HOLOMORPHIC MOTIONS OF WEIGHTED PERIODIC POINTS

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Abstract. We study the holomorphic motions of repelling periodic points in stable families of endomorphisms of $\mathbb{P}^k(\mathbb{C})$. In particular, we establish an asymptotic equidistribution of the graphs associated to such periodic points with respect to natural measures in the space of all holomorphic motions of points in the Julia sets.

1. Introduction

A *holomorphic family of endomorphims of* $\mathbb{P}^k$ is a pair (M, f), where M is a complex manifold and $f: M \times \mathbb{P}^k \to M \times \mathbb{P}^k$ is a holomorphic map of the form $f(\lambda, z) = (\lambda, f_\lambda(z))$, where each f_λ is an endomorphism of $\mathbb{P}^k$ of the same algebraic degree d. We always assume that M is connected and simply connected, and that $d \geq 2$. The following fundamental result due by Lyubich [26], Mañé-Sad-Sullivan [28], and DeMarco [20] defines and characterizes *stability* within such families when $k = 1$, see also [25, 29, 31] for further characterizations and previous results in the polynomial case. Recall that Freire-Lopes-Mañé [23] and Lyubich [27] proved that each rational map f_λ admits a unique invariant measure of maximal entropy μ_λ, whose support, denoted as J_λ, is the Julia set of f_λ.

Theorem-Definition 1.1. Let (M, f) be a holomorphic family of rational maps as above. The following conditions are equivalent:

1. the Julia sets J_λ move holomorphically with λ;
2. $dd^c L(\lambda) \equiv 0$, where $L(\lambda) := \int \log|f'_\lambda| \mu_\lambda$ is the Lyapunov exponent of the measure of maximal entropy μ_λ of f_λ;
3. the repelling periodic points of f_λ move holomorphically with λ.

We say that the family is *stable* if any (hence, all) of the above conditions hold.

Recall that a family of Borel sets $E_\lambda \subset \mathbb{P}^1$ *move holomorphically with* λ if there exists a set $\mathcal{L}$ of holomorphic functions $\gamma : M \to \mathbb{P}^1$ such that the graphs $\Gamma_{\gamma_1}, \Gamma_{\gamma_2} \subset M \times \mathbb{P}^1$ of two distinct functions $\gamma_1, \gamma_2 \in \mathcal{L}$ do not intersect, and for any parameter $\lambda \in M$, we have $\mathcal{L}_\lambda := \{\gamma(\lambda), \gamma \in \mathcal{L}\} = E_\lambda$. A crucial point here is that, if the sets E_λ move holomorphically with λ, the same is true for $\overline{E_\lambda}$. This fact, usually referred to as the λ–lemma, is a consequence of the Hurwitz theorem for (one-dimensional) holomorphic maps.

A generalization of the above result for families in any dimension $k \geq 1$ was proved by Berteloot, Dupont, and the first author in [3]. As the Hurwitz theorem fails in higher dimensions, the approach replaces the holomorphic motion of the Julia sets by a *measurable holomorphic motion*, namely a family $\mathcal{L}$ of non-intersecting graphs $\gamma: M \to \mathbb{P}^k$ such that for any parameter $\lambda \in M$, we have $\mu_\lambda(\mathcal{L}_\lambda) = 1$, where μ_λ is the unique measure of maximal entropy of the system $(\mathbb{P}^k, f_\lambda)$ (see [15, 18, 22]). We still denote its support as J_λ. The result is as follows, see also [3, 4, 8] for details and further characterizations and [2] for an explanation of the strategy of the proof.

Theorem-Definition 1.2. Let (M, f) be a holomorphic family of endomorphisms of $\mathbb{P}^k$. The following conditions are equivalent:

1. there exists a measurable holomorphic motion for the Julia sets J_λ;
2. $dd^c L(\lambda) \equiv 0$, where $L(\lambda) := \int \log|Df_\lambda| \mu_\lambda$ is the sum of the Lyapunov exponents of the measure of maximal entropy μ_λ of f_λ.

We say that the family is *stable* if any (hence, all) of the above conditions hold.

Moreover, the following condition

(3) the repelling periodic points of f_λ move holomorphically with λ

implies the above two, and is equivalent to them if $k = 2$, or if M is an open connected and simply connected subset of the space $\mathcal{H}_d(\mathbb{P}^k)$ of all holomorphic endomorphisms of $\mathbb{P}^k$ of algebraic degree d.

It is still an open question whether the stability of a general family of endomorphisms in any dimension is equivalent to the motion of all the repelling periodic cycles. On the other hand, it was proved in [7, 8] that the stability of a family implies at least a weaker version of the motion of the repelling cycles. Notice that, in turn, this weaker notion is still sufficient to imply stability, see also [1]. The definition is as follows.

Definition 1.3. We say that *asymptotically all* repelling cycles move holomorphically on M if there exists a countable set $\mathcal{P} = \cup_{n\in\mathbb{N}^*}\mathcal{P}_n$ of holomorphic functions $\gamma : M \to \mathbb{P}^k$ satisfying the following properties:

(1) $\operatorname{Card} \mathcal{P}_n = d^{kn} + o(d^{kn})$;
(2) the point $\gamma(\lambda)$ belongs to J_λ and is a fixed point of f^n_λ, for every $\lambda \in M$ and $\gamma \in \mathcal{P}_n$;
(3) for all open subsets $M' \Subset M$, we have

$$\frac{\operatorname{Card}\{\gamma \in \mathcal{P}_n : \gamma(\lambda) \text{ is repelling for all } \lambda \in M'\}}{d^{kn}} \underset{n\to\infty}{\longrightarrow} 1.$$

The cycles in Definition 1.3 can be seen, in some sense, as generic with respect to the measure of maximal entropy. More precisely, consider the space $\mathcal{J}$, defined as

$$\mathcal{J} := \{\gamma : M \to \mathbb{P}^k : \gamma(\lambda) \in J_\lambda \quad \forall\lambda \in M\}, \tag{1.1}$$

where the maps γ are holomorphic. $\mathcal{J}$ is a metric space, see Section 2 for details. We can turn $\mathcal{J}$ into a topological dynamical system by defining a natural map $\mathcal{F}$ as

$$\mathcal{F}(\gamma)(\lambda) := f_\lambda(\gamma(\lambda)). \tag{1.2}$$

A *web* is a probability measure $\mathcal{M}$ compactly supported on $\mathcal{J}$ which is invariant under $\mathcal{F}$. An *equilibrium web* [3] is a web such that

$$(p_\lambda)_*\mathcal{M} = \mu_\lambda \quad \forall\lambda \in M,$$

where we denote by $p_\lambda\colon \mathcal{J} \to \mathbb{P}^k$ the natural map $\gamma \mapsto \gamma(\lambda)$. The properties of equilibrium webs are crucial in the approach in [3].

With this terminology, and owing to the equidistribution of the repelling periodic cycles with respect to the measure of maximal entropy [14], we see in particular that the fact that the existence of $\mathcal{P}$ as in Definition 1.3 is equivalent to stability as in Theorem-Definition 1.2 leads to the following equidistribution result for periodic graphs in any stable family. This can be seen as a weak version of the implication (1)⇒(3).

Theorem 1.4. *Let (M, f) be a stable family of endomorphisms of $\mathbb{P}^k$. Then, for every $M' \Subset M$ and every $n \in \mathbb{N}^*$, there exists a non-empty subset $\mathcal{P}_n \subset \mathcal{J}$ of motions γ of n-periodic points such that $\gamma(\lambda)$ is repelling for all $\lambda \in M'$ and*

$$\lim_{n\to\infty} d^{-kn} \sum_{\gamma\in\mathcal{P}_n} \delta_\gamma = \mathcal{M},$$

where $\mathcal{M}$ is an equilibrium web.

In this paper, we address the question of the motion of cycles which similarly equidistribute invariant measures in a much larger class, that we now introduce.

Let f be an endomorphism of $\mathbb{P}^k$ of algebraic degree $d \geq 2$ and let ϕ be a real continuous function on $\mathbb{P}^k$ (usually called a *weight*). The *pressure* of ϕ is defined as

$$P(\phi) := \sup_{\nu} (h_\nu + \int \phi\nu)$$

where the supremum is over all f-invariant measures ν and h_ν denotes the measure-theoretic entropy of ν. An *equilibrium state* for the weight ϕ is defined as a maximizer of the pressure function, i.e., as an invariant measure μ_ϕ satisfying

$$h_{\mu_\phi} + \int \phi\mu = P(\phi).$$

Assume that f satisfies the following condition:

(A) the local degree of the iterate f^n satisfies

$$\lim_{n\to\infty} \frac{1}{n} \log \max_{a\in\mathbb{P}^k} \deg(f^n, a) = 0.$$

Here, $\deg(f^n, a)$ is the multiplicity of a as a solution of the equation $f^n(z) = f^n(a)$. Note that generic endomorphisms of $\mathbb{P}^k$ satisfy this condition, see [19]. Let ϕ satisfy

(B) $\|\phi\|_{\log^q} < \infty$ for some $q > 2$ and $\Omega(\phi) < \log d$,

where we define $\Omega(\phi) := \max(\phi) - \min(\phi)$,

$$\|\phi\|_{\log^q} := \sup_{a,b\in\mathbb{P}^k} |\phi(a) - \phi(b)| \cdot (\log^\star \operatorname{dist}(a, b))^q,$$

and $\log^\star(\cdot) = 1 + |\log(\cdot)|$. The existence and uniqueness of the equilibrium state μ_ϕ for f under the assumptions **(A)** and **(B)** have been proved in [33, 9] see also [21, 32, 10] for further properties of these measures, and [30] and references therein for previous results in dimension 1. The case $\phi = 0$ corresponds to the case of the measure of maximal entropy, see for instance [18] and references therein for an account of this case.

By the definition of pressure and the assumption on $\Omega(\phi)$, all these equilibrium states satisfy $h_{\mu_\phi} > \log d^{k-1}$. It is proved in [11] that, given a stable family $(f_\lambda)_{\lambda\in M}$ and any $\lambda_0 \in M$, for any f_{λ_0}-invariant measure ν satisfying $h_\nu > \log d^{k-1}$, it is possible to construct an associated web $\mathcal{M}_{\lambda_0,\nu}$ with the property that

$$(p_{\lambda_0})_* \mathcal{M}_{\lambda_0,\nu} = \nu,$$

as well as the associated lamination. This in particular applies to the equilibrium states as above. The following is our main result.

Theorem 1.5. *Let (M, f) be a stable family of endomorphisms of $\mathbb{P}^k$. Take $\lambda_0 \in M$ and assume that f_{λ_0} satisfies condition* **(A)***. Let $\phi : \mathbb{P}^k \to \mathbb{R}$ satisfy* **(B)** *and let μ_ϕ be the equilibrium state for f_{λ_0} associated to ϕ. Then, for every $M' \Subset M$ and every $n \in \mathbb{N}$, there exists a non-empty subset $\mathcal{P}_{\phi,n} \subset \mathcal{J}$ of motions γ of n-periodic points such that $\gamma(\lambda)$ is repelling for all $\lambda \in M'$ and*

$$\lim_{n\to\infty} e^{-nP(\phi)} \sum_{\gamma\in\mathcal{P}_{\phi,n}} e^{\phi(\gamma(\lambda_0)) + \cdots + \phi(f_{\lambda_0}^{n-1}(\gamma(\lambda_0)))} \delta_\gamma = \mathcal{M}_{\lambda_0,\mu_\phi}.$$

Observe that, in particular, Theorem 1.5 generalizes to general weights ϕ Theorem 1.4 , the latter corresponding to the case $\phi = 0$.

At the parameter λ_0, the equidistribution of repelling periodic points with respect to the equilibrium state μ_ϕ has been established in [10, Theorem 4.10]. The proof follows the now classical strategy by Briend-Duval [14], who showed this result for the measure of maximal entropy, which corresponds to the case $\phi = 0$. On the other hand, when $\phi \neq 0$, as the Jacobian of μ_ϕ is not constant, the proof requires more precise estimates on the contraction along generic inverse branches for μ_ϕ, which in turn follow from delicate distortion estimates along inverse branches due to Berteloot-Dupont-Molino [5, 6]. In the current paper, we adapt this strategy in the setting of the dynamical system $(\mathcal{J}, \mathcal{F}, \mathcal{M}_{\lambda_0,\phi})$. This requires to precisely control the contraction of f on tubes (i.e., tubular neighbourhoods, of uniform radius in λ, of the graphs in $M' \times \mathbb{P}^k$ of elements of $\mathcal{J}$) centered at $\mathcal{M}_{\lambda_0,\phi}$-generic elements of $\mathcal{J}$. As a result, we get the motions of repelling periodic points as repelling periodic elements for $\mathcal{F}$.

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2. Nice inverse branches for expanding webs

2.1. **Transfer operators on** $(\mathcal{J}, \mathcal{F})$**.** We consider in this section a holomorphic family (M, f) of endomorphisms of $\mathbb{P}^k$. We let $\mathcal{J}$ be as in (1.1) (observe that stability is not required to define this set). We can turn $\mathcal{J}$ into a Polish (i.e., separable complete metric) space $(\mathcal{J}, \mathrm{dist}_{\mathcal{J}})$ as a closed subset of the space $\mathcal{O}(M, \mathbb{P}^k)$ endowed with the metric of local uniform convergence. More precisely, the distance between two elements $\gamma_1, \gamma_2 \in \mathcal{J}$ is given by

$$\mathrm{dist}_{\mathcal{J}}(\gamma_1, \gamma_2) := \sum_{n=0}^{+\infty} 2^{-n} \max \left(1, \sup_{\lambda \in K_n} \mathrm{dist}_{\mathbb{P}^k}(\gamma_1(\lambda), \gamma_2(\lambda))\right),$$

where the family $(K_n)_{n\in\mathbb{N}}$ is an exhaustion of M, namely a nested sequence of compact sets whose union is M and such that K_n is a subset of the interior of K_{n+1} for every $n \in \mathbb{N}$. The resulting topology is independent of the choice of the exhaustion.

Let $\mathcal{F}: \mathcal{J} \to \mathcal{J}$ be as in (1.2). For $\gamma \in \mathcal{J}$, we denote by Γ_γ the graph of γ in $M \times \mathbb{P}^k$. We denote by $\mathcal{J}_s$ the subset of $\mathcal{J}$ given by

$$\mathcal{J}_s := \{\gamma \in \mathcal{J} : \Gamma_\gamma \cap GO(C_f) \neq \emptyset\},$$

and set $\mathcal{X} := \mathcal{J} \setminus \mathcal{J}_s$. Observe that every element $\gamma \in \mathcal{X}$ admits d^k well defined inverse elements by $\mathcal{F}$ in $\mathcal{X}$, i.e., elements $\gamma' \in \mathcal{X}$ such that $\mathcal{F}(\gamma') = \gamma$.

We let $\psi: \mathcal{J} \to \mathbb{R}$ be a continuous function and define the (transfer) operator Λ_ψ acting on measurable real functions on $\mathcal{J}$ as

$$\Lambda_\psi(g)(\gamma) = \sum_{\mathcal{F}(\gamma')=\gamma} e^{\psi(\gamma')} g(\gamma'). \tag{2.1}$$

Observe that the operator Λ_ψ preserves positivity. However, even when g is continuous, $\Lambda_\psi(g)$ needs not be continuous (as, for example, the system $\mathcal{F}: \mathcal{J} \to \mathcal{J}$ may not have a well defined degree). On the other hand, as every $\gamma \in \mathcal{X}$ has precisely d^k preimages under $\mathcal{F}$, which are also in $\mathcal{X}$, the operator Λ_ψ defines a continuous operator from $\mathcal{C}^0(\mathcal{X})$ to $\mathcal{C}^0(\mathcal{X})$.

Given a positive measure $\mathcal{N}$ satisfying $\mathcal{N}(\mathcal{J}_s) = 0$, the measure $\mathcal{N}$ integrates any continuous functions on $\mathcal{X}$, and we can define $\Lambda_\psi^* \mathcal{N}$ by the relation

$$\langle \Lambda_\psi^* \mathcal{N}, g \rangle = \langle \mathcal{N}, \Lambda_\psi(g) \rangle,$$

where g is any continuous function on $\mathcal{J}$. We will use the operator Λ_ψ^* only on measures vanishing on $\mathcal{J}_s$.

Lemma 2.1. *The following assertions are equivalent:*

(1) *there exists a continuous and strictly positive function $\theta: \mathcal{X} \to \mathbb{R}$ such that $\Lambda_\psi^n(g) \to c_g \theta$ for every continuous function $g: \mathcal{J} \to \mathbb{R}$, where the constant c_g depends linearly and continuously on g;*

(2) *there exists a positive measure $\mathcal{N}$ on $\mathcal{J}$, satisfying $\mathcal{N}(\mathcal{J}_s) = 0$, such that $(\Lambda_\psi^*)^n \delta_\gamma \to c_\gamma \mathcal{N}$ for every $\gamma \in \mathcal{X}$, where c_γ is a strictly positive constant depending continuously on $\gamma \in \mathcal{X}$.*

Proof. (1)⇒(2) Define a measure $\mathcal{N}$ on $\mathcal{X}$ by setting $\langle \mathcal{N}, g\rangle := c_g$ for every continuous function $g: \mathcal{X} \to \mathbb{R}$, where c_g is as in (1). We can extend $\mathcal{N}$ to a measure on $\mathcal{J}$ by setting $\mathcal{N}(\mathcal{J} \setminus \mathcal{X}) = 0$. Such measure is positive since $c_g \geq 0$ for every non-negative g.

For every $\gamma \in \mathcal{X}$ and continuous function $g: \mathcal{X} \to \mathbb{R}$, we have

$$\langle (\Lambda_\psi^*)^n \delta_\gamma, g\rangle = \langle \delta_\gamma, \Lambda_\psi^n(g)\rangle \to \langle \delta_\gamma, c_g\theta\rangle = \theta(\gamma) c_g.$$

By the definition of $\mathcal{N}$, and setting $c_\gamma := \theta(\gamma)$, this shows that $(\Lambda_\psi^*)^n \delta_\gamma \to c_\gamma \mathcal{N}$.

(2)⇒(1) Define a function $\theta: \mathcal{X} \to \mathbb{R}$ by $\theta(\gamma) := c_\gamma$, for every $\gamma \in \mathcal{X}$, where c_γ is as in (2). For every $\gamma \in \mathcal{X}$ and continuous function $g: \mathcal{J} \to \mathbb{R}$ we have

$$\Lambda_\psi^n(g)(\gamma) = \langle \delta_\gamma, \Lambda_\psi^n(g)\rangle = \langle (\Lambda_\psi^*)^n \delta_\gamma, g\rangle \to \langle c_\gamma \mathcal{N}, g\rangle = c_\gamma \langle \mathcal{N}, g\rangle.$$

By the definition of θ, and setting $c_g := \langle \mathcal{N}, g\rangle$, this shows that $\Lambda_\psi^n(g) \to c_g\theta$. □

Remark 2.2. The two assertions in Lemma 2.1 stay equivalent if θ is assumed to just be non-negative in (1) and c_γ is assumed to just be non-negative in (2).

For every graph $\gamma \in \mathcal{X}$ and every integer $n \in \mathbb{N}$, define the measure

$$\mathcal{M}_{\gamma,n} := \theta \cdot \theta(\gamma)^{-1} \cdot (\Lambda_\psi^*)^n \delta_\gamma = \theta(\gamma)^{-1} \sum_{\mathcal{F}^n(\gamma')} e^{\psi(\gamma') + \ldots + \psi(\mathcal{F}^{n-1}(\gamma'))} \theta(\gamma') \delta_{\gamma'}. \tag{2.2}$$

Lemma 2.3. *If the conditions in Lemma 2.1 are satisfied and θ and $\mathcal{N}$ are as in that lemma, the measure $\mathcal{M} := \theta\mathcal{N}$ is well defined and the following properties hold:*

(1) $\Lambda_\psi \theta = \theta$;
(2) $\Lambda_\psi^* \mathcal{N} = \mathcal{N}$;
(3) *$\mathcal{M}$ is a probability measure and $\mathcal{M}(\mathcal{J}_s) = 0$;*
(4) *$\mathcal{M}$ is $\mathcal{F}$-invariant;*
(5) *$\mathcal{M}_{\gamma,n} \to \mathcal{M}$ for every $\gamma \in \mathcal{X}$;*
(6) $\mathcal{F}^{-1}(\operatorname{Supp} \mathcal{M}) \subseteq \operatorname{Supp} \mathcal{M}$.

Proof. (1) This is a consequence of the first condition in Lemma 2.1. Indeed, as Λ_ψ is continuous on $\mathcal{C}^0(\mathcal{X})$, we have

$$c_\theta \theta = \lim_{n\to\infty} \Lambda_\psi^{n+1}\theta = \Lambda_\psi(\lim_{n\to\infty} \Lambda_\psi^n \theta) = \Lambda_\psi(c_\theta\theta) = c_\theta \Lambda_\psi \theta.$$

As $c_\theta > 0$, we deduce that $\theta = \Lambda_\psi(\theta)$, as desired. Observe in particular that this implies that $c_\theta = 1$, since $\Lambda_\psi^n \theta = \theta$ for every $n \in \mathbb{N}$ and $\Lambda_\psi^n \theta \to c_\theta \theta$.

(2) Observe that $\Lambda_\psi^* \mathcal{N}$ is well defined since $\mathcal{N}(\mathcal{J}_s) = 0$, and still satisfies $(\Lambda_\psi^* \mathcal{N})(\mathcal{J}_s) = 0$. It is enough to check that $\langle \Lambda_\psi^* \mathcal{N}, g\rangle = \langle \mathcal{N}, g\rangle$ for every $g \in \mathcal{C}^0(\mathcal{J})$. With the notation of Lemma 2.1, we also have $\langle \mathcal{N}, g\rangle = c_g$, where c_g is characterized by the convergence $\Lambda_\psi^n g \to c_g \rho$. Observe in particular that $c_g = c_{\Lambda_\psi g}$. It follows that, for every g as above, we have

$$\langle \Lambda_\psi^* \mathcal{N}, g\rangle = \langle \mathcal{N}, \Lambda_\psi g\rangle = c_{\Lambda_\psi g} = c_g = \langle \mathcal{N}, g\rangle.$$

(3) As θ is positive on $\mathcal{X}$, $\mathcal{N}(\mathcal{J}_s) = 0$, and $\langle \mathcal{N}, \theta\rangle = c_\theta = 1$, we have $\theta \in L^1(\mathcal{N})$. As $\mathcal{N}(\mathcal{J}_s) = 0$, we have $\mathcal{M}(\mathcal{J}_s) = 0$. As θ is non-negative and $\mathcal{N}$ is a positive measure, $\mathcal{M}$ is a positive measure. It is a probability measure since

$$\mathcal{M}(\mathcal{X}) = \langle \theta\mathcal{N}, \mathbb{1}_{\mathcal{X}}\rangle = \langle \mathcal{N}, \theta\rangle = c_\theta = 1.$$

(4) We need to check that $\langle \mathcal{M}, g\rangle = \langle \mathcal{M}, g \circ \mathcal{F}\rangle$ for every continuous function g on $\mathcal{J}$. As $\mathcal{M}(\mathcal{J}_s) = 0$, the pairing can be computed on $\mathcal{X}$. A direct computation gives that

$$\langle \mathcal{M}, g \circ \mathcal{F}\rangle = \langle \theta\mathcal{N}, g \circ \mathcal{F}\rangle = \langle \mathcal{N}, \theta \cdot g \circ \mathcal{F}\rangle = \langle \mathcal{N}, \Lambda_\psi(\theta \cdot g \circ \mathcal{F})\rangle,$$

where in the last step we used the equality $\Lambda_\psi^*\mathcal{N}=\mathcal{N}$. It follows from the definition of Λ_ψ and the Λ_ψ-invariance of θ that

$$\Lambda_\psi(\theta\cdot g\circ F)=g\cdot\Lambda_\psi(\theta)=g\theta.$$

Together with the previous equalities, this gives

$$\langle\mathcal{M},g\circ\mathcal{F}\rangle=\langle\mathcal{N},\theta g\rangle=\langle\mathcal{M},g\rangle,$$

as desired.

(5) Take $g\in\mathcal{C}^0(\mathcal{J})$. By the second assertion of Lemma 2.1, and recalling that $c_\gamma=\theta(\gamma)$ for every $\gamma\in\mathcal{X}$, we have

$$\langle\mathcal{M}_{\gamma,n},g\rangle=\theta(\gamma)^{-1}\langle\theta\cdot(\Lambda_\psi^*)^n\delta_\gamma,g\rangle\to\theta(\gamma)^{-1}\langle c_\gamma\mathcal{N},\theta g\rangle=\langle\mathcal{M},g\rangle.$$

The assertion follows.

(6) Since θ is positive, we have $\operatorname{Supp}\mathcal{M}=\operatorname{Supp}\mathcal{N}$. Take $\gamma\in\operatorname{Supp}\mathcal{N}$ and $\gamma'\in\mathcal{J}$ such that $\mathcal{F}(\gamma')=\gamma$. Fix a small open set U containing γ and let U' be the connected component of $\mathcal{F}^{-1}(U)$ containing γ' (which is open since $\mathcal{F}$ is continuous). Since $\gamma\in\operatorname{Supp}\mathcal{N}$, we have $\mathcal{N}(U)>0$. By the definition of Λ_ψ, we have that $\Lambda_\psi^*\mathcal{N}(U')>0$. This gives that $\gamma'\in\operatorname{Supp}\Lambda_\psi^*\mathcal{N}$. By (2), it follows that $\gamma'\in\operatorname{Supp}\mathcal{N}$, as desired.

□

2.2. **Estimates of contraction along inverse branches.** We assume in the following that the family (M,f) admits a web $\mathcal{M}$, i.e., an $\mathcal{F}$-invariant compactly supported probability measure on $\mathcal{J}$, satisfying the following properties:

- **(M1)** $\mathcal{M}$ is *acritical*, i.e., we have $\mathcal{M}(\mathcal{J}_s)=0$;
- **(M2)** there exists a constant $A_1>0$ such that, for every $\lambda\in M$, the probability measure $(p_\lambda)_*(\mathcal{M})$ is ergodic and the Lyapunov exponents of $(p_\lambda)_*(\mathcal{M})$ are strictly larger than A_1;
- **(M3)** there exists a continuous function $\psi\colon\mathcal{X}\to\mathbb{R}$, a strictly positive continuous function $\theta\colon\mathcal{X}\to\mathbb{R}$, and a positive measure $\mathcal{N}$ on $\mathcal{J}$ with $\mathcal{N}(\mathcal{J}_s)=0$ such that
  1. $\mathcal{M}=\theta\mathcal{N}$,
  2. $\Lambda_\psi\theta=\theta$,
  3. $\Lambda_\psi^*\mathcal{N}=\mathcal{N}$;
  4. for $\mathcal{M}$-almost every $\gamma\in\mathcal{J}$, we have $\mathcal{M}_{\gamma,n}\to\mathcal{M}$, where $\mathcal{M}_{\gamma,n}$ is defined as in (2.2).

Observe that condition **(M1)** is sufficient to imply the stability of the family (M,f) in the sense of Theorem-Definition 1.2, see [3, Theorem 4.1]. Conversely, a stable family always admits at least a web $\mathcal{M}$ satisfying the above assumptions. To this purpose, it is enough to consider an acritical equilibrium web $\mathcal{M}_0$, as constructed in [3]. This web corresponds to the case of $(p_\lambda)_*\mathcal{M}=\mu_\lambda$ (the measure of maximal entropy) for all $\lambda\in M$, and $\psi\equiv-k\log d$. An asymptotic contraction property along generic inverse branches for $\mathcal{M}_0$ is proved in [3, Proposition 4.2 and 4.3]. This property is the key to getting the measurable holomorphic motion in Theorem-Definition 1.2. It was generalized in [11] for the larger class of webs satisfying **(M1)** and **(M2)**, see Proposition 2.10 below. We aim here at establishing a more quantitative version of such results, under the extra condition **(M3)**. In particular, all this will also apply to the equilibrium web $\mathcal{M}_0$ as above.

Given $\Omega\subset M$, $\gamma\in\mathcal{X}$ and $\eta>0$, we denote by $T_\Omega(\gamma,\eta)$ the η-neighbourhood of the graph Γ_γ of γ in $\Omega\times\mathbb{P}^k$, i.e.,

$$T_\Omega(\gamma,\eta):=\{(\lambda,z)\in\Omega\times\mathbb{P}^k\colon \operatorname{dist}_{\mathbb{P}^k}(z,\gamma(\lambda))<\eta\}.$$

We call such neighbourhood a *tube* at γ over Ω. Observe that a tube $T_\Omega(\gamma,\eta)$ corresponds to the ball $\mathcal{B}_\Omega(\gamma,\eta)$ in the metric space $(\mathcal{J},\operatorname{dist}_\Omega)$, where the distance $\operatorname{dist}_\Omega$ is given by

$$\operatorname{dist}_\Omega(\gamma_1,\gamma_2):=\sup_{\lambda\in\Omega}\operatorname{dist}_{\mathbb{P}^k}(\gamma_1(\lambda),\gamma_2(\lambda)).\tag{2.3}$$

Given a tube $T = T_\Omega(\gamma, \eta)$, the *slice* $T_{|\lambda}$ is the ball $B(\gamma(\lambda), \eta) = T \cap (\{\lambda\} \times \mathbb{P}^k)$. More generally, given the image of a tube T by a holomorphic map $g: T \to \Omega \times \mathbb{P}^k$ fibered over M, we define the slice $g(T)_{|\lambda}$ of $g(T)$ at λ as $g(T) \cap \{\lambda\} \times \mathbb{P}^k$.

We fix in what follows a constant $0 < A_0 < A_1$, where A_1 is given in **(M2)**.

Definition 2.4. Given $\Omega \subset M$, $\gamma \in \mathcal{X}$, a tube T at γ over Ω, and $n \in \mathbb{N}$, we say that a map $g: T \to g(T)$ is a *m-good inverse branch of f of order n on T* if

1. $g \circ f^n = id_{g(T)}$;
2. for all $\lambda \in \Omega$, diam $f_\lambda^l(g(T)_{|\lambda}) \le e^{-m-(n-l)A_0}$ for all $0 \le l \le n$.

Observe that, by definition, we have diam $T_{|\lambda} \le e^{-m}$ for every tube T admitting a m-good inverse branch as above.

Given an inverse branch g of f^m defined on a tube T, given any $\gamma \in \mathcal{J}$ with $\Gamma_\gamma \subset T$ we can in particular associate to such inverse branch a map γ_g such that $\Gamma_{\gamma_g} \subset g(T)$ and $\mathcal{F}(\gamma_g) = \gamma$. In particular, the association $\gamma \mapsto \gamma_g$ defines a map $\mathcal{G}$ on the ball $\mathcal{B}_\Omega(\gamma, \eta)$, that we can see as an inverse branch for $\mathcal{F}$ over such ball.

Given $\Omega \subset M$ and a tube T at $\gamma \in \mathcal{X}$ over Ω, we denote by $\mathcal{M}_{T,n}^{(m)}$ the measure

$$\mathcal{M}_{T,n}^{(m)} := \theta(\gamma)^{-1} \sum_{\gamma_g} e^{\psi(\gamma_g)+\ldots+\psi(\mathcal{F}^{n-1}(\gamma_g))} \theta(\gamma_g) \delta_{\gamma_g}, \tag{2.4}$$

where the sum is over the preimages γ_g of γ associated to m-good inverse branches g of f of order n on T.

Remark 2.5. We have $\mathcal{M}_{T,n}^{(m)} \le \mathcal{M}_{\gamma,n}$ for all $n \ge 0$. Hence, the condition **(M3)** ensures that any limit value $\mathcal{M}'_T$ of the sequence $\{\mathcal{M}_{T,n}^{(m)}\}_n$ satisfies $\|\mathcal{M}'_T\| \le 1$.

Definition 2.6. Given $\Omega \subset M$ and $m > 1$, we say that a tube T at $\gamma \in \mathcal{X}$ over Ω is *m-nice* if $\|\mathcal{M}_{T,n}^{(m)}\| \ge 1 - 1/m$ for all n sufficiently large. We say that a ball $\mathcal{B}_\Omega(\gamma, \eta)$ is m-nice if the tube $T_\Omega(\gamma, \eta)$ is m-nice.

Remark 2.7. Every m-nice tube $T(\gamma, \eta)$ satisfies $\eta \le e^{-m}$, and $\|\mathcal{M}'_T\| \ge 1 - 1/m$ for every limit value $\mathcal{M}'_T$ of the sequence $\mathcal{M}_{T,n}^{(m)}$.

The following is the main result of this section, giving a quantitative control on the contraction along $\mathcal{M}$-generic inverse branches of $\mathcal{F}$.

Proposition 2.8. *Let $\mathcal{M}$ be a compactly supported and ergodic web satisfing* **(M1)**, **(M2)**, *and* **(M3)**. *For every $\Omega \subset M$ and $\gamma \in \mathcal{X}$, the tube $T_\Omega(\gamma, \eta)$ is m-nice if η is sufficiently small.*

In order to prove Proposition 2.8, we will make use of the natural extension $(\hat{\mathcal{J}}, \hat{\mathcal{F}}, \hat{\mathcal{M}})$ of the system $(\mathcal{J}, \mathcal{F}, \mathcal{M})$. We refer to [17, Section 10.4] and [30, Theorem 2.7.1] for the general setting of Lebesgue spaces. The main difficulty lies in proving that $\hat{\mathcal{M}}$ is σ-additive. We recall here how to construct the extension of the Polish space $(\mathcal{J}, \mathrm{dist}_\mathcal{J})$, introducing some notations that will be used later.

We denote by $\hat{\mathcal{J}}$ the subspace

$$\hat{\mathcal{J}} := \{\hat{\gamma} = (\gamma_n)_{n\in\mathbb{Z}} \in \mathcal{J}^\mathbb{Z} : \mathcal{F}(\gamma_n) = \gamma_{n+1}, n \in \mathbb{Z}\},$$

by $\pi_n: \hat{\mathcal{J}} \to \mathcal{J}$ the projection defined as $\pi_n(\hat{\gamma}) = \gamma_n$ and by $\hat{\mathcal{F}}: \hat{\mathcal{J}} \to \hat{\mathcal{J}}$ the shift map

$$\hat{\mathcal{F}}(\hat{\gamma}) = (\mathcal{F}(\gamma_n))_{n\in\mathbb{Z}} = (\gamma_{n+1})_{n\in\mathbb{Z}}.$$

The maps satisfy $\pi_n \circ \hat{\mathcal{F}} = \mathcal{F} \circ \pi_n$ for all $n \in \mathbb{Z}$. We endow $\mathcal{J}$ with its Borel σ-algebra. If $\mathcal{B} \subset \mathcal{J}$ is a Borel set, define $\mathcal{A}_{n,\mathcal{B}} := \pi_n^{-1}(\mathcal{B}) = \{\hat{\gamma} \in \hat{\mathcal{J}} : \gamma_n \in \mathcal{B}\} \subset \hat{\mathcal{J}}$. We may consider the set $\mathcal{A}_{n,\mathcal{B}}$ as a subset of $\hat{\mathcal{J}}^{\{-n,\ldots,n\}} := \{(\gamma_k)_{-n\le k\le n} \in \mathcal{J}^{\{-n,\ldots,n\}} : \mathcal{F}(\gamma_k) = \gamma_{k+1}, -n \le k < n\}$ as well.

A *cylinder* is a finite intersection of subsets of $\hat{\mathcal{J}}$ of the form $\mathcal{A}_{n,\mathcal{B}}$, for some $n \in \mathbb{Z}$ and open set $\mathcal{B} \subseteq \mathcal{J}$. The family of all the cylinders forms a basis for the product topology. It is also a π-system (see [24, Chapter 1]) whose generated σ-algebra is the Borel σ-algebra of $\hat{\mathcal{J}}$. By the monotone classes theorem (see for instance [24, Theorem 1.1]), any measure on $\hat{\mathcal{J}}^{\{-n,\dots,n\}}$ is thus defined by its values on the cylinders. Given $\mathcal{B}_{-n},\dots,\mathcal{B}_n \subset \mathcal{J}$ open sets, define

$$\hat{\mathcal{M}}_n(\mathcal{A}_{-n,\mathcal{B}_{-n}} \cap \ldots \cap \mathcal{A}_{n,\mathcal{B}_n}) := \mathcal{M}(\mathcal{B}_{-n} \cap \mathcal{F}^{-1}(\mathcal{B}_{-(n-1)}) \cap \ldots \cap \mathcal{F}^{-2n}(\mathcal{B}_n)). \tag{2.5}$$

The probability measure $\hat{\mathcal{M}}_n$ is well defined on $\hat{\mathcal{J}}^{\{-n,\dots,n\}}$. The invariance of $\mathcal{M}$ and the fact that $\gamma_k \in \mathcal{B}$ if and only if $\gamma_{k-1} \in \mathcal{F}^{-1}(\mathcal{B})$ yield the following consistency condition

$$\hat{\mathcal{M}}_n(\mathcal{A}_{k,\mathcal{B}}) = \hat{\mathcal{M}}_{n+1}(\mathcal{A}_{k,\mathcal{B}}),$$

for every Borel set $\mathcal{B} \subset \mathcal{J}$ and integers $n \in \mathbb{N}$, $k \in \{-n,\dots,n\}$. By Kolmogorov extension theorem, the measure $\hat{\mathcal{M}}$ then admits a unique extension on $\hat{\mathcal{J}}$.

Observe that $\hat{\mathcal{M}}$ is then $\hat{\mathcal{F}}$-invariant and satisfies $(\pi_n)_*\hat{\mathcal{M}} = \mathcal{M}$ for every $n \in \mathbb{Z}$.

In the following, we will use the disintegration of $\hat{\mathcal{M}}$ with respect to $\mathcal{M}$ and the projection π_0, and more precisely the *conditional measures* $\hat{\mathcal{M}}^\gamma$ of $\hat{\mathcal{M}}$ on $\{\gamma_0 = \gamma\}$. These measures are uniquely defined for $\mathcal{M}$-almost every $\gamma \in \mathcal{J}$ and are characterized by the property that

$$\langle \hat{\mathcal{M}}, g \rangle = \int_{\mathcal{X}} \langle \hat{\mathcal{M}}^\gamma, g \rangle \mathcal{M}(\gamma)$$

for all non-negative bounded measurable functions $g\colon \mathcal{J} \to \mathbb{R}$ (see [12] for more details about the construction). We now describe a useful approximation of the conditional measures $\hat{\mathcal{M}}^\gamma$, that we will need later.

For every $n > 0$, consider the projection $\pi^n\colon \hat{\mathcal{J}} \to \mathcal{J}^{n+1}$ defined as

$$\pi^n := (\pi_{-n},\dots,\pi_0).$$

By **(M1)**, $\mathcal{X}$ satisfies $\mathcal{M}(\mathcal{X}) = 1$ and the map $\mathcal{F}\colon \mathcal{X} \to \mathcal{X}$ is well defined and surjective. For every $(\gamma_{-n},\dots,\gamma_0) \in \mathcal{X}^{n+1}$ with $\mathcal{F}(\gamma_j) = \gamma_{j+1}$ for every $-n \le j \le -1$, we choose a representative $\hat{\beta} \in \hat{\mathcal{J}}$ such that $\beta_j = \gamma_j$ for all $-n \le j \le 0$. Observe that, given any $\gamma_0 \in \mathcal{X}$, there are d^{kn} elements as above in $\mathcal{X}^{n+1}$, hence d^{kn} representatives. We denote by $\hat{\mathcal{Z}}_n$ the collection of such representatives.

For every $\gamma \in \mathcal{X}$, define

$$\hat{\mathcal{M}}_n^\gamma := \theta(\gamma)^{-1} \sum_{\hat{\beta} \in \hat{\mathcal{Z}}_n : \beta_0 = \gamma} e^{\psi(\beta_{-n}) + \ldots + \psi(\beta_{-1})} \theta(\beta_{-n}) \delta_{\hat{z}}.$$

Observe in particular that, by **(M1)** and the fact that θ is positive on $\mathcal{X}$, $\hat{\mathcal{M}}_n^\gamma$ is defined for $\mathcal{M}$-almost every $\gamma \in \mathcal{J}$.

Lemma 2.9. *For every $\gamma \in \mathcal{X}$, we have*

$$\lim_{n\to\infty} \hat{\mathcal{M}}_n^\gamma = \hat{\mathcal{M}}^\gamma.$$

Proof. It is enough to show that, for $\gamma \in \mathcal{X}$, we have

$$\lim_{n\to\infty} \hat{\mathcal{M}}_n^\gamma(\mathcal{A}_{-i,\mathcal{B}}) = \hat{\mathcal{M}}^\gamma(\mathcal{A}_{-i,\mathcal{B}}) \text{ for all } i \ge 0 \text{ and Borel set } \mathcal{B}.$$

Hence, the assertion is a consequence of the following two identities:

(1) $\hat{\mathcal{M}}_n^\gamma(\mathcal{A}_{-i,\mathcal{B}}) = \hat{\mathcal{M}}_i^\gamma(\mathcal{A}_{-i,\mathcal{B}})$ for every $\gamma \in \mathcal{X}$ and all $n > i \ge 0$;
(2) $\int \hat{\mathcal{M}}_i^\gamma(\mathcal{A}_{-i,\mathcal{B}})\mathcal{M}(\gamma) = \mathcal{M}(\mathcal{B})$ for all $i \ge 0$.

We start proving the first identity above. This is a consequence of the identity $\Lambda_\psi \theta = \theta$. Indeed, for every $n > i \geq 0$ and Borel set $\mathcal{B}$, we have

$$\begin{aligned}\hat{\mathcal{M}}_n^\gamma(\mathcal{A}_{-i,\mathcal{B}}) &= \theta(\gamma)^{-1} \sum_{\hat\beta \in \hat{\mathcal{Z}}_n : \beta_0 = \gamma} e^{\psi(\beta_{-n}) + \ldots + \psi(\beta_{-1})} \theta(\beta_{-n}) \delta_{\hat\beta}(\mathcal{A}_{-i,\mathcal{B}}) \\ &= \theta(\gamma)^{-1} \sum_{\hat\beta \in \hat{\mathcal{Z}}_i : \beta_0 = \gamma} (\Lambda_\psi^{n-i}\theta)(\beta_{-i}) e^{\psi(\beta_{-i}) + \ldots + \psi(\beta_{-1})} \delta_{\hat\beta}(\mathcal{A}_{-i,\mathcal{B}}) \\ &= \theta(\gamma)^{-1} \sum_{\hat\beta \in \hat{\mathcal{Z}}_i : \beta_0 = \gamma} \theta(\beta_{-i}) e^{\psi(\beta_{-i}) + \ldots + \psi(\beta_{-1})} \delta_{\hat\beta}(\mathcal{A}_{-i,\mathcal{B}}) \\ &= \hat{\mathcal{M}}_i^\gamma(\mathcal{A}_{-i,\mathcal{B}}),\end{aligned}$$

which proves the first identity.

Let us now prove the second identity. This time, we will use the fact that $\mathcal{N}$ is a fixed point for Λ_ψ^*. For all $i \geq 0$, we have

$$\begin{aligned}\int \hat{\mathcal{M}}_i^\gamma(\mathcal{A}_{-i,\mathcal{B}}) \mathcal{M}(\gamma) &= \int \Big(\theta(\gamma)^{-1} \sum_{\hat\beta \in \hat{\mathcal{Z}}_i : \beta_0 = \gamma} \theta(\beta_{-i}) e^{\psi(\beta_{-i}) + \ldots + \psi(\beta_{-1})} \delta_{\hat\beta}(\mathcal{A}_{-i,\mathcal{B}}) \Big) \mathcal{M}(\gamma) \\ &= \int \Big(\theta(\gamma)^{-1} \sum_{\mathcal{F}^i(\gamma') = \gamma} \theta(\gamma') e^{\psi(\gamma') + \ldots + \psi(\mathcal{F}^{i-1}(\gamma'))} \mathbb{1}_{\mathcal{B}}(\gamma) \Big) \mathcal{M}(\gamma) \\ &= \langle \mathcal{M}, \theta^{-1} \Lambda_\psi^i (\theta \mathbb{1}_{\mathcal{B}}) \rangle = \langle \mathcal{N}, \Lambda_\psi^i(\theta \mathbb{1}_{\mathcal{B}}) \rangle \\ &= \langle (\Lambda_\psi^i)^* \mathcal{N}, \theta \mathbb{1}_{\mathcal{B}} \rangle = \langle \mathcal{N}, \theta \mathbb{1}_{\mathcal{B}} \rangle = \mathcal{M}(\mathcal{B}),\end{aligned}$$

where we denoted by $\mathbb{1}_{\mathcal{B}}$ the indicatrix function of $\mathcal{B}$ and in the last step we used the identity $\mathcal{M} = \theta \mathcal{N}$. The proof is complete. □

For every $n \geq 0$, we denote by $f_{\hat\gamma}^{-n}$ the inverse branch of f^n, defined in a neighbourhood of Γ_γ, and such that $f_{\hat\gamma}^{-n}(\gamma(\lambda)) = \gamma_{-n}(\lambda)$ for all $\lambda \in M$. Such branch exists for every $\gamma \in \mathcal{X}$, but a priori it is only defined in some (non-controlled) neighbourhood of Γ_γ. The following proposition (see [3, Propositions 4.2 and 4.3] for the case of the equilibrium web and [11, Proposition A.1] for the general case) gives a uniform control in λ on the size of the neighbourhood of $\gamma(\lambda)$ where such inverses are defined.

Proposition 2.10. *Let (M, f) be a stable holomorphic family of endomorphisms of $\mathbb{P}^k$. Let $\mathcal{M}$ be a web satisfying* **(M1)** *and* **(M2)**. *Then, for every open set $\Omega \Subset M$ and $0 < A < A_1$, there exists a Borel subset $\hat{\mathcal{Y}} \subseteq \hat{\mathcal{J}}$ with $\hat{\mathcal{M}}(\hat{\mathcal{Y}}) = 1$, and two measurable functions $\hat\eta_A \colon \hat{\mathcal{Y}} \to]0, 1]$ and $\hat{l}_A \colon \hat{\mathcal{Y}} \to [1, +\infty[$ which satisfy the following properties.*

For every $\hat\gamma \in \hat{\mathcal{Y}}$ and every $n \in \mathbb{N}^*$ the iterated inverse branch $f_{\hat\gamma}^{-n}$ is defined and Lipschitz on the tubular neighbourhood $T_\Omega(\gamma_0, \hat\eta_A(\hat\gamma))$ of the graph $\Gamma_{\gamma_0} \cap (\Omega \times \mathbb{P}^k)$ of γ_0, and we have

$$f_{\hat\gamma}^{-n}(T_\Omega(\gamma_0, \hat\eta_A(\hat\gamma))) \subset T_\Omega(\gamma_{-n}, e^{-nA}) \quad \text{and} \quad \widetilde{\mathrm{Lip}}(f_{\hat\gamma}^{-n}) \leq \hat{l}_A(\hat\gamma) e^{-nA},$$

where $\widetilde{\mathrm{Lip}}(f_{\hat\gamma}^{-n}) := \sup_{\lambda \in \Omega} \mathrm{Lip}\,((f_{\hat\gamma}^{-n})_{|B(\gamma_0(\lambda), \hat\eta_A)})$.

We can now prove Proposition 2.8.

Proof of Proposition 2.8. Fix $m > 0$, a constant $A_0 < A < A_1$ and a positive integer r. For every $N \in \mathbb{N}$, define

$$\hat{\mathcal{Y}}_N := \{\hat\gamma \in \hat{\mathcal{Y}} \colon \eta_A(\hat\gamma) \geq N^{-1} \text{ and } l_A(\hat\gamma) \leq N\}.$$

By Proposition 2.10, we have $\lim_{N \to \infty} \hat{\mathcal{M}}(\hat{\mathcal{Y}}_N) = 1$. Fix $N_0 = N_0(m, r)$ such that $\hat{\mathcal{M}}(\hat{\mathcal{Y}}_N) > 1 - 1/(2m^{r+1})$ for all $N \geq N_0$. Then, by Markov inequality and the definition of the measures $\hat{\mathcal{M}}^\gamma$, we see that there exists a subset $\mathcal{X}_r \subset \mathcal{X}$ such that $\mathcal{M}(\mathcal{X}_r) > 1 - 1/m^r$ and

$$\hat{\mathcal{M}}^\gamma(\hat{\mathcal{Y}}_N \cap \{\gamma_0 = \gamma\}) > 1 - 1/(2m) \quad \text{for all } \gamma \in \mathcal{X}_r \quad \text{and} \quad N \geq N_0. \tag{2.6}$$

In order to prove the statement, it is enough to prove the assertion for all $\gamma \in \mathcal{X}_r$. Fix one such γ. It follows from Proposition 2.10 that all inverse branches defined on $T_\Omega(\gamma, e^{-m}/(2N))$ and corresponding to $\hat{\gamma} \in \hat{\mathcal{Y}}_N \cap \{\gamma_0 = \gamma\}$ are m-good for all n. It follows from Lemma 2.9 and (2.6) that

$$\hat{\mathcal{M}}_n^\gamma(\hat{\mathcal{Y}}_N \cap \{\gamma_0 = \gamma\}) > 1 - 1/m \quad \text{for all } n \text{ large enough.}$$

This implies that, for all n sufficiently large, we have $\|\mathcal{M}_{T,n}^{(m)}\| > 1 - 1/m$, for $T = T_\Omega(\gamma, e^{-m}/(2N))$. By definition, this means that T is m-nice. $\square$

3. Proof of Theorem 1.5

3.1. **Equilibrium states and associated webs.** We work here in the assumptions of Theorem 1.2. In particular, we assume that f_{λ_0} satisfies condition **(A)** and ϕ satisfies the conditions in **(B)** in the Introduction. Up to replacing ϕ by $\phi - P(\phi)$, we can assume that $P(\phi) = 0$. We denote by Λ_ϕ the (Ruelle-Perron-Frobenius) transfer operator

$$\Lambda_\phi(g)(y) := \sum_{f_{\lambda_0}(x)=y} e^{\phi(x)} g(x)$$

acting on $\mathcal{C}^0(\mathbb{P}^k)$, where the elements in the sum are counted with multiplicity. By [9, Theorem 1.1], there exists a unique probability measure m_ϕ and a unique (up to multiplicative constant) strictly positive continuous function $\rho_\phi \colon \mathbb{P}^k \to \mathbb{R}$ such that

$$\Lambda_\phi^n(g) \to \langle m_\phi, g \rangle \rho_\phi \quad \text{and} \quad (\Lambda_\phi^n)^* \delta_x \to \rho_\phi(x) m_\phi$$

for all continuous function $g \colon \mathbb{P}^k \to \mathbb{R}$ and all $x \in \mathbb{P}^k$. We normalize ρ_ϕ so that $\langle m_\phi, \rho_\phi \rangle = 1$. The probability measure $\mu_\phi := \rho_\phi m_\phi$ is then f_{λ_0}-invariant, and is the unique equilibrium state for f_{λ_0} associated to ϕ. In particular, for every $x \in \mathbb{P}^k$, the measures

$$\mu_{x,n} := \rho_\phi \cdot \rho_\phi(x)^{-1} (\Lambda_\phi^n)^* \delta_x = \rho_\phi^{-1}(x) \sum_{f^n(y)=x} e^{\phi(y)+\ldots+\phi(f^{n-1}(y))} \rho_\phi(y) \delta_y$$

satisfy $\mu_{x,n} \to \mu_\phi$ as $n \to \infty$.

By [9, Proposition 4.9], μ_ϕ satisfies $h_{\mu_\phi} > \log d^{k-1}$. Hence, by [11], there exists an ergodic web $\mathcal{M}_{\lambda_0,\mu_\phi}$ on $\mathcal{J}$ with the property that $(p_{\lambda_0})_* \mathcal{M}_{\lambda_0,\mu_\phi} = \mu_\phi$ and which satisfies **(M1)** and **(M2)**. The following lemma, proved in [13], gives in particular the uniqueness of such web.

Lemma 3.1. *Let (M, f) be a stable family of endomorphisms of $\mathbb{P}^k$. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two probability measures compactly supported on $\mathcal{J}$. Assume that $\mathcal{M}_1$ is an acritical web and that there exists $\lambda_0 \in M$ such that $(p_{\lambda_0})_* \mathcal{M}_1 = (p_{\lambda_0})_* \mathcal{M}_2$. Then $\mathcal{M}_1 = \mathcal{M}_2$.*

Define the functions $\theta, \psi \colon \mathcal{J} \to \mathbb{R}$ as

$$\theta(\gamma) := \rho_\phi(\gamma(\lambda_0)) \quad \text{and} \quad \psi(\gamma) := \phi(\gamma(\lambda_0)).$$

Observe that these functions are continuous on $\mathcal{J}$. In particular, for every $m \in \mathbb{N}^*$, any graph $\gamma \in \mathcal{J}$ admits an open neighbourhood $\mathcal{A} \subset \mathcal{J}$ such that

$$\inf_{\mathcal{A}} \theta \geq (1 - 1/m) \sup_{\mathcal{A}} \theta. \tag{3.1}$$

To see this, take an open neighbourhood $\mathcal{A}' \subset \mathcal{J}$ of γ. By the continuity of θ at γ, and the fact that ρ_ϕ is positive on the compact set $\mathbb{P}^k$, there exists an open neighbourhood $\mathcal{A} \subset \mathcal{A}'$ such that

$$\theta(\gamma_1) - \theta(\gamma_2) \leq \frac{\inf_{\mathcal{A}'} \theta}{m} \quad \text{for any } \gamma_1, \gamma_2 \in \mathcal{A}.$$

Observe that $\inf_{\mathcal{A}'} \theta \leq \inf_{\mathcal{A}} \theta \leq \sup_{\mathcal{A}} \theta$. Taking respectively the supremum over γ_1 and the infimum over γ_2 proves the claim.

Consider the operator Λ_ψ defined as in (2.1). For $\gamma \in \mathcal{X}$, set $\mathcal{M}_{\gamma,n}$ as defined in (2.2). It follows from the above and Lemmas 2.1, 2.3, and 3.1, that θ satisfies $\Lambda_\psi(\theta) = \theta$ and that there exists a unique probability measure $\mathcal{N}$ on $\mathcal{X}$ which is a fixed point for Λ_ψ^*. Furthermore, the web $\mathcal{M}_{\lambda_0,\mu_\phi}$ satisfies $\mathcal{M}_{\lambda_0,\mu_\phi} = \theta\mathcal{N}$, and for any $\gamma \in \mathcal{X}$ we have

$$\begin{aligned}\mathcal{M}_{\gamma,n} &= \theta(\gamma)^{-1} \sum_{\mathcal{F}^n(\gamma')=\gamma} e^{\psi(\gamma')+\ldots+\psi(\mathcal{F}^{n-1}(\gamma'))}\theta(\gamma')\delta_{\gamma'} \\ &= \theta(\gamma(\lambda_0))^{-1} \sum_{\mathcal{F}^n(\gamma')=\gamma} e^{\phi(\gamma'(\lambda_0))+\ldots+\phi(\mathcal{F}^{n-1}(\gamma')(\lambda_0))}\theta(\gamma'(\lambda_0))\delta_{\gamma'} \\ &\to \mathcal{M}_{\lambda_0,\mu_\phi} \text{ as } n \to \infty.\end{aligned} \tag{3.2}$$

In particular, $\mathcal{M}_{\lambda_0,\mu_\phi}$ satisfies condition **(M3)**.

We conclude this section with the following lemma, that we will need in the next section. Recall that $q > 2$ and $\|\phi\|_{\log^q} < \infty$.

Lemma 3.2. *There exists a positive constant $C = C(A_0, q)$ such that, for all $n \in \mathbb{N}$, $m \geq 0$, and every m-good inverse branch $g: T \to g(T)$ of f of order n on a tube T, and for all sequences of graphs $\{\gamma_l^1\}_{0\leq l\leq n-1}$ and $\{\gamma_l^2\}_{0\leq l\leq n-1}$ with $\Gamma_{\gamma_l^1}, \Gamma_{\gamma_l^2} \subset f^l(g(T))$ for all $0 \leq l \leq n-1$, we have*

$$\sum_{l=0}^{n-1} |\psi(\gamma_l^1) - \psi(\gamma_l^2)| \leq C m^{-(q-1)}.$$

Proof. It follows from the definition of ψ that

$$\sum_{l=0}^{n-1} |\psi(\gamma_l^1) - \psi(\gamma_l^2)| = \sum_{l=0}^{n-1} |\phi(\gamma_l^1(\lambda_0)) - \phi(\gamma_l^2(\lambda_0))|.$$

The assertion is a consequence of [9, Lemma 4.13]. □

3.2. **Construction of repelling graphs.** For simplicity, we denote in this section by $\mathcal{M}$ the web $\mathcal{M}_{\lambda_0,\mu_\phi}$ defined in the previous section. We also recall that we are assuming, with no loss of generality, that $P(\phi) = 0$. We will fix a relatively compact open subset $M' \Subset M$, and only consider the graphs of elements of $\mathcal{J}$ as graphs over M'. In particular, all distances and balls are with respect to the distance $\text{dist}_{M'}$, see (2.3).

The following is the main result of this section and the key estimate to prove Theorem 1.5. Thanks to the results proved in Section 2, we can follow the strategy of [10, Lemma 4.14]. In particular we also employ a trick due to Buff [16] which simplifies the original proof of the equidistribution of repelling points with respect to the measure of maximal entropy in [14].

Proposition 3.3. *Let $\mathfrak{U}$ be a finite collection of disjoint open subsets of $\mathcal{J}$. For every $m > 0$ there exists $n(m, U) > m$ and, for every $n > N(m, \mathfrak{U})$, a set $\mathcal{Q}_{m,n}$ of motions of repelling periodic points of period n such that, for all $\mathcal{U} \in \mathfrak{U}$ we have*

$$(1 - 1/m)\mathcal{M}(\mathcal{U}) \leq \sum_{\gamma\in\mathcal{Q}_{m,n}\cap U} e^{\psi(\gamma)+\ldots+\psi(\mathcal{F}^{n-1}(\gamma))} \leq (1 + 1/m)\mathcal{M}(\mathcal{U})$$

Before proving Proposition 3.3, we make some preliminary simplifications. First of all, it is enough to prove the statement for a single open set $\mathcal{U}$. The general case follows setting $n(m, \mathfrak{U}) := \max_{\mathcal{U}\in\mathfrak{U}} n(n, U)$.

Assume that $\mathcal{M}(\mathcal{U}) = 0$. In that case, it is enough to take $n(m, \mathcal{U}) = m + 1$ and $\mathcal{Q}_{m,n} = \emptyset$. Hence, we can assume that $\mathcal{M}(\mathcal{U}) > 0$.

Fix integers $m_2 \gg m_1 \gg m$. By Proposition 2.8, for $\mathcal{M}$-almost every γ, the tube $T_{M'}(\gamma, \eta)$ (and hence the ball $\mathcal{B}_{M'}(\gamma, \eta)$ of $\mathcal{J}$) is m_2-nice if η is sufficiently small (depending on γ). It follows that, for $\mathcal{U}$ as above, we can find a finite union of m_2-nice balls $\mathcal{B}_i \subset \mathcal{J}$ with $\mathcal{B}_i \Subset U$, satisfying (3.1), whose centers belong to $\text{Supp}\,\mathcal{M}$, and with the property that $\mathcal{M}(\mathcal{U} \setminus \cup_i\mathcal{B}_i) < \mathcal{M}(\mathcal{U})/m_2$. Hence, it is enough to prove the statement for each of the balls $\mathcal{B}_i$. Namely, the above gives that, in order to prove Proposition 3.3, it is enough to to prove the following statement.

Proposition 3.4. *For every $m_1 > 0$ there exists $\bar{m}_2(m_1) > m_1$ such that, for all $m_2 > \bar{m}_2$ and every m_2-nice ball $\mathcal{T} = \mathcal{B}(\gamma, \eta)$ with $\gamma \in \operatorname{Supp} \mathcal{M}$ and satisfying (3.1) (with $\mathcal{A} = \mathcal{T}$ and $m = m_2$), there exists $n(m_2) > m_2$ and, for all $n \geq n(m_2)$ a set $\mathcal{Q}_n \subset \mathcal{T} \cap \operatorname{Supp} \mathcal{M}$ of motions of repelling n-periodic points with the property that*

$$(1 - 1/m_1)\mathcal{M}(\mathcal{T}) \leq \sum_{\gamma \in \mathcal{Q}_n} e^{\psi(\gamma) + \ldots + \psi(\mathcal{F}^{n-1}(\gamma))} \leq (1 + 1/m_1)\mathcal{M}(\mathcal{T}).$$

Since $\mathcal{M}(\mathcal{T}) > 0$, we fix an integer $m_3 \geq m_2/\mathcal{M}(\mathcal{T})$ and a ball $\mathcal{T}^\star := \mathcal{B}_{M'}(\gamma, \eta^\star)$, where $\eta^\star < \eta$ is such that $\mathcal{M}(\mathcal{T}^\star) > (1 - 1/m_2)\mathcal{M}(\mathcal{T})$. We also denote by T and $T^\star$ the tubes corresponding to $\mathcal{T}$ and $\mathcal{T}^\star$. Recalling that the support of $\mathcal{M}$ is compact, we also choose a finite family of disjoint m_3-nice balls $\mathcal{D}_i := \mathcal{B}_{M'}(\gamma_i, \eta_i)$ such that $\mathcal{M}(\cup_i \mathcal{D}_i) > 1 - 1/m_3$ and satisfying (3.1) (with $\mathcal{A} = \mathcal{D}_i$ and $m = m_3$). Set $\mathcal{D} := \cup_i \mathcal{D}_i$. We also fix balls $\mathcal{D}_i^\star := \mathcal{B}_{M'}(\gamma_i, \eta_i^\star) \subset \mathcal{D}_i$ with $\eta_i^* < \eta_i$, $\mathcal{M}(\cup_i \mathcal{D}_i^\star) > 1 - 1/m_3$ and set $\mathcal{D}^\star := \cup_i \mathcal{D}_i^\star$. We will denote by D_i the tube corresponding to $\mathcal{D}_i$ and set $D := \cup D_i$.

Lemma 3.5. *There exists an integer $M_1 = M_1(m_2, \mathcal{T}, \mathcal{T}^\star, \mathcal{D}_i)$ such that*

$$(1 - 4/m_2)\mathcal{M}(\mathcal{T}) \leq \mathcal{M}_{D_i,N}^{(m_3)}(\mathcal{T}^\star) \leq (1 + 4/m_2)\mathcal{M}(\mathcal{T}) \text{ for all } i \text{ and } N \geq M_1.$$

Recall that $\mathcal{M}_{D_i,N}^{(m_3)}$ is defined in (2.4).

Proof. Since the balls $\mathcal{D}_i$ are m_3-nice and $m_3 \geq m_2/\mathcal{M}(\mathcal{T})$, we have

$$\mathcal{M}_{D_i,N}^{(m_3)} \geq (1 - \mathcal{M}(\mathcal{T})/m_2) \quad \text{for all } i \text{ and all } N \text{ large enough.}$$

As $\mathcal{M}_{D_i,N}^{(m_3)} \leq \mathcal{M}_{\gamma_i,N}$ and $\|\mathcal{M}_{\gamma_i,N}\| = 1 + o(1)$, it follows that

$$\|\mathcal{M}_{\gamma_i,N} - \mathcal{M}_{D_i,N}^{(m_3)}\| \leq \mathcal{M}(\mathcal{T})/m_2 + o(1).$$

Hence, it is enough to prove that

$$(1 - 2/m_2)\mathcal{M}(\mathcal{T}) \leq \mathcal{M}_{\gamma_i,N}(\mathcal{T}^\star) \leq (1 + 2/m_2)\mathcal{M}(\mathcal{T}) \text{ for all } i \text{ and all } N \text{ large enough.}$$

As $\mathcal{M}(\mathcal{T}^\star) \geq (1 - 1/m_2)\mathcal{M}(\mathcal{T})$, this is a consequence of (3.2). □

Lemma 3.6. *There exists an integer $M_2 = M_2(m_2, \mathcal{T}, \mathcal{D}^\star)$ such that*

$$(1 - 4/m_2) \leq \mathcal{M}_{T,N}^{(m_2)}(\mathcal{D}^\star) \leq (1 + 4/m_2) \text{ for all } i \text{ and } N \geq M_2.$$

Proof. Since $\mathcal{T}$ is m_2-nice, we have

$$\|\mathcal{M}_{T,N}^{(m_2)}\| \geq (1 - 1/m_2) \text{ for all } N \text{ large enough.}$$

Recall also that $\mathcal{M}_{T,N}^{(m_2)} \leq \mathcal{M}_{\gamma,N}$ and that $\|\mathcal{M}_{\gamma,N}\| \leq 1 + o(1)$. So, in order to prove the statement, it is enough to prove that

$$1 - 2/m_2 \leq \mathcal{M}_{\gamma,N}(\mathcal{D}^\star) \leq 1 + 1/m_2.$$

Since $\mathcal{M}(\mathcal{D}^\star) \geq 1 - 1/m_3$ and $m_3 \geq m_2$, this follows from (3.2). □

We now construct the set $\mathcal{Q}$ of motions of repelling periodic points as in Proposition 3.4.

For every N_1 sufficiently large, every element in the support of $\mathbb{1}_{\mathcal{T}^\star}\mathcal{M}_{D_i,N_1}^{(m_3)}$ corresponds to an m_3-good inverse branch of $f : M \times \mathbb{P}^k \to M \times \mathbb{P}^k$ of order N_1 on the tube D_i, which maps D_i to a subset of the tube T whose slice at any λ is relatively compact in $T_{|\lambda}$. In the same way, for every N_2 sufficiently large, every element in the support of $\mathbb{1}_{\mathcal{D}^\star}\mathcal{M}_{T,N_2}^{(m_2)}$ corresponds to an m_2-good inverse branch of order N_2 sending the tube T to a subset of $D = \cup_i D_i$.

We define the collection $\{g_j\}$ to be the compositions of such inverse branches, giving inverse branches for $f^{N_1+N_2}$ sending the tube T to a subset of T whose slice at any λ is relatively compact in $T_{|\lambda}$. Given any such g_j, we write $g_j = g_j^{(1)} \circ g_j^{(2)}$, where $g_j^{(2)}$ is an inverse branch of

f^{N_2} on T with image in D and $g_j^{(1)}$ is an inverse branch of f^{N_1} on some D_i with image in T. For every j, we also set $i = i(j)$ where i is defined by $g_j^{(2)}(T) \subset D_i$.

By definition, the inverse branch g_j is of the form $g_j(\lambda, z) = (\lambda, g_j^\lambda(z))$, where $\lambda \mapsto g_j^\lambda(z)$ is holomorphic. For $\gamma' \in \mathcal{T}$, set $\mathcal{G}_j(\gamma')(\lambda) := g_j^\lambda(\gamma'(\lambda))$, so that $\mathcal{G}_j(\mathcal{T}) \subseteq \mathcal{T}^*$. One can define in a similar way $\mathcal{G}_j^{(1)} : \mathcal{D}_i \to \mathcal{T}^*$ and $\mathcal{G}_j^{(2)} : \mathcal{T} \to \mathcal{D}_i$ and remark that $\mathcal{G}_j = \mathcal{G}_j^{(1)} \circ \mathcal{G}_j^{(2)}$.

Lemma 3.7. *The space* $(\operatorname{Supp} \mathcal{M}, \operatorname{dist}_{M'})$ *is complete.*

Proof. Let $(\gamma_n)_{n\in\mathbb{N}}$ be a Cauchy sequence. As $\operatorname{Supp} \mathcal{M}$ is compact with respect to the local uniform convergence, there exists a subsequence γ_{n_j} such that γ_{n_j} converges locally uniformly to a graph $\gamma \in \operatorname{Supp} \mathcal{M}$. In particular, γ_{n_j} converges to γ on M' and the Cauchy sequence $(\gamma_n)_{n\in\mathbb{N}}$ admits a cluster value, hence converges. □

Lemma 3.8. *For $N = N_1 + N_2$ large enough, the operator $\mathcal{G}_j$ admits a fixed point $\gamma_j^\infty \in \mathcal{T} \cap \operatorname{Supp} \mathcal{M}$. Furthermore, $\gamma_j^\infty(\lambda)$ is a periodic point of period N of f_λ, for any parameter $\lambda \in M$, and it is repelling for any $\lambda \in M'$.*

Proof. Since $\gamma \in \operatorname{Supp} \mathcal{M}$, by Lemma 2.3 (6) and Lemma 3.7, it is sufficient to prove that $\mathcal{G}_j$ is a contraction on a closed subset of $\mathcal{T}$ containing γ. Take $\gamma_1, \gamma_2 \in \mathcal{T}$. By Proposition 2.10, we have

$$\operatorname{dist}_{M'}(\mathcal{G}_j(\gamma_1), \mathcal{G}_j(\gamma_2)) \lesssim e^{-NA} \operatorname{dist}_{M'}(\gamma_1, \gamma_2), \tag{3.3}$$

where the implicit constant is independent of N. Hence, for all N sufficiently large, $\mathcal{G}_j$ is a contraction on the closed set $\overline{\mathcal{T}^*} = \{\gamma' \in \mathcal{J} \colon \operatorname{dist}_{\mathcal{J}}(\gamma, \gamma') \leq \eta^*\}$. By the Banach fixed point theorem, applied to the complete metric space $\overline{\mathcal{T}^*} \cap \operatorname{Supp} \mathcal{M}$, the map $\mathcal{G}_j$ admits a fixed point $\gamma_j^\infty \in \mathcal{T} \cap \operatorname{Supp} \mathcal{M}$. In particular, observe that $\gamma_j^\infty(\lambda)$ lies in J_λ for any $\lambda \in M$. To conclude, it remains to prove that $\gamma_j^\infty(\lambda)$ is an N-periodic point for every $\lambda \in M$ and is repelling for every $\lambda \in M'$.

Recall that $T = T(\gamma, \eta)$ is the tube associated to the ball $\mathcal{T}$. Remark that $f^N \colon g_j(T) \to T$ is onto, hence $f^N \circ g_j = id_T$. For all $\lambda \in M'$, the equality $\gamma_j^\infty(\lambda) = \mathcal{G}_j(\gamma_j^\infty)(\lambda) = g_j(\gamma_j^\infty(\lambda))$ then leads to $f_\lambda^N(\gamma_j^\infty(\lambda)) = \gamma_j^\infty(\lambda)$. This shows that γ_j^∞ is indeed the motion of an N-periodic point over M'. In order to extend it to a motion over all M, it is sufficient to show that $\gamma_j^\infty(\lambda)$ is repelling for any $\lambda \in M'$. Indeed, since $\gamma_j^\infty \in \operatorname{Supp} \mathcal{M}$, [3, Lemma 2.5] then allows us to conclude.

Observe that (3.3) also gives a positive constant c such that

$$\operatorname{dist}_{M'}(\mathcal{F}^{qN}(\gamma_1), \gamma_j^\infty) \gtrsim e^{qNA} \operatorname{dist}_{M'}(\gamma_1, \gamma_j^\infty),$$

for every positive integer $q \in \mathbb{N}^*$ and any graph $\gamma_1 \in \mathcal{T}\left(\mathcal{G}_j^q(\gamma_1), c \cdot e^{-qNA}\right)$, where the implicit constant is independent of q. This proves that $\gamma_j^\infty(\lambda)$ is repelling for any $\lambda \in M'$, and concludes the proof. □

We can now conclude the proof of Proposition 3.4. As explained above, this also completes the proof of Proposition 3.3.

End of the proof of Proposition 3.4. We continue to use the notations introduced above. Up to possibly increasing M_1 and M_2 in Lemmas 3.5 and 3.6, we can assume that we can take $N_1 = M_1$ and $N_2 = M_2$ in the construction before Lemma 3.8. We set $n(m_2) := M_1(m_2) + M_2(m_2)$, for a fixed choice of sufficiently large m_2 and m_3 as above. For every $n > n(n_2)$, we denote by $\mathcal{Q}_n$ the set of the motions of n-repelling point σ_j given by Lemma 3.8 applied with $N_1 = M_1(m_2)$ and $N_2 = n - M_2(m_2)$. Every element $\sigma \in \mathcal{Q}_n$ is then the motion of a repelling n-periodic point.

Set

$$\mathcal{M}_n := \sum_{\sigma \in \mathcal{Q}_n} e^{\psi(\gamma) + \ldots + \psi(\mathcal{F}^{n-1}(\gamma))} \delta_\gamma = \sum_j e^{\psi(\sigma_j) + \ldots + \psi(\mathcal{F}^{n-1}(\sigma_j))} \delta_{\sigma_j},$$

where $\mathcal{Q}_n = \{\sigma_j\}$, and

$$\tilde{\mathcal{M}}_n := \sum_j \Big(e^{\psi(\mathcal{G}_j^{(1)}(\gamma_i(j)))+\ldots+\psi(\mathcal{F}^{N_1-1}\circ\mathcal{G}_j^{(1)}(\gamma_{i(j)}))} \cdot \frac{\theta(\mathcal{G}_j^{(1)}(\gamma_{i(j)}))}{\theta(\gamma_{i(j)})} \cdot e^{\psi(\mathcal{G}_j^{2)}(\gamma)+\ldots+\psi(\mathcal{F}^{N_2-1}\circ\mathcal{G}_j^{(2)}(\gamma))} \frac{\theta(\mathcal{G}_j^{(2)}(\gamma))}{\theta(\gamma)} \Big) \delta_{\mathcal{G}_j(\gamma)},$$

where we recall that $\mathcal{T} = B(\gamma, \eta)$ and $\mathcal{D}_i = B(\gamma_i, \eta_i)$. Observe in particular that

$$\tilde{\mathcal{M}}_n(\mathcal{T}) = \sum_i \mathcal{M}_{T,N_2}^{(m_2)}(\mathcal{D}_i^\star) \cdot \mathcal{M}_{D_i,N_1}^{(m_3)}(\mathcal{T}^\star). \tag{3.4}$$

By Lemma 3.5 and the fact that $\sum_i \mathcal{M}_{T,N_2}^{(m_2)}(\mathcal{D}_i^\star) = \mathcal{M}_{T,N_2}^{(m_2)}(\mathcal{D}^\star)$, we have

$$(1-4/m_2)\mathcal{M}(\mathcal{T})\mathcal{M}_{T,N_2}^{(m_2)}(\mathcal{D}^\star) \le \tilde{\mathcal{M}}_n(\mathcal{T}) \le (1+4/m_2)\mathcal{M}(\mathcal{T})\mathcal{M}_{T,N_2}^{(m_2)}(\mathcal{D}^\star).$$

By Lemma 3.6, this gives

$$(1-10/m_2)\mathcal{M}(\mathcal{T}) \le \tilde{\mathcal{M}}_n(\mathcal{T}) \le (1+10/m_2)\mathcal{M}(\mathcal{T}). \tag{3.5}$$

Hence, up to taking m_2 sufficiently large, the desired estimate holds if we replace $\mathcal{M}_n(\mathcal{T})$ with $\tilde{\mathcal{M}}_n(\mathcal{T})$. It is then enough to compare $\mathcal{M}_n(\mathcal{T})$ with $\tilde{\mathcal{M}}_n(\mathcal{T})$.

Observe that, for all i and j, the graphs of γ and $\mathcal{G}_j^{(1)}(\gamma_{i(j)})$ belong to T and those of $\gamma_{i(j)}$ and $\mathcal{G}_j^{(2)}(\gamma)$ belong to D_i. Hence, since the tubes T and D_i are m_2-nice and satisfy (3.1) (with $\mathcal{A}$ replaced by $\mathcal{T}$ and $\mathcal{D}_i$, respectively), we have

$$\Big|\frac{\theta(\mathcal{G}_j^{(1)}(\gamma_{i(j)}))}{\theta(\gamma)} - 1\Big| \le \frac{2}{m_2} \text{ and } \Big|\frac{\theta(\mathcal{G}_j^{(2)}(\gamma))}{\theta(\gamma_{i(j)})} - 1\Big| \le \frac{2}{m_2} \text{ for all } j.$$

It follows from these inequalities and Lemma 3.2 that $|\mathcal{M}_n(\mathcal{T}) - \tilde{\mathcal{M}}_n(\mathcal{T})| \lesssim \tilde{\mathcal{M}}_n(\mathcal{T})/m_2$. This, together with (3.5), concludes the proof. ☐

3.3. **Proof of Theorem 1.5.** We can now conclude the proof of Theorem 1.5. Recall that we are assuming that $P(\phi) = 0$ and that we denote by $\mathcal{M}$ the web $\mathcal{M}_{\lambda_0,\mu_\phi}$, whose construction is detailed in Section 3.1.

We first build a family of measurable partitions of $(\mathcal{J}, \mathcal{M})$ whose diameter converges to 0.

Lemma 3.9. *There exists a sequence $(\mathfrak{U}_i)_{i\in\mathbb{N}}$ of finite families of disjoint open sets $\mathfrak{U}_i = \{\mathcal{U}_{i,j}\}_{1\le j\le J_i}$ satisfying the following properties:*
- **(U1)** *for all $i \in \mathbb{N}$, we have $\mathcal{M}(\cup_{1\le j\le J_i}\mathcal{U}_{i,j}) = 1$;*
- **(U2)** $\sup_{i\in\mathbb{N}}(i \cdot \max_{1\le j\le J_i} \operatorname{diam}_{\mathcal{J}} \mathcal{U}_{i,j}) \le 1$*;*
- **(U3)** *for all $i \ge 2$ and $1 \le j \le J_i$ there exists $1 \le j' \le J_{i-1}$ such that $\mathcal{U}_{i,j} \subset \mathcal{U}_{i-1,j'}$.*

Proof. We prove the lemma by induction on $i \in \mathbb{N}$. We set $J_0 = 1$ and $\mathcal{U}_{0,1} = \mathcal{J}$. The family $\mathfrak{U}_0 := \{U_{0,1}\}$ satisfies conditions **(U1)** and **(U2)**, and condition **(U3)** is empty in this case.

We now do the induction step. Assume that we have built, for some $i \in \mathbb{N}$, a finite family $\mathfrak{U}_i = \{\mathcal{U}_{i,j}\}_{1\le j\le J_i}$ satisfying **(U1)**, **(U2)**, and **(U3)**. Consider the interval $I := (\frac{1}{2(i+2)}, \frac{1}{2(i+1)})$. As $\operatorname{Supp}\mathcal{M} \subseteq \cup_{\gamma\in\operatorname{Supp}\mathcal{M}}\mathcal{B}_{\mathcal{J}}(\gamma, \frac{1}{2(i+2)})$ and $\operatorname{Supp}\mathcal{M}$ is compact, there exists $m_i \in \mathbb{N}$ and a collection $\{\gamma_{i,m}\}_{1\le m\le m_i} \subset \operatorname{Supp}\mathcal{M}$ such that, for any $\varepsilon \in I$, $\operatorname{Supp}\mathcal{M} \subseteq \cup_{m=1}^{m_i}\mathcal{B}_{\mathcal{J}}(\gamma_{i,m}, \varepsilon)$. Set $B_{i,m}^\varepsilon := \mathcal{B}_{\mathcal{J}}(\gamma_{i,m}, \varepsilon)$ and $D_{i,m}^\varepsilon := \partial\mathcal{B}_{i,m}^\varepsilon$.

Claim. The set $A_i := \{\varepsilon \in I \,:\, \mathcal{M}(\cup_{1\le m\le m_i} D_{i,m}^\varepsilon) > 0\}$ is countable.

Proof. As the family $\{D^{\varepsilon}_{i,m}\}_{1\le m\le m_i}$ is finite, it suffices to prove that $A_{i,m} := \{\varepsilon \in I : \mathcal{M}(D^{\varepsilon}_{i,m}) > 0\}$ is countable for every $1 \le m \le m_i$. Assume this is not true for some $A_{i,\bar m}$.

For $\alpha \in \mathbb{N}$, set $A^{\alpha}_{i,\bar m} = \{\varepsilon \in I : \mathcal{M}(D^{\varepsilon}_{i,\bar m}) > \frac{1}{\alpha+1}\}$. Observe that $A_{i,\bar m} = \cup_{\alpha\in\mathbb{N}} A^{\alpha}_{i,\bar m}$. Hence, there exists $\alpha \in \mathbb{N}$ such that $\mathrm{Card}(A^{\alpha}_{i,\bar m}) = +\infty$. As $D^{\varepsilon}_{i,\bar m} \cap D^{\varepsilon'}_{i,\bar m} = \emptyset$ for all $\varepsilon \ne \varepsilon' \in I$, we have

$$1 \ge \mathcal{M}(\cup_{\varepsilon\in A^{\alpha}_{i,\bar m}} D^{\varepsilon}_{i,\bar m}) = \sum_{\varepsilon\in A^{\alpha}_{i,\bar m}} \mathcal{M}(D^{\varepsilon}_{i,\bar m}) \ge \frac{\mathrm{Card}(A^{\alpha}_{i,\bar m})}{\alpha+1} = +\infty.$$

This gives a contradiction, and the proof of the claim is complete. □

Fix now $\varepsilon \in I$ such that $\mathcal{M}(D^{\varepsilon}_{i,m}) = 0$ for all $1 \le m \le m_i$. Denote the connected components of $\cup_{j=1}^{J_i} \mathcal{U}_{i,j} \cap B^{\varepsilon}_{i,1}$ by $\mathcal{U}_{i+1,1}, \dots, \mathcal{U}_{i+1,l_1}$. By induction, for $1 < m \le m_i$, define also $\mathcal{U}_{i+1,l_{m-1}+1}, \dots, \mathcal{U}_{i+1,l_m}$ as the connected components of

$$\bigcup_{j=1}^{J_i} \mathcal{U}_{i,j} \cap \left(B^{\varepsilon}_{i,m} \setminus \bigcup_{l=1}^{l_{m-1}} \overline{\mathcal{U}_{i+1,l}} \right).$$

Set $J_{i+1} := l_{m_i}$. By definition, the sets $\mathcal{U}_{i+1,l}$ are open and pairwise disjoint. We claim that

$$\bigcup_{l=1}^{J_{i+1}} \mathcal{U}_{i+1,l} \supset \left(\bigcup_{m=1}^{m_i} B^{\varepsilon}_{i,m} \setminus \bigcup_{l=1}^{J_{i+1}} \partial\mathcal{U}_{i+1,l} \right) \cap \bigcup_{j=1}^{J_i} \mathcal{U}_{i,j}.$$

Fix $m \le m_i$ and take $\gamma \in B^{\varepsilon}_{i,m} \setminus \cup_{1\le l\le J_{i+1}} \partial\mathcal{U}_{i+1,l}$, for some $m \le m_i$. We can assume that γ does not satisfy this property for all $m' < m$. By the definition of the sets $\mathcal{U}_{i+1,l}$, this implies that $\gamma \notin \mathcal{U}_{i+1,l}$, for every $l \le l_{m-1}$. Hence $\gamma \in B^{\varepsilon}_{i,m} \setminus \bigcup_{l=1}^{l_m-1} \overline{\mathcal{U}_{i+1,l}}$. This proves the claim.

Remark that $\bigcup_{l=1}^{J_{i+1}} \partial\mathcal{U}_{i+1,l} \subset \bigcup_{m=1}^{m_i} D^{\varepsilon}_{i,m}$. As $\mathfrak{U}_i$ satisfies **(U1)**, the choice of ε yields

$$\mathcal{M}\left(\bigcup_{l=1}^{J_{i+1}} \mathcal{U}_{i+1,l}\right) \ge \mathcal{M}\left(\bigcup_{m=1}^{m_i} B^{\varepsilon}_{i,m} \setminus \bigcup_{l=1}^{J_{i+1}} \partial\mathcal{U}_{i+1,l}\right) \ge \mathcal{M}\left(\bigcup_{m=1}^{m_i} B^{\varepsilon}_{i,m}\right) - \mathcal{M}\left(\bigcup_{m=1}^{m_i} D^{\varepsilon}_{i,m}\right) = 1.$$

This proves that $\mathfrak{U}_{i+1}$ satisfies **(U1)**. The property **(U2)** comes from the definition of the $B^{\varepsilon}_{i,m}$'s and the fact that, by construction, for every l there exists $m(l)$ such that $\mathcal{U}_{i+1,l} \subset B^{\varepsilon}_{i,m(l)}$. Finally, the fact that $\mathcal{U}_{i+1,l}$ is a connected subset of $\cup_{j=1}^{J_i} \mathcal{U}_{i,j}$ gives **(U3)**. The proof is complete. □

For every $n \in \mathbb{N}$, define $i_n := \max\{m \le n : n \ge n(m, \mathfrak{U}_m)\}$, where $n(m, \mathfrak{U}_m)$ is given by Proposition 3.3. For every $\alpha > 0$, set $m_\alpha := \lfloor \alpha + 1 \rfloor$. Then, for any $n \ge \max(m, n(m, \mathcal{U}_m))$, we have $i_n \ge m_\alpha > \alpha$. In particular, we have $i_n \to \infty$ as $n \to \infty$.

For every $n \in \mathbb{N}$, we apply Proposition 3.3 with $\mathfrak{U}_{i_n}$ instead of $\mathfrak{U}$. This gives a collection $\mathcal{P}_{\phi,n} \subset \cup_{1\le j\le J_{i_n}} \mathcal{U}_{i_n,j}$ of motions of repelling n-periodic points, satisfying the properties in that statement. We define

$$\mathcal{M}'_n := \sum_{\sigma\in\mathcal{P}_{\phi,n}} e^{\psi(\sigma)+\ldots+\psi(\mathcal{F}^{n-1}\sigma)} \delta_\sigma. \tag{3.6}$$

Lemma 3.10. *Any limit $\mathcal{M}'$ of the sequence $\{\mathcal{M}'_n\}_{n\in\mathbb{N}}$ has mass 1.*

Proof. By definition, for every $n \in \mathbb{N}$ we have

$$\|\mathcal{M}'_n\| = \sum_{\sigma\in\mathcal{P}_{\phi,n}} e^{\psi(\sigma)+\ldots+\psi(\mathcal{F}^{n-1}\sigma)} = \sum_{j=1}^{J_{i_n}} \sum_{\sigma\in\mathcal{P}_{\phi,n}\cap\mathcal{U}_{i_n,j}} e^{\psi(\sigma)+\ldots+\psi(\mathcal{F}^{n-1}\sigma)}.$$

Since the family $\mathfrak{U}_n$ is pairwise disjoint, Proposition 3.3 yields

$$\left(1-\frac{1}{i_n}\right)\mathcal{M}_\phi\left(\bigcup_{j=1}^{J_{i_n}}\mathcal{U}_{i_n,j}\right) \leq \|\mathcal{M}'_n\| \leq \left(1+\frac{1}{i_n}\right)\mathcal{M}_\phi\left(\bigcup_{j=1}^{J_{i_n}}\mathcal{U}_{i_n,j}\right).$$

We conclude by Property **(U1)** and letting $n \to \infty$. □

Lemma 3.11. *For all $i^\star \in \mathbb{N}$ and $1 \leq j^\star \leq J_{i^\star}$ we have*

$$\liminf_{n\to\infty} \mathcal{M}'_n(\mathcal{U}_{i^\star,j^\star}) \geq \mathcal{M}(\mathcal{U}_{i^\star,j^\star}).$$

Proof. Fix $i^\star, j^\star$ as in the statement and $\varepsilon > 0$. It is enough to prove that, for all n sufficiently large, we have

$$\mathcal{M}'_n(\mathcal{U}_{i^\star,j^\star}) \geq \mathcal{M}(\mathcal{U}_{i^\star,j^\star}) - \varepsilon.$$

It is enough to consider only those n for which $i_n > i^\star$. For all such n, by **(U1)** and pairwise disjointness, we have

$$\mathcal{M}(\mathcal{U}_{i^\star,j^\star}) = \sum_{j'} \mathcal{M}(\mathcal{U}_{i_n,j'}) \quad \text{and} \quad \mathcal{M}'_n(\mathcal{U}_{i^\star,j^\star}) \geq \sum_{j'} \mathcal{M}'_n(\mathcal{U}_{i_n,j'}),$$

where the two sums are over the j' such that $U_{i_n,j'} \subset U_{i^\star,j^\star}$. Using the same convention for the sums, by the definition of $\mathcal{M}'_n$ and Proposition 3.3, we have

$$\mathcal{M}(\mathcal{U}_{i^\star,j^\star}) - \mathcal{M}'_n(\mathcal{U}_{i^\star,j^\star}) \leq \sum_{j'} \mathcal{M}(\mathcal{U}_{i_n,j'}) - \mathcal{M}'_n(\mathcal{U}_{i_n,j'}) \leq i_n^{-1} \sum_{j'} \mathcal{M}(\mathcal{U}_{i_n,j'}) = i_n^{-1}\mathcal{M}(\mathcal{U}_{i^*,j^*}).$$

The assertion follows. □

End of the proof of Theorem 1.5. We show that the sequence $\{\mathcal{M}'_n\}_{n\in\mathbb{N}}$ as in (3.6) converges to $\mathcal{M}$. Let $\mathcal{A} \subset \mathcal{J}$ be a closed set. Consider $\mathcal{A}' := \bigcap_{n\in\mathbb{N}} \bigcup_{\mathcal{U}\in\mathfrak{U}_{i_n}^{\mathcal{A}}} \mathcal{U}$, where $\mathfrak{U}_i^{\mathcal{A}} := \{\mathcal{U} \in \mathfrak{U}_i : \mathcal{U} \cap \mathcal{A} \neq \emptyset\}$. By **(U1)**, for every $n \in \mathbb{N}$ we have

$$\mathcal{M}\left(\bigcup_{\mathcal{U}\in\mathfrak{U}_{i_n}^{\mathcal{A}}} \mathcal{U}\right) = 1 - \mathcal{M}\left(\bigcup_{\mathcal{U}\in\mathfrak{U}_{i_n}\setminus\mathfrak{U}_{i_n}^{\mathcal{A}}} \mathcal{U}\right) \geq 1 - \mathcal{M}(\mathcal{A}^c) = \mathcal{M}(\mathcal{A}).$$

Remark that property **(U3)** ensures that $\mathcal{A}'$ is a countable non-increasing intersection. Letting $n \to \infty$, we deduce that $\mathcal{M}(\mathcal{A}') \geq \mathcal{M}(\mathcal{A})$.

Claim. We have $\mathcal{A}' \subseteq \mathcal{A}$. In particular, we have $\mathcal{M}(\mathcal{A}') = \mathcal{M}(\mathcal{A})$.

Proof. Since $\mathcal{M}(\mathcal{A}') \leq \mathcal{M}(A)$, it is enough to show that $\mathcal{A}' \subseteq \mathcal{A}$. Take $\gamma \in \mathcal{A}'$. For any $n \in \mathbb{N}$, there exist some open set $\mathcal{U} \in \mathfrak{U}_{i_n}$ and $\gamma_n \in \mathcal{J}$ with $\gamma \in \mathcal{U}$ and $\gamma_n \in \mathcal{U} \cap \mathcal{A}$. By property **(U2)**, we have $\mathrm{dist}_{\mathcal{J}}(\gamma, \gamma_n) < 1/i_n$. We deduce that $\gamma = \lim_{n\to\infty} \gamma_n \in \overline{\mathcal{A}} = \mathcal{A}$. □

Let now $\mathcal{M}'$ be any limit of the sequence $\{\mathcal{M}'_n\}_{n\in\mathbb{N}}$. Lemma 3.11 and the Claim above give that

$$\mathcal{M}(\mathcal{A}) = \mathcal{M}(\mathcal{A}') = \lim_{n\to\infty} \sum_{\mathcal{U}\in\mathfrak{U}_{i_n}^{\mathcal{A}}} \mathcal{M}(\mathcal{U}) \leq \lim_{n\to\infty} \sum_{\mathcal{U}\in\mathfrak{U}_{i_n}^{\mathcal{A}}} \mathcal{M}'(\mathcal{U}) = \mathcal{M}'(\mathcal{A}') \leq \mathcal{M}'(\mathcal{A}).$$

As the closed set $\mathcal{A}$ was chosen arbitrarily, Lemma 3.10 and the fact that $\|\mathcal{M}\|= 1$ imply that that $\mathcal{M} = \mathcal{M}'$. This shows that $\mathcal{M}'_n \to \mathcal{M}$ as $n \to \infty$, and completes the proof. □

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