-n_1)}(U) \quad \text{and} \quad \omega_0 = \|\Omega'_{n_1,\ldots,n_k}\|^{-1}_{f^{-(N-n_1)}(U)} \cdot (\Omega'_{n_1,\ldots,n_k})_{|f^{-(N-n_1)}(U)}$$

there exists a psh function u_λ such that

$$\begin{aligned}\int_{f^{-N}(U)} \psi \Omega_{n_1,\ldots,n_k} &= \Big| \Big\langle (f^{n_1})^*(\Omega'_{n_1,\ldots,n_k}) - d_t^{n_1} \|\Omega'_{n_1,\ldots,n_k}\|_{f^{-(N-n_1)}(U)} \mu, \psi \Big\rangle \Big| \\ &\le A \|\psi\|_{\mathcal{C}^2(U)} \lambda^{n_1} \int_{f^{-(N-n_1)}(U)} |u_\lambda| \Omega'_{n_1,\ldots,n_k},\end{aligned}$$

where $A > 0$ is a constant independent of $n_1, \ldots, n_k, N$, and ψ. We deduce from the above inequality and Lemma 3.4 (applied with $m_j = n_j - n_1$ for all j, so that $\Omega_{m_1,\ldots,m_k} = \Omega'_{n_1,\ldots,n_k}$) that there exists a positive constant A_1 (depending on λ, but independent of $n_1, \ldots, n_k$, N, and ψ) such that

$$\int_{f^{-N}(U)} \psi \Omega_{n_1,\ldots,n_k} \le A \|\psi\|_{\mathcal{C}^2(U)} \lambda^{n_1} \Big(A_1 + N^2 \int_{f^{-N+n_1}(U)} \Omega'_{n_1,\ldots,n_k} \Big).$$

Recalling the definitions of $\Omega_{n_1,\ldots,n_k}$ and ψ, we deduce from the above expression that we have

$$\int_{W \cap f^{-N}(U)} (f^{n_1})^*\omega \wedge \ldots \wedge (f^{n_k})^*\omega \le A \|\psi\|_{\mathcal{C}^2(U)} \lambda^{n_1} \Big(A_1 + N^2 \int_{f^{-N+n_1}(U)} \Omega'_{n_1,\ldots,n_k} \Big). \tag{3.7}$$

By Lemma 3.3 (applied with $m'_j = n_j - n_1$) we have

$$\int_{f^{-N+n_1}(U)} \Omega'_{n_1,\ldots,n_k} = \int_{f^{-N+n_1}(U)} \Omega_{0,n_2-n_1,\ldots,n_k-n_1} \le \eta(N)(d_{k-1})^{N-n_1}, \tag{3.8}$$

where the function η satisfies $\lim_{n\to\infty} \eta(n)^{1/n} = 1$. Combining (3.7) and (3.8), we obtain

$$\begin{aligned}\int_{W \cap f^{-N}(U)} (f^{n_1})^*\omega \wedge \ldots \wedge (f^{n_k})^*\omega &\le A \|\psi\|_{\mathcal{C}^2(U)} \lambda^{n_1} (A_1 + N^2 \eta(N)(d_{k-1})^{N-n_1}) \\ &\le \tilde{\eta}(N) \lambda^{n_1} (d_{k-1})^{N-n_1} \\ &\le \tilde{\eta}(N) \lambda^N,\end{aligned}$$

where the function $\tilde{\eta}$ (which can depend on λ) satisfies $\lim_{n\to\infty} \tilde{\eta}(n)^{1/n} = 1$ and in the last step we used the inequality $d_{k-1} \le \lambda$. Consequently, we have

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathrm{vol}(\Gamma_N^W) \le \limsup_{N\to\infty} \frac{1}{N} \log \big(N^k \tilde{\eta}(N) \lambda^N \big) \le \log \lambda,$$

which gives $h_t(f, F) \le \log \lambda$, as desired. This concludes the proof. □

4. Further results and remarks

4.1. **Hausdorff dimension of the Julia set.** In this section we fix a polynomial-like map $f : U \to V$ with dominant topological degree. Let ν be an ergodic probability measure with $h_\nu(f) > \log \beta$ (where β is as in (2.3)) and denote by

$$0 < L_k(\nu) \leq L_{k-1}(\nu) \leq \ldots \leq L_1(\nu)$$

its Lyapunov exponents, counting multiplicities. The inequality $0 < L_k(\nu)$ is proved in [7, Theorem 4.1] (see [10, 19] for the case of endomorphisms). We also have the following property, originally proved by Dupont for endomorphisms [18, Theorem A]. As the proof is local, it also applies in our setting.

Theorem 4.1. *Let f, ν, and $L_j, 1 \leq j \leq k$ be as above. Then for ν-almost all $z \in V$ we have*

$$\liminf_{r \to 0} \frac{\log \nu(B(z,r))}{\log r} \geq \frac{\log d_{k-1}}{L_1(\nu)} + \frac{h_\nu(f) - \log d_{k-1}}{L_k(\nu)},$$

where $B(z,r)$ is the ball of radius $r > 0$ and centred at z. In particular, for every Borel set $E \subset V$ with $\nu(E) > 0$, the Hausdorff dimension $\dim_H E$ of E satisfies

$$\dim_H E \geq \frac{\log d_{k-1}}{L_1(\nu)} + \frac{h_\nu(f) - \log d_{k-1}}{L_k(\nu)}.$$

As in [18], the following consequence of Theorem 4.1 gives a lower bound for the Hausdorff dimension of the Julia set J of f.

Corollary 4.2. *Let f be as above and β be as in (2.3). Then, for every ergodic measure ν whose measure-theoretic entropy satisfies $h_\nu(f) > \log \beta$, we have*

$$\dim_{\mathcal{H}} J \geq \frac{\log d_{k-1}}{L_1(\nu)} + \frac{h_\nu(f) - \log d_{k-1}}{L_k(\nu)}.$$

Proof. Theorem 3.5 implies that ν is supported on J, and hence $\nu(J) = 1$. Therefore, the assertion follows from Theorem 4.1. ☐

4.2. **Strong stability in families of polynomial-like maps.** We consider in this section a holomorphic family of polynomial-like maps $(f_\tau)_{\tau \in M}$ with dominant topological degree parametrized by a complex manifold M, see for instance [13, Section 3.8] and [15, Section 2.5]. Recall that all the f_τ's have the same topological degree d_t, which we assume to be at least 2, and that the map $\tau \mapsto d_{k-1}(f_\tau)$ is upper semicontinuous. We will denote by μ_τ the equilibrium measure of f_τ. The dynamical stability for such families has been studied in [3], as a generalization of the theory developed for families of endomorphisms of $\mathbb{P}^k$ in [1], see also [23] and [11, 21, 22] for the case $k = 1$. Following [1, 3], let us denote by $\mathcal{J}$ the set of all holomorphic maps $\gamma : M \to \mathbb{C}^k$ such that $\gamma(\tau)$ belongs to the Julia set of f_τ for all $\tau \in M$. Define $\mathcal{F} : \mathcal{J} \to \mathcal{J}$ as $\mathcal{F}\gamma(\tau) := f_\tau(\gamma(\tau))$.

Definition 4.3. A *dynamical lamination* for the family $(f_\tau)_{\tau \in M}$ is an $\mathcal{F}$-invariant subset $\mathcal{L}$ of $\mathcal{J}$ such that

(1) $\Gamma_\gamma \cap \Gamma_{\gamma'} = \emptyset$ for every $\gamma \neq \gamma' \in \mathcal{L}$, where Γ_γ is the graph of γ in $M \times \mathbb{C}^k$;
(2) $\Gamma_\gamma \cap GO(C_f) = \emptyset$ for every $\gamma \in \mathcal{L}$, where C_f is the critical set of the map $f : (\tau, z) \mapsto (\tau, f_\tau(z))$, and $GO(f) := \cup_{n,m \geq 0} f^{-m}(f^n(C_f))$;
(3) $\mathcal{F} : \mathcal{L} \to \mathcal{L}$ is d_t-to-1.

The dynamical stability of the family $(f_\tau)_{\tau\in M}$ is defined and characterized in [1, 3] by a number of equivalent conditions, among which there is the existence of a dynamical lamination $\mathcal{L}$ such that $\mu_\tau(\{\gamma(\tau) : \gamma \in \mathcal{L}\}) = 1$ for all $\tau \in M$. It was proved in [7] that stability implies (and is then equivalent to) the existence of a dynamical lamination associated to any ergodic measure for some f_{τ_0} whose entropy in larger than $\log d_{k-1}(f_{\tau_0})$ and supported on the Julia set. The following is another corollary of our main results, which permits to remove the assumption on the support of the measure in [7, Section 4.3] when the measures satisfy the stronger bound on their measure-theoretic entropy as in Theorem 3.5.

Corollary 4.4. *Let M be a connected and simply connected complex manifold and $(f_\tau)_{\tau\in M}$ a stable family of polynomial-like maps with dominant topological degree. Fix $\tau_0 \in M$ and let $\beta(f_{\tau_0})$ be as in (2.3). Then, there exists a dynamical lamination $\mathcal{L}$ such that $\nu(\{\gamma(\tau_0) : \gamma \in \mathcal{L}\}) = 1$ for every ergodic f_{τ_0}-invariant probability measure ν with $h_\nu(f_{\tau_0}) > \log\beta(f_{\tau_0})$.*

Proof. By Theorem 3.5, every ergodic f_{τ_0}-invariant probability measure ν with $h_\nu(f_{\tau_0}) > \log\beta(f_{\tau_0})$ is supported in the Julia set J_{τ_0} of f_{τ_0}. Thus, the conclusion follows from [7, Corollary 4.5]. □

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