An optimal uniqueness result for Riccati equations arising in abstract parabolic control problems

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Abstract

An abstract nonautonomous parabolic linear-quadratic regulator problem with very general final cost operator $\boldsymbol{P_T}$ is considered, subject to the same assumptions under which a classical solution of the associated differential Riccati equation was shown to exist, in two papers appeared in 1999 and 2000, by Terreni and the first named author. We prove an optimal uniqueness result for the integral Riccati equation in in the largest possible class, filling a gap existing in the autonomous case, too. In addition, we give a regularity result for the optimal state.

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1 Introduction

Let H, U be Hilbert spaces. For fixed $T > 0$ we consider, for every $s \in [0, T[$, the following regulator problem: minimize the functional

$$J_s(u) = \int_s^T \|M(r)^{1/2} y(r)\|_H^2 \, dr + \int_s^T \|N(r)^{1/2} u(r)\|_U^2 \, dr + \|P_T^{1/2} y(T)\|_H^2 \qquad (1)$$

among all controls $u \in L^2(s,T;U)$, constrained by the following state equation, whose strong form is

$$\begin{cases} y'(t) = A(t)y(t) - A(t)G(t)u(t), & t \in]s,T], \\ y(s) = x, \end{cases} \tag{2}$$

and whose mild form is

$$y(t) = U(t,s)x - \int_s^t U(t,r)A(r)G(r)u(r)\,dr, \qquad t \in [s,T]. \tag{3}$$

Here the operators $\{M(\cdot)\}$ and P_T are linear, non-negative, bounded, selfadjoint in H, the operators $\{N(\cdot)\}$ are linear, positive, bounded, selfadjoint in U, x is an element of H, the operators $\{A(r)\}_{r\in[0,T]}$ are generators of analytic semigroups $\{e^{tA(r)}\}_{t\geq 0}$ in H, for every $r \in [s,T]$ the family $\{U(r,s)\}_{0\leq s\leq r}$ is the evolution operator associated to $A(r)$ and the operator $G(r)$ is the "Green map" associated to $A(r)$, having the property that $[-A(r)]^\alpha G(r)$ is a bounded operator from U into H for some $\alpha \in]0,1/2[$.
The pair (1)-(3) provides an abstract model for a large class of purely parabolic boundary control problems: the realization of $G(t)$ in concrete problems yields the lifting of the nonzero datum at the boundary, i.e. transforms the nonhomogeneous initial-boundary value problem into a homogeneous one by a modification of the right member of the evolution equation. Typical examples are given in [1, Section 9].
Following and generalizing the methods employed in the autonomous case by [2], [3] (see also the full description in the book [4]), the above problem was analyzed in [1] and [5], proving existence and uniqueness of the optimal pair $(\widehat{y},\widehat{u})$, and studying the associated Riccati equation, whose differential form is

$$\begin{cases} P'(t) + A(t)^*P(t) + P(t)A(t) \\ \qquad = -M(t) + P(t)A(t)G(t)N(t)^{-1}G(t)^*A(t)^*P(t), \quad t \in [s,T[\,, \\ P(T) = P_T, \end{cases} \tag{4}$$

and whose integral form is

$$\begin{aligned} P(t) &= U(T,t)^*P_TU(T,t) \\ &+ \int_t^T U(r,t)^*[M(r) - P(r)A(r)G(r)N(r)^{-1}G(r)^*A(r)^*P(r)]U(r,t)\,dr, \quad t \in [s,T]\,. \end{aligned} \tag{5}$$

In particular, existence and representation of a classical solution $P(\cdot)$ of equation (5) was proved; its uniqueness was guaranteed only under further assumptions on the operator P_T. More precisely, it was shown, among other things, that:

(i) the constructed solution $P(\cdot)$ of the Riccati equation (5) is classical, i.e. $P(\cdot)$ is continuously differentiable as an $\mathcal{L}(H)$-valued function and satisfies (4) in the sense of $\mathcal{L}(H)$, provided the operator $A(t)^*P(t)+P(t)A(t)$ is replaced by its bounded extension $\Lambda(t)P(t)$ (see [1, Section 7] for details);
(ii) the optimal pair enjoys some weighted Hölder continuity properties;

(iii) the final datum P_T belongs to the largest possible class, according to the counterexample in [6]: namely, the composition $P_T^{1/2} L_{sT}$ is a closed operator in H, where L_{sT} is defined in Hypothesis 7 below (see also (10)). In addition the Riccati operator $P(\cdot)$ is strongly continuous in $[s,T]$, and a necessary and sufficient condition is given in order that $P(t) \to P_T$ in $\mathcal{L}(H)$ as $t \to T^-$;
(iv) uniqueness holds provided $P_T \in \mathcal{L}(H, D([-A(T)^*]^{2\beta}))$ for some $\beta \in \left]\frac{1}{2} - \alpha, 1\right[$.

The main purpose of this paper is the proof, under the same assumptions made in [1] and [5], of an optimal uniqueness result for the Riccati equation (5): we drop the assumption written in (iv) above, and the solution turns out to be unique in a very general and natural class: exactly the same where the existence of a solution was established. In addition we prove a regularity result for the optimal state $\widehat{y}$: although in the parabolic case, as it is well known, such function may be unbounded as $t \to T^-$, nevertheless we show that $t \mapsto P(t)^{1/2}\widehat{y}(t)$ is continuous in the whole interval $[s,T]$, a result that seems to be new even in the autonomous case of [2], [3]. The precise assumptions of the paper are listed in Section 2 below.
The question of uniqueness for Riccati equations is a delicate issue. We confine ourselves to the finite horizon theory: in the autonomous setting, uniqueness was proved in the parabolic case under the assumption that $[-A(T)^*]^{2\beta} P_T$, or $[-A(T)^*]^{\beta} P_T [-A(T)]^{\beta}$ as in [7, Part IV, Chapter 2, Section 2.3, Theorem 2.2], is bounded for some $\beta > \frac{1}{2} - \alpha$, by means of a suitable *a priori* bound for $G(\cdot)^* A(\cdot)^* Q(\cdot)$ (in our notation), where Q is the difference between two solutions; see [2], [3], [8], [4, Theorem 1.5.3.3]. In the nonautonomous setting, after the pioneering paper [9], the same method is used in [10]. A similar argument works in the autonomous hyperbolic case, see [11, Theorems 8.3.7.1 and 9.3.5.4] or [12]. In the intermediate case of the so called "singular estimate control systems" most papers use the same method: see [12], [13], where $P_T = 0$, and [14], [15], [16] where the relevant parameter in the singular estimate assumption is less than 1/2. A similar condition is required in [17], where the method for proving uniqueness is based on a "fundamental identity" which resembles our Lemma 10 below. Finally we mention the case of abstract systems arising from composite PDEs systems with boundary control: the existence of a solution of the differential Riccati equation was proved in [18], while the proof of its uniqueness in the case $P_T = 0$, using the fundamental identity, is in [19].
We shortly describe our method: on one hand, the fundamental identity of Lemma 10 allows to prove that for any solution Q of the integral Riccati equation it must be $P \geq Q$; on the other hand, an accurate analysis of the terms appearing in the three equivalent formulations in $[s, T-\varepsilon]$ of the integral Riccati equation (see (15) and Lemma 12 below), as well as the careful evaluation of their limits as $\varepsilon \to 0^+$, leads us to show that the opposite inequality $P \leq Q$ also holds. Our method could perhaps be useful in improving the uniqueness results in some of the papers quoted above, too.
Remark 1. This paper should have been written more than 20 years ago, signed by the first author and his dear friend and colleague Brunello Terreni. But after the death of Brunello, for several years the first author felt unable to complete the work and even to think about this topic. Now, finally, it is time to end the whole story: we dedicate this article to the memory of Brunello.

2 Assumptions and main results

First of all we list our assumptions (the same as in [1], [5]). Let H and U be complex Hilbert spaces. We denote by $\Sigma^+(H)$ the class of linear, bounded, selfadjoint, non-negative operators from H into itself.

Hypothesis 1. *For each $t \in [0,T]$, $A(t) : D(A(t)) \subseteq H \to H$ is a closed linear operator generating an analytic semigroup $\{e^{rA(t)}\}_{r\geq 0}$, such that $0 \in \rho(A(t))$; in particular, there exist $M > 0$ and $\vartheta \in \left]\frac{\pi}{2}, \pi\right[$ such that*

$$\|(\lambda - A(t))^{-1}\|_{\mathcal{L}(H)} \leq M(1+|\lambda|)^{-1} \qquad \forall \lambda \in \overline{S(\vartheta)}, \quad \forall t \in [0,T],$$

where $S(\vartheta) = \{z \in \mathbb{C} : |\arg z| < \vartheta\}$.

Hypothesis 2. *There exist $N > 0$ and $\rho, \mu \in]0,1]$ with $\delta := \rho + \mu - 1 \in]0, \frac{1}{2}[$, such that*

$$\begin{aligned} &\left\|A(t)[\lambda - A(t)]^{-1}\left[A(t)^{-1} - A(s)^{-1}\right]\right\|_{\mathcal{L}(H)} \\ &\qquad + \left\|A(t)^*[\lambda - A(t)^*]^{-1}\left[[A(t)^*]^{-1} - [A(s)^*]^{-1}\right]\right\|_{\mathcal{L}(H)} \\ &\qquad \leq N|t-s|^{\mu}(1+|\lambda|)^{-\rho} \qquad \forall \lambda \in \overline{S(\vartheta)}, \quad \forall t,s \in [0,T]. \end{aligned}$$

By the results of [20] and [21], the family $\{A(t)\}_{t\in[0,T]}$ generates in H a strongly continuous evolution operator $U(t,s)$.

Hypothesis 3. *The family $\{U(t,s)\}_{0\leq s\leq t\leq T}$ is the evolution operator of $\{A(t)\}_{t\in[0,T]}$ and satisfies*

$$\begin{aligned} &\|[-A(t)]^{\eta}U(t,s)[-A(s)]^{-\gamma}\|_{\mathcal{L}(H)} + \|[-A(s)^*]^{\eta}U(t,s)^*[-A(t)^*]^{-\gamma}\|_{\mathcal{L}(H)} \\ &\qquad \leq M_{\eta\gamma}[1+(t-s)^{\eta-\gamma}] \qquad \text{for } 0 \leq s < t \leq T, \quad \eta,\gamma \in [0,1]. \end{aligned}$$

Hypothesis 4. *The number $\delta = \rho + \mu - 1$ is such that $0 < \delta < \frac{1}{2}$ and*

$$\begin{aligned} &\|[-A(t)]^{\eta}U(t,s)[-A(s)]^{-\gamma} - [-A(\tau)]^{\eta}U(\tau,s)[-A(s)]^{-\gamma}\|_{\mathcal{L}(H)} \\ &\qquad \leq N_{\eta\gamma}(t-\tau)^{\delta}[1+(\tau-s)^{\gamma-\eta-\delta}] \qquad \text{for } 0 \leq s < \tau \leq t \leq T, \quad \eta,\gamma \in [0,1], \end{aligned}$$

$$\begin{aligned} &\|[-A(\sigma)^*]^{\eta}U(t,\sigma)^*[-A(t)^*]^{-\gamma} - [-A(s)^*]^{\eta}U(t,s)^*[-A(t)^*]^{-\gamma}\|_{\mathcal{L}(H)} \\ &\qquad \leq N_{\eta\gamma}(\sigma-s)^{\delta}[1+(t-\sigma)^{\gamma-\eta-\delta}] \qquad \text{for } 0 \leq s \leq \sigma < t \leq T, \quad \eta,\gamma \in [0,1], \end{aligned}$$

all operators being strongly continuous with respect to t, τ, σ, s.

The properties of Hypothesis 3 were proved in [21] and follow by the results of [20] and [22]. The statement of Hypothesis 4 was never proved explicitly, but it follows essentially by estimates contained in [21] and [22].

Hypothesis 5. *$G(t) \in \mathcal{L}(U,H)$ for every $t \in [0,T]$, and there exists $\alpha \in \left]\delta, \frac{1}{2}\right[$ such that*

$$t \mapsto [-A(t)]^{\alpha}G(t) \in C^{\delta}([0,T], \mathcal{L}(U,H)).$$

The verification of the abstract Hypotheses 3 and 5 in several concrete initial boundary value problems for parabolic systems was made in [10].

By Hypotheses 4 and 5 we can give a precise meaning to the expression $U(t,s)A(s)G(s)$, $0 \leq s \leq t \leq T$, often appearing in the sequel: namely

$$U(t,s)A(s)G(s) = -[[-A(s)^*]^{1-\alpha}U(t,s)^*]^*[-A(s)]^\alpha G(s), \tag{6}$$

and this operator is estimated by

$$|[[-A(s)^*]^{1-\alpha}U(t,s)^*]^*[-A(s)]^\alpha G(s)\|_{\updownarrow(U,H)} \leq c(t-s)^{\alpha-1}. \tag{7}$$

Hypothesis 6. *$M(\cdot) \in C^\delta([0,T],\Sigma^+(H))$, $N(\cdot) \in C^\delta([0,T],\Sigma^+(U))$, and there exists $\nu > 0$ such that*

$$\langle N(t)u,u\rangle_U \geq \nu\|u\|_U^2 \qquad \forall t \in [0,T], \quad \forall u \in U.$$

This assumption is essential in the proof of the existence of a unique optimal control for the functional (1) under the constraint (3).
Before stating Hypothesis 7, it is useful to introduce some notations. We rewrite equation (3) as

$$y(t) = U(t,s)x + [L_s u](t) \qquad t \in [s,T]. \tag{8}$$

where, of course (recalling (6)),

$$L_s u(t) = -\int_s^t U(t,r)A(r)G(r)u(r)\,dr, \qquad t \in [s,T[\,, \qquad u \in L^2(s,T;U). \tag{9}$$

The properties of the operator L_s and of its adjoint L_s^* are listed in [1, Lemma 4.4]. We also introduce the operator L_{sT}, $0 \leq s < T$, as

$$\begin{cases} D(L_{sT}) = \left\{u \in L^2(s,T;U) : \displaystyle\int_s^T A(T)^{-1}U(T,r)A(r)G(r)u(r)dr \in D(A(T))\right\} \\ L_{sT}u = -A(T)\displaystyle\int_s^T A(T)^{-1}U(T,r)A(r)G(r)u(r)dr \qquad \forall u \in D(L_{sT}). \end{cases} \tag{10}$$

Clearly it holds $D(L_{sT}) = D(L_{0T})$ for every $s \in]0,T[$, since

$$\int_s^T A(T)^{-1}U(T,r)A(r)G(r)u(r)dr - \int_0^T A(T)^{-1}U(T,r)A(r)G(r)u(r)dr$$
$$= -A(T)^{-1}\int_0^s U(T,r)A(r)G(r)u(r)dr \in D(A(T)),$$

and for every $u \in D(L_{sT})$

$$L_{0T}u = L_{sT}u - \int_0^s U(T,r)A(r)G(r)u(r)\,dr = L_{sT}u + U(T,s)L_0u(s). \tag{11}$$

Let us finally state the crucial Hypothesis 7.

Hypothesis 7. *$P_T \in \Sigma^+(H)$, and the linear operator*

$$P_T^{1/2} L_{0T} : D(L_{0T}) \subseteq L^2(0,T;U) \to H$$

is closed.

We cannot drop this assumption, due to the counterexample in [6] (see also [4, Section 1.7]).

Remark 2. As said before, the above assumptions are exactly the same as in the papers [1] and [5]. □

As a consequence of Hypothesis 7, since the admissible controls are precisely the elements of $D(L_{sT})$, we can rewrite more exactly the cost functional as

$$J_s(u) = \begin{cases} \displaystyle\int_s^T \|M(r)^{1/2} y(r)\|_H^2\, dr + \int_s^T \|N(r)^{1/2} u(r)\|_U^2\, dr + \|P_T^{1/2} y(T)\|_H^2 \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } u \in D(L_{sT}) \\ +\infty \qquad\qquad\qquad\qquad\qquad\qquad \text{if } u \in L^2(s,T;U) \setminus D(L_{sT}), \end{cases} \tag{12}$$

where $P_T^{1/2} y(T)$ means $P_T^{1/2} U(T,s)x + P_T^{1/2} L_{sT} u$ for every $u \in D(L_{sT})$.
For further use (see Lemma 21 below), we also recall the definition of L_{sT}^* (see [1, formula (2.8)],

$$\begin{cases} D(L_{sT}^*) = \{y \in H : \; G(\cdot)^* A(\cdot)^* U(T,\cdot)^* y \in L^2(s,T;U)\} \\ L_{sT}^* y = -G(\cdot)^* A(\cdot)^* U(T,\cdot)^* y; \end{cases} \tag{13}$$

note that, by definition, $D(L_{tT}^*) \subseteq D(L_{sT}^*)$ for every $s \in [0,t]$, and that $L_{tT}^* y = L_{sT}^* y|_{[t,T[}$ for every $y \in D(L_{sT})$.
We now state our main results.

Theorem 8. *The Riccati equation (5) has a unique solution P within the class*

$$\begin{aligned} \mathcal{Q} = \Big\{ & Q \in L^\infty(0,T;\Sigma^+(H)) \cap C([0,T[\,,\Sigma^+(H)) : \\ & \lim_{t \to T^-} \|Q(t)x - P_T x\|_H = 0 \quad \forall x \in H, \\ & \left\| [-A(t)^*]^{1-\alpha} Q(t) \right\|_{\mathcal{L}(H)} \le c(T-t)^{\alpha-1} \quad \forall t \in [0,T[\Big\}. \end{aligned} \tag{14}$$

The proof is postponed in Section 4.

Theorem 9. *The optimal state $\widehat{y}$ of problem (12)-(8) satisfies*

$$t \mapsto P(t)^{1/2} \widehat{y}(t) \in C([0,T],H).$$

This theorem will follow as a simple consequence of Theorem 8, at the end of Section 4.

3 Auxiliary results

Our proof needs some auxiliary facts. In the next statements we consider a fixed operator Q, belonging to the class $\mathcal{Q}$ introduced in (14), which satisfies the differential Riccati equation (4). As we do not know yet whether the singularity of the quadratic term in (4) is integrable at T or not, we rewrite (4) as an integral equation in the smaller interval $[s, T-\varepsilon]$, where $0 \le s < T-\varepsilon < T$. We write it in its weak form, valid for all $x, y \in H$:

$$\begin{aligned}\langle Q(s)x, y\rangle_H &= \langle Q(T-\varepsilon)U(T-\varepsilon, s)x, U(T-\varepsilon, s)y\rangle_H \\ &\quad + \int_s^{T-\varepsilon} \langle M(r)U(r,s)x, U(r,s)y\rangle_H \, dr \\ &\quad - \int_s^{T-\varepsilon} \langle N(r)^{-1}G(r)^*A(r)^*Q(r)U(r,s)x, G(r)^*A(r)^*Q(r)U(r,s)y\rangle_U \, dr,\end{aligned} \tag{15}$$

We recall explicitly that in (15) the operators $Q(\cdot)$, $U(\cdot,\cdot)$, $M(\cdot)$, $N(\cdot)^{-1}$ are uniformly bounded in $[0, T]$ by some constant K, while the operator $G(\cdot)^*A(\cdot)^*Q(\cdot)$, taking into account (6) and (7), is well defined and uniformly bounded in $[s, T-\varepsilon]$ by some constant C_ε , since its singularity is concentrated at T.
We observe now that we may let $\varepsilon \to 0^+$ in equation (15): indeed all terms converge, but (possibly) the last. Hence, by difference, the last one converges, too, and when $x = y$ we deduce

$$N(\cdot)^{-1/2}G(\cdot)^*A(\cdot)^*Q(\cdot)U(\cdot, s)x \in L^2(s, T; U).$$

Thus, we obtain from (15), for all $x, y \in H$, as $\varepsilon \to 0^+$:

$$\begin{aligned}\langle Q(s)x, y\rangle_H &= \langle P_T U(T,s)x, U(T,s)y\rangle_H + \int_s^T \langle M(r)U(r,s)x, U(r,s)y\rangle_H \, dr \\ &\quad - \int_s^T \langle N(r)^{-1}G(r)^*A(r)^*Q(r)U(r,s)x, G(r)^*A(r)^*Q(r)U(r,s)y\rangle_U \, dr.\end{aligned} \tag{16}$$

Our first lemma concerns a basic identity which is well known in classical control theory.
Lemma 10 (Fundamental identity)**.** *Let $\mathcal{Q}$ be given by (14), and let $Q \in \mathcal{Q}$ be a solution of* (15)*. In addition, fix $s \in [0, T[$, $x \in H$, a control $u \in L^2(s, T; U)$ and the corresponding state $y \in L^2(s, T; H)$, given by* (8)*. Then the following identity holds*

for every $\varepsilon \in]0, T-s]$*:*

$$\begin{aligned}\langle Q(T-\varepsilon)y(T-\varepsilon), y(T-\varepsilon)\rangle_H - \langle Q(s)x, x\rangle_H \\ = -\int_s^{T-\varepsilon} \|M(r)^{1/2}y(r)\|_H^2\, dr - \int_s^{T-\varepsilon} \|N(r)^{1/2}u(r)\|_U^2\, dr \\ + \int_s^{T-\varepsilon} \|N(r)^{1/2}u(r) - N(r)^{-1/2}G(r)^*A(r)^*Q(r)y(r)\|_U^2\, dr.\end{aligned} \tag{17}$$

The proof is in Appendix A.

Our second lemma concerns the so called "closed loop equation".

Lemma 11. *Let* $Q \in \mathcal{Q}$*, with* $\mathcal{Q}$ *given by* (14)*. For fixed* $s \in [0, T[$*, the closed loop equation*

$$\Phi(t,s) = U(t,s) - \int_s^t U(t,r)A(r)G(r)N(r)^{-1}G(r)^*A(r)^*Q(r)\Phi(r,s)\, dr, \quad t \in [s, T[, \tag{18}$$

has a unique solution $\Phi(\cdot, s)$*, belonging to* $C([s, T-\varepsilon], \mathcal{L}(H))$ *for every* $\varepsilon \in]0, T-s[$*.*

Proof. Fix $\varepsilon \in]0, T-s[$. The integral operator, acting on functions $g \in L^2(s, T; U)$,

$$[K_s g](t) := \int_s^t U(t,r)A(r)G(r)N(r)^{-1}G(r)^*A(r)^*Q(r)g(r)\, dr, \quad t \in [s, T-\varepsilon],$$

with kernel $K(t,r)$ given by

$$K(t,r) := U(t,r)A(r)G(r)N(r)^{-1}G(r)^*A(r)^*Q(r), \quad 0 \le r < t < T,$$

is continuous in the region $\{(t,r) : 0 \le r < t < T\}$ with values in $\mathcal{L}(H)$ and satisfies the estimate (see [1, formula (6.6)])

$$\|K(t,r)\|_{\mathcal{L}(H)} \le c(t-r)^{\alpha-1}(T-r)^{\alpha-1}, \qquad 0 \le r < t < T,$$

so that, in particular,

$$\|K(t,r)\|_{\mathcal{L}(H)} \le c_\varepsilon (t-r)^{\alpha-1}, \quad 0 \le r < t \le T-\varepsilon. \tag{19}$$

It is shown in [5], among other things, that $(1+K_s)^{-1}$ is well defined and belongs to the space $\mathcal{L}(C([s, T-\varepsilon], \mathcal{L}(H)))$. Thus if we set $\Phi(t,s) = [(1+K_s)^{-1}U(\cdot, s)](t)$, we immediately obtain that (18) holds in $[s, T-\varepsilon]$, with arbitrary $\varepsilon \in]0, T-s[$, i.e. in $[s, T[$. □

We remark that, by uniqueness of the solution of (18),

$$\Phi(t,s) = \Phi(t,q)\Phi(q,s), \qquad 0 \le s \le q \le t < T. \tag{20}$$

The next lemma is basic: it expresses the integral Riccati equation (5) in two equivalent forms, from which we will deduce all the relevant informations for our proof.

Lemma 12. *Let* $\Phi(t,s)$ *be the unique solution of* (18)*, as stated in Lemma 11. The Riccati integral equation* (15) *is equivalent to both equations below, which hold for all* $x,y \in H$ *and* $0 \le s \le T-\varepsilon < T$*:*

$$\begin{aligned}\langle Q(s)x,y\rangle_H \\ = \langle Q(T-\varepsilon)\Phi(T-\varepsilon,s)x, U(T-\varepsilon,s)y\rangle_H + \int_s^{T-\varepsilon} \langle M(r)\Phi(r,s)x, U(r,s)y\rangle_H \, dr;\end{aligned} \tag{21}$$

$$\begin{aligned}\langle Q(s)x,y\rangle_H = \\ = \langle Q(T-\varepsilon)\Phi(T-\varepsilon,s)x, \Phi(T-\varepsilon,s)y\rangle_H + \int_s^{T-\varepsilon} \langle M(r)\Phi(r,s)x, \Phi(r,s)y\rangle_H \, dr + \\ + \int_s^{T-\varepsilon} \langle N(r)^{-1}G(r)^*A(r)^*Q(r)\Phi(r,s)x, G(r)^*A(r)^*Q(r)\Phi(r,s)y\rangle_U \, dr.\end{aligned} \tag{22}$$

The proof of Lemma 12 is in Appendix B. Equation (22) is specially useful for our purposes: indeed, when $y = x$ it can be written as

$$\begin{aligned}\|Q(s)^{1/2}x\|_H^2 = \|Q(T-\varepsilon)^{1/2}\Phi(T-\varepsilon,s)x\|_H^2 + \int_s^{T-\varepsilon} \|M(r)^{1/2}\Phi(r,s)x\|_H^2 \, dr \\ + \int_s^{T-\varepsilon} \|N(r)^{-1/2}G(r)^*A(r)^*Q(r)\Phi(r,s)x\|_U^2 \, dr, \qquad x \in H.\end{aligned} \tag{23}$$

Since the integrals are bounded and monotonically increasing as $\varepsilon \to 0^+$, the first term in the right member is bounded and decreasing. Thus, if $Q \in \mathcal{Q}$ is a solution of (15), then by (23) we can define

$$\langle B(s)x,x\rangle_H := \lim_{\varepsilon\to 0^+} \langle Q(T-\varepsilon)\Phi(T-\varepsilon,s)x, \Phi(T-\varepsilon,s)x\rangle_H\,, \quad x \in H. \tag{24}$$

We have $B(s) \in \Sigma^+(H)$ and equation (23) becomes, as $\varepsilon \to 0^+$,

$$\begin{aligned}\langle Q(s)x,x\rangle_H = \langle B(s)x,x\rangle_H + \int_s^T \|M(r)^{1/2}\Phi(r,s)x\|_H^2 \, dr \\ + \int_s^T \|N(r)^{-1/2}G(r)^*A(r)^*Q(r)\Phi(r,s)x\|_U^2 \, dr, \quad x \in H.\end{aligned} \tag{25}$$

Then we can pass to the limit as $\varepsilon \to 0^+$ in (22), too; using polarization in (24), we get

$$\langle B(s)x,y\rangle_H = \lim_{\varepsilon\to 0^+} \langle Q(T-\varepsilon)\Phi(T-\varepsilon,s)x, \Phi(T-\varepsilon,s)y\rangle_H\,, \quad x,y \in H, \tag{26}$$

and

$$
\begin{aligned}
\langle Q(s)x, y\rangle_H &= \langle B(s)x, y\rangle_H + \int_s^T \langle M(r)\Phi(r,s)x, \Phi(r,s)y\rangle_H \, dr \\
&+ \int_s^T \langle N(r)^{-1}G(r)^*A(r)^*Q(r)\Phi(r,s)x, G(r)^*A(r)^*Q(r)\Phi(r,s)y\rangle_U \, dr, \quad x, y \in H.
\end{aligned}
\tag{27}
$$

We now recall some properties of the optimal pair $(\widehat{y}, \widehat{u})$ with initial point $x \in H$ and initial time s, whose existence was proved in [1] and [5]. By [1, Proposition 5.4], it holds

$$
\begin{aligned}
\langle P(s)x, x\rangle_H &= J_s(\widehat{u}) \\
&= \int_s^T \|M(r)^{1/2}\widehat{y}(r)\|_H^2 \, dr + \int_s^T \|N(r)^{1/2}\widehat{u}(r)\|_U^2 \, dr + \|P_T^{1/2}\widehat{y}(T)\|_H^2 \, ;
\end{aligned}
\tag{28}
$$

here $P(\cdot)$ is the classical solution, constructed in [1], of the differential Riccati equation (4). The optimal control $\widehat{u}$ is given in feedback form by

$$
\widehat{u}(t; s, x) = N(t)^{-1}G(t)^*A(t)^*P(t)\widehat{y}(t), \quad t \in [s, T[\,. \tag{29}
$$

By optimality, $\widehat{u} = \widehat{u}(\cdot; s, x)$ belongs to $D(L_{sT})$. The optimal state $\widehat{y} = \widehat{y}(\cdot; s, x)$ is

$$
\widehat{y}(t) = \widehat{\Phi}(t, s)x = U(t, s)x + [L_s\widehat{u}](t), \quad t \in [s, T[\,. \tag{30}
$$

Now, let $Q(\cdot)$ be any solution of the Riccati equation (15), belonging to the class $\mathcal{Q}$ defined by (14). In order to prove that $P = Q$, we introduce the pair $(\bar{y}, \bar{u})$ where, similarly to (29), $\bar{u}$ is defined in feedback form in terms of $\bar{y}$ and Q: namely, for fixed $s \in [0, T[\,$, with $\Phi(t, s)$ defined by (18), we set

$$
\bar{u}(t) = \bar{u}(t; s, x) = N(t)^{-1}G(t)^*A(t)^*Q(t)\Phi(t, s)x, \quad s \le t < T, \tag{31}
$$

and

$$
\bar{y}(t) = \bar{y}(t; s, x) = \Phi(t, s)x = U(t, s)x + [L_s\bar{u}(\cdot; s, x)](t), \quad s \le t < T. \tag{32}
$$

By (25) we have $\bar{u} \in L^2(s, T; U)$ and hence, by [1, Lemma 4.4(i)], $\bar{y} \in L^{\frac{2}{1-2\alpha}}(s, T; H)$; we note explicitly that, by (20),

$$
\bar{u}(r; s, x) = \bar{u}(r; t, \Phi(t, s)x) \qquad \forall r \in [t, T[\,, \quad \forall t \in [s, T[\,. \tag{33}
$$

Since we do not know whether or not $\bar{u} \in D(L_{sT})$, we define suitable approximations of $\bar{u}$ and $\bar{y}$:

$$
u_\varepsilon = u_\varepsilon(\cdot; s, x) = \begin{cases} \bar{u}(\cdot; s, x) & \text{in } [s, T-\varepsilon] \\ 0 & \text{in }]T-\varepsilon, T], \end{cases} \qquad y_\varepsilon = U(\cdot, s)x + L_s u_\varepsilon; \tag{34}
$$

then in particular

$$y_\varepsilon(t) = y_\varepsilon(t; s, x) = \begin{cases} \Phi(t,s)x & \text{if } t \in [s, T-\varepsilon[\\ U(t, T-\varepsilon)\Phi(T-\varepsilon, s)x & \text{if } t \in [T-\varepsilon, T]. \end{cases} \tag{35}$$

Note that $u_\varepsilon \in D(L_{sT})$, with $L_{sT} u_\varepsilon = U(T, T-\varepsilon)[L_s \bar{u}](T-\varepsilon)$; moreover, $u_\varepsilon \to \bar{u}$ in $L^2(s, T; U)$ and, using [1, Lemma 4.4(i)],

$$y_\varepsilon \to \bar{y} \quad \text{in} \quad L^{\frac{2}{1-2\alpha}}(s, T; H) \quad \text{as} \quad \varepsilon \to 0^+. \tag{36}$$

4 Proof of the main results

Let $Q(\cdot)$ be any solution of the Riccati equation, belonging to the class $\mathcal{Q}$ defined by (14). We will prove the two inequalities

$$\begin{aligned} &\textbf{(a)}\ \langle P(s)x, x\rangle_H \geq \langle Q(s)x, x\rangle_H \quad \forall x \in H, \\ &\textbf{(b)}\ \langle P(s)x, x\rangle_H \leq \langle Q(s)x, x\rangle_H \quad \forall x \in H. \end{aligned} \tag{37}$$

Theorem 8 will then follow by the arbitrariness of $s \in [0, T[$ and a simple polarization argument.

4.1 Proof of Theorem 8(a)

We start with the following proposition:
Proposition 13. *Let $(\widehat{u}, \widehat{y}) \in D(L_{sT}) \times L^2(s, T; H)$ be the optimal pair, given by (32)-(31), and set for every $\varepsilon \in]0, T-s[$*

$$\widehat{u}_\varepsilon = \begin{cases} \widehat{u} & \text{in } [s, T-\varepsilon] \\ 0 & \text{in }]T-\varepsilon, T], \end{cases} \qquad \widehat{y}_\varepsilon = U(\cdot, s)x + L_s \widehat{u}_\varepsilon;$$

then we have $\widehat{y}_\varepsilon \in C([s,T], H)$ and

$$\lim_{\varepsilon\to 0^+} \langle P_T \widehat{y}_\varepsilon(T), \widehat{y}_\varepsilon(T)\rangle_H = \langle P_T \widehat{y}(T), \widehat{y}(T)\rangle_H = \lim_{\varepsilon \to 0^+} \langle P(T-\varepsilon)\widehat{y}(T-\varepsilon), \widehat{y}(T-\varepsilon)\rangle_H\,. \tag{38}$$

Proof. It is well known that $\widehat{y} \in C([s, T[\,, H)$, so that $\widehat{y}_\varepsilon \in C([s, T-\varepsilon], H)$. On the other hand

$$\widehat{y}_\varepsilon(t) = U(t,s)x + [L_s \widehat{u}_\varepsilon](t) = U(t,s)x + U(t, T-\varepsilon)[L_s \widehat{u}](T-\varepsilon) \qquad \forall t \in [T-\varepsilon, T];$$

this implies $\widehat{y}_\varepsilon \in C([s,T], H)$. Next, we note that, by the definition of $\widehat{u}_\varepsilon$,

$$L_{sT}(\widehat{u}_\varepsilon - \widehat{u}) = A(T)\int_{T-\varepsilon}^T A(T)^{-1} U(T,r)A(r)G(r)\widehat{u}(r)\,dr = -L_{T-\varepsilon\, T}\widehat{u};$$

Now we have obviously

$$\widehat{u}_\varepsilon \to \widehat{u} \quad \text{in } L^2(s,T;U) \quad \text{as } \varepsilon \to 0^+;$$

moreover, by [1, Proposition 4.2(ii)],

$$P_T^{1/2} L_{sT}(\widehat{u}_\varepsilon - \widehat{u}) = -P_T^{1/2} L_{T-\varepsilon\, T}\widehat{u} \to 0 \quad \text{in } H \quad \text{as } \varepsilon \to 0^+.$$

Hence, as $\varepsilon \to 0^+$,

$$P_T^{1/2}\widehat{y}_\varepsilon(T) = P_T^{1/2}U(T,s)x + P_T^{1/2}L_{sT}\widehat{u}_\varepsilon \to P_T^{1/2}U(T,s)x + P_T^{1/2}L_{sT}\widehat{u} = P_T^{1/2}\widehat{y}(T) \quad \text{in } H,$$

which implies the first equality in (38).
Next, choosing in Lemma 10 $Q = P$, $u = \widehat{u}$, $y = \widehat{y}$, we obtain by (17) and (29):

$$\begin{aligned}
&\langle P(T-\varepsilon)\widehat{y}(T-\varepsilon), \widehat{y}(T-\varepsilon)\rangle_H - \langle P(s)x, x\rangle_H \\
&\qquad = -\int_s^{T-\varepsilon} \|M(r)^{1/2}\widehat{y}(r)\|_H^2\, dr - \int_s^{T-\varepsilon} \|N(r)^{1/2}\widehat{u}(r)\|_U^2\, dr + 0,
\end{aligned}$$

and letting $\varepsilon \to 0^+$ we get, using (28),

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+} \langle P(T-\varepsilon)\widehat{y}(T-\varepsilon), \widehat{y}(T-\varepsilon)\rangle_H \\
&= \langle P(s)x, x\rangle_H - \int_s^T \|M(r)^{1/2}\widehat{y}(r)\|_H^2\, dr - \int_s^T \|N(r)^{1/2}\widehat{u}(r)\|_U^2\, dr \\
&= J_s(\widehat{u}) - \int_s^T \|M(r)^{1/2}\widehat{y}(r)\|_H^2\, dr - \int_s^T \|N(r)^{1/2}\widehat{u}(r)\|_U^2\, dr = \langle P_T\widehat{y}(T), \widehat{y}(T)\rangle_H\,,
\end{aligned}$$

which gives the second equality in (38). □

We now complete the proof of **(a)**. For fixed $\varepsilon \in]0, T-s[$, replace in (17) ε by δ, with $0 < \delta < \varepsilon$, and choose $u = \widehat{u}_\varepsilon$, $y = \widehat{y}_\varepsilon$. Then we find

$$\begin{aligned}
\langle Q(s)x, x\rangle_H &= \langle Q(T-\delta)\widehat{y}_\varepsilon(T-\delta), \widehat{y}_\varepsilon(T-\delta)\rangle_H \\
&\quad + \int_s^{T-\delta} \langle M(r)\widehat{y}_\varepsilon(r), \widehat{y}_\varepsilon(r)\rangle_H\, dr + \int_s^{T-\delta} \langle N(r)\widehat{u}_\varepsilon(r), \widehat{u}_\varepsilon(r)\rangle_U\, dr \\
&\quad - \int_s^{T-\delta} \|N(r)^{1/2}\widehat{u}_\varepsilon(r) - N(r)^{-1/2}G(r)^*A(r)^*Q(r)\widehat{y}_\varepsilon(r)\|_U^2\, dr \\
&\le \langle Q(T-\delta)\widehat{y}_\varepsilon(T-\delta), \widehat{y}_\varepsilon(T-\delta)\rangle_H \\
&\quad + \int_s^{T-\delta} \langle M(r)\widehat{y}_\varepsilon(r), \widehat{y}_\varepsilon(r)\rangle_H\, dr + \int_s^{T-\delta} \langle N(r)\widehat{u}_\varepsilon(r), \widehat{u}_\varepsilon(r)\rangle_U\, dr.
\end{aligned}$$

As $\delta \to 0^+$, since we know that $\widehat{y}_\varepsilon \in C([s,T],H)$ we deduce

$$\langle Q(s)x,x\rangle_H \le \langle P_T\widehat{y}_\varepsilon(T),\widehat{y}_\varepsilon(T)\rangle_H + \int_s^T \langle M(r)\widehat{y}_\varepsilon(r),\widehat{y}_\varepsilon(r)\rangle_H\,dr + \int_s^T \langle N(r)\widehat{u}_\varepsilon(r),\widehat{u}_\varepsilon(r)\rangle_U\,dr.$$

Finally, we let $\varepsilon \to 0^+$: noting that $\widehat{y}_\varepsilon \to \widehat{y}$ in $L^2(s,T;H)$ and $\widehat{u}_\varepsilon \to \widehat{u}$ in $L^2(s,T;U)$, and using the first equality in (38), we get

$$\begin{aligned}
&\langle Q(s)x,x\rangle_H \\
&\quad\le \langle P_T\widehat{y}(T),\widehat{y}(T)\rangle_H + \int_s^T \langle M(r)\widehat{y}(r),\widehat{y}(r)\rangle_H\,dr + \int_s^T \langle N(r)\widehat{u}(r),\widehat{u}(r)\rangle_U\,dr \\
&\quad= J_s(\widehat{u}) = \langle P(s)x,x\rangle_H\,.
\end{aligned}$$

This proves **(a)** in (37).

4.2 Proof of Theorem 8(b)

This proof is longer.
First, we list some statements where certain terms of equations (16), (21) and (22) are analyzed. The first one is easy and concerns two integral terms in (16) (with $y = x$) and (25).
Lemma 14. *We have*

$$\lim_{\varepsilon\to 0^+}\int_{T-\varepsilon}^T \|M(r)^{1/2}U(r,T-\varepsilon)x\|_H^2\,dr = \lim_{\varepsilon\to 0^+}\int_{T-\varepsilon}^T \|M(r)^{1/2}\Phi(r,T-\varepsilon)x\|_H^2\,dr = 0.$$

Proof. The first limit is trivial, since $M(\cdot)^{1/2}U(\cdot,T-\varepsilon)$ is uniformly bounded in $\mathcal{L}(H)$, independently on ε. Concerning the second one, we have, using Hölder inequality and (36):

$$\int_{T-\varepsilon}^T \|M(r)^{1/2}\Phi(r,T-\varepsilon)x\|_H^2\,dr \le K_1\,\varepsilon^{2\alpha}\|\bar{y}(\cdot;T-\varepsilon,x)\|^2_{L^{\frac{2}{1-2\alpha}}(T-\varepsilon,T;H)} \le K_2\,\varepsilon^{2\alpha}\|x\|_H^2$$

for some absolute constants $K_1, K_2 > 0$. The result follows. □

The second lemma analyzes the last integral term of (16) with $y = x$.
Lemma 15. *We have*

$$\lim_{\varepsilon\to 0^+}\int_{T-\varepsilon}^T \|N(r)^{-1/2}G(r)^*A(r)^*Q(r)U(r,T-\varepsilon)x\|_U^2\,dr = 0.$$

Proof. By (16) with $y = x$ and $s = T-\varepsilon$ we have

$$\int_{T-\varepsilon}^T \|N(r)^{-1/2}G(r)^*A(r)^*Q(r)U(r,T-\varepsilon)x\|_U^2\,dr$$

$$= \|P_T^{1/2}U(T,T-\varepsilon)x\|_H^2 + \int_{T-\varepsilon}^{T} \|M(r)^{1/2}U(r,T-\varepsilon)x\|_H^2\,dr - \|Q(T-\varepsilon)^{1/2}x\|_H^2\,.$$

As $\varepsilon \to 0^+$ the result follows, since in the right-hand side the integral term goes to 0 by Lemma 14, while both $P_T^{1/2}U(T,T-\varepsilon)x$ and $Q(T-\varepsilon)^{1/2}x$ converge in H to $P_T^{1/2}x$ as $\varepsilon \to 0^+$. □

The next result concerns the last integral term of (25).

Lemma 16. *We have*

$$\lim_{\varepsilon\to 0^+} \int_{T-\varepsilon}^{T} \|N(r)^{-1/2}G(r)^*A(r)^*Q(r)\Phi(r,T-\varepsilon)x\|_U^2\,dr = 0.$$

Proof. With $y = x$, we sum equation (16), minus twice equation (21), plus equation (22). The result is

$$\begin{aligned} 0 = {} & \|Q(T-\varepsilon)^{1/2}U(T-\varepsilon,s)x - Q(T-\varepsilon)^{1/2}\Phi(T-\varepsilon,s)x\|_H^2 \\ & + \int_s^{T-\varepsilon} \|M(r)^{1/2}U(r,s)x - M(r)^{1/2}\Phi(r,s)x\|_H^2\,dr \\ & - \int_s^{T-\varepsilon} \|N(r)^{-1/2}G(r)^*A(r)^*Q(r)U(r,s)x\|_U^2\,dr \\ & + \int_s^{T-\varepsilon} \|N(r)^{-1/2}G(r)^*A(r)^*Q(r)\Phi(r,s)x\|_U^2\,dr. \end{aligned} \tag{39}$$

Replace ε by δ. It follows that

$$\begin{aligned} & \int_s^{T-\delta} \|N(r)^{-1/2}G(r)^*A(r)^*Q(r)\Phi(r,s)x\|_U^2\,dr \\ & \qquad \le \int_s^{T-\delta} \|N(r)^{-1/2}G(r)^*A(r)^*Q(r)U(r,s)x\|_U^2\,dr. \end{aligned}$$

Letting $\delta \to 0^+$ and replacing s by $T-\varepsilon$, the result is a consequence of Lemma 15. □

In the next lemma we introduce a linear operator, whose importance is basic in the sequel. To this purpose it is useful to define:

$$\Pi : H \to H, \quad \Pi = \text{orthogonal projection onto } \overline{R(P_T^{1/2})}. \tag{40}$$

Lemma 17. *Let $s \in [0,T[$. There exists $C(s) \in \mathcal{L}(H)$, with range contained in $\overline{R(P_T^{1/2})}$, such that $\Pi Q(T-\varepsilon)^{1/2}\Phi(T-\varepsilon,s)x \rightharpoonup C(s)x$ in H for every $x \in H$, i.e.*

$$\lim_{\varepsilon\to 0^+} \langle Q(T-\varepsilon)^{1/2}\Phi(T-\varepsilon)x, y\rangle_H = \langle C(s)x, y\rangle_H \qquad \forall x \in H, \quad \forall y \in \overline{R(P_T^{1/2})}. \tag{41}$$

Proof. We start from (21). Taking into account Lemma 14, we get

$$\exists \lim_{\varepsilon\to 0^+} \langle Q(T-\varepsilon)^{1/2}\Phi(T-\varepsilon,s)x, Q(T-\varepsilon)^{1/2}U(T-\varepsilon,s)y\rangle_H$$
$$= \langle Q(s)x,y\rangle_H - \int_s^T \langle M(r)\Phi(r,s)x, U(r,s)y\rangle_H\, dr.$$

Replacing s by $T-\vartheta$, with $0<\varepsilon<\vartheta<T-s$, and x by $\Phi(T-\vartheta,s)x$, we obtain by (20)

$$\begin{aligned}&\exists \lim_{\varepsilon\to 0^+} \langle Q(T-\varepsilon)^{1/2}\Phi(T-\varepsilon,s)x, Q(T-\varepsilon)^{1/2}U(T-\varepsilon,T-\vartheta)y\rangle_H\\ &= \langle Q(T-\vartheta)\Phi(T-\vartheta,s)x,y\rangle_H - \int_{T-\vartheta}^T \langle M(r)\Phi(r,s)x, U(r,T-\vartheta)y\rangle_H\, dr.\end{aligned} \tag{42}$$

Now note that, by (24) and (25),

$$\|Q(T-\varepsilon)^{1/2}\Phi(T-\varepsilon,s)\|_{\mathcal{L}(H)} \le K \qquad \forall x\in H, \quad \forall s\in[0,T[\,, \quad \forall \varepsilon\in\,]0,T-s]; \tag{43}$$

hence, for each $x\in H$ and $s\in[0,T[$ we can find a sequence $\sigma=\{\sigma_k\}_{k\in\mathbb{N}}$, decreasing monotonically to 0, and an element $v(s,x,\sigma)\in H$ such that

$$\langle v(s,x,\sigma),y\rangle_H = \lim_{k\to\infty} \langle Q(T-\sigma_k)^{1/2}\Phi(t-\sigma_k,s)x,y\rangle_H \qquad \forall y\in H. \tag{44}$$

Going back to (42), as $Q(T-\varepsilon)^{1/2}U(T-\varepsilon,T-\vartheta)x \to P_T^{1/2}U(T,T-\vartheta)x$ in H when $\varepsilon\to 0^+$, we get for every $y\in H$

$$\begin{aligned}&\langle v(s,x,\sigma), P_T^{1/2}U(T,T-\vartheta)y\rangle_H\\ &\quad= \lim_{k\to\infty} \langle Q(T-\sigma_k)^{1/2}\Phi(t-\sigma_k)x, Q(T-\sigma_k)^{1/2}U(T-\sigma_k,T-\vartheta)y\rangle_H\\ &\quad= \lim_{\varepsilon\to 0^+} \langle Q(T-\varepsilon)^{1/2}\Phi(T-\varepsilon,s)x, Q(T-\varepsilon)^{1/2}U(T-\varepsilon,T-\vartheta)y\rangle_H\\ &\quad= \langle Q(T-\vartheta)\Phi(T-\vartheta,s)x,y\rangle_H - \int_{T-\vartheta}^T \langle M(r)\Phi(r,s)x, U(r,T-\vartheta)y\rangle_H\, dr.\end{aligned}$$

As a consequence, since the integral term, by Lemma 14, goes to 0 as $\vartheta\to 0^+$,

$$\exists \lim_{\vartheta\to 0^+} \langle Q(T-\vartheta)^{1/2}\Phi(T-\vartheta,s)x, Q(T-\vartheta)^{1/2}y\rangle_H = \langle v(s,x,\sigma), P_T^{1/2}y\rangle_H \qquad \forall y\in H,$$

and also, using (43) and strong convergence of $Q(\cdot)^{1/2}$ to $P_T^{1/2}$,

$$\exists \lim_{\vartheta\to 0^+} \langle Q(T-\vartheta)^{1/2}\Phi(T-\vartheta,s)x, P_T^{1/2}y\rangle_H = \langle v(s,x,\sigma), P_T^{1/2}y\rangle_H \qquad \forall y\in H.$$

By density, recalling again (43), we obtain

$$\exists \lim_{\vartheta \to 0^+} \langle Q(T-\vartheta)^{1/2}\Phi(T-\vartheta, s)x, z\rangle_H = \langle v(s,x,\sigma), z\rangle_H \qquad \forall z \in \overline{R(P_T^{1/2})}.$$

As the limit in the left-hand side is independent of the sequence σ and is linear with respect to x, we deduce that there is a bounded (by (43)), linear operator $\Gamma(s) : H \to H$ such that

$$\langle \Gamma(s)x, z\rangle_H = \langle v(s,x,\sigma), z\rangle_H = \lim_{\vartheta \to 0^+} \langle Q(T-\vartheta)^{1/2}\Phi(T-\vartheta, s)x, z\rangle_H \quad \forall z \in \overline{R(P_T^{1/2})}.$$

Finally, setting $C(s)x = \Pi\Gamma(s)x$ for every $x \in H$, it is clear that $\langle C(s)x, z\rangle_H = \langle \Gamma(s)x, z\rangle_H$ for every $z \in \overline{R(P_T^{1/2})}$, so that

$$\langle C(s)x, z\rangle_H = \langle v(s,x,\sigma), z\rangle_H = \lim_{\vartheta \to 0^+} \langle Q(T-\vartheta)^{1/2}\Phi(T-\vartheta, s)x, z\rangle_H \quad \forall z \in \overline{R(P_T^{1/2})}.$$

This proves the result. □

We now prove two basic lemmas relative to the behaviour of $C(\cdot)$ near T.

Lemma 18. *Let $C(s)$ be defined by (41). Then:*

(i) $\lim_{t \to T^-} \|C(t)x - P_T^{1/2}x\|_H = 0$ *for every* $x \in H$*;*

(ii) $\overline{R(P_T^{1/2})} = \overline{\bigcup_{0 \le t < T} R(C(t))}$.

Proof. In order to prove (i), as in the proof of Lemma 16, we arrive to (39). With $x = y$, δ in place of ε and $T-\varepsilon$ in place of s (with $\varepsilon > \delta$), we obtain

$$\begin{aligned}
0 &= \|Q(T-\delta)^{1/2}\Phi(T-\delta, T-\varepsilon)x - Q(T-\delta)^{1/2}U(T-\delta, T-\varepsilon)x\|_H^2 \\
&\quad + \int_{T-\varepsilon}^{T-\delta} \|M(r)^{1/2}[\Phi(r, T-\varepsilon)x - U(r, T-\varepsilon)x]\|_H^2 \, dr \\
&\quad + \int_{T-\varepsilon}^{T-\delta} \Big[\|N(r)^{-1/2}G(r)^*A(r)^+Q(r)\Phi(r, T-\varepsilon)x\|_U^2 \\
&\qquad - \|N(r)^{-1/2}G(r)^*A(r)^+Q(r)U(r, T-\varepsilon)x\|_U^2\Big] \, dr,
\end{aligned}$$

Letting $\delta \to 0^+$, we see that

$$\begin{aligned}
\exists \lim_{\delta\to 0^+} &\|Q(T-\delta)^{1/2}\Phi(T-\delta,T-\varepsilon)x - Q(T-\delta)^{1/2}U(T-\delta,T-\varepsilon)x\|_H^2 \\
&= -\int_{T-\varepsilon}^{T} \|M(r)^{1/2}[\Phi(r,T-\varepsilon)x - U(r,T-\varepsilon)x]\|_H^2\,dr \\
&\quad - \int_{T-\varepsilon}^{T} \Big[\|N(r)^{-1/2}G(r)^*A(r)^+Q(r)\Phi(r,T-\varepsilon)x\|_U^2 \\
&\qquad - \|N(r)^{-1/2}G(r)^*A(r)^+Q(r)U(r,T-\varepsilon)x\|_U^2\Big]\,dr.
\end{aligned}$$

By strong convergence of $Q(T-\delta)^{1/2}U(T-\delta,T-\varepsilon)x$ to $P_T^{1/2}U(T,T-\varepsilon)x$, we also have (the operator Π is defined in (40))

$$\begin{aligned}
&\limsup_{\delta\to 0^+} \|\Pi Q(T-\delta)^{1/2}\Phi(T-\delta,T-\varepsilon)x - P_T^{1/2}U(T,T-\varepsilon)x\|_H^2 \\
&\le \lim_{\delta\to 0^+} \|Q(T-\delta)^{1/2}\Phi(T-\delta,T-\varepsilon)x - P_T^{1/2}U(T,T-\varepsilon)x\|_H^2 \\
&\quad = -\int_{T-\varepsilon}^{T} \|M(r)^{1/2}[\Phi(r,T-\varepsilon)x - U(r,T-\varepsilon)x]\|_H^2\,dr \\
&\quad\quad - \int_{T-\varepsilon}^{T} \Big[\|N(r)^{-1/2}G(r)^*A(r)^+Q(r)\Phi(r,T-\varepsilon)x\|_U^2 \\
&\qquad\quad - \|N(r)^{-1/2}G(r)^*A(r)^+Q(r)U(r,T-\varepsilon)x\|_U^2\Big]\,dr.
\end{aligned}$$

By (41) we get

$$\begin{aligned}
&\|C(T-\varepsilon)x - P_T^{1/2}U(T,T-\varepsilon)x\|_H^2 \\
&\quad \le -\int_{T-\varepsilon}^{T} \|M(r)^{1/2}[\Phi(r,T-\varepsilon)x - U(r,T-\varepsilon)x]\|_H^2\,dr \\
&\quad\quad - \int_{T-\varepsilon}^{T} \Big[\|N(r)^{-1/2}G(r)^*A(r)^+Q(r)\Phi(r,T-\varepsilon)x\|_U^2 \\
&\qquad\quad - \|N(r)^{-1/2}G(r)^*A(r)^+Q(r)U(r,T-\varepsilon)x\|_U^2\Big]\,dr.
\end{aligned}$$

Finally, we let $\varepsilon \to 0^+$: by Lemmas 14, 16 and 15 we educe that

$$\lim_{\varepsilon\to 0^+} \|C(T-\varepsilon)x - P_T^{1/2}U(T,T-\varepsilon)x\|_H^2 = 0;$$

as $P_T^{1/2}U(T,T-\varepsilon)x \to P_T^{1/2}x$ in H, we conclude that $C(T-\varepsilon)x \to P_T^{1/2}x$, thus proving (i).
Concerning (ii), the inclusion $\supseteq$ is easy, since Lemma 17 implies that

$$\bigcup_{0\le t<T} R(C(t)) \subseteq \overline{R(P_T^{1/2})}.$$

To prove the reverse inclusion, fix $z \in \overline{R(P_T^{1/2})}$ and $\varepsilon > 0$: then there exists $x \in H$ such that $\|P_T^{1/2}x - z\|_H < \varepsilon$. By (i), $C(t)x \to P_T^{1/2}x$ as $t \to T^-$, so that there is $t_0 \in [0, T[$ such that

$$\|C(t)x - P_T^{1/2}x\|_H < \varepsilon \qquad \forall t \in [t_0, T[\,.$$

This shows that

$$\overline{R(P_T^{1/2})} \subseteq \overline{\bigcup_{0 \le t_0 < T} \bigcap_{t_o \le t < T} R(C(t))}.$$

However, if $t_0 \le t < t' < T$ it holds, by (20),

$$\begin{aligned} C(t)x &= \text{w-}\lim_{\varepsilon \to 0^+} Q(T-\varepsilon)^{1/2}\Phi(T-\varepsilon, t)x \\ &= \text{w-}\lim_{\varepsilon \to 0^+} Q(T-\varepsilon)^{1/2}\Phi(T-\varepsilon, t')\Phi(t', t)x = C(t')\Phi(t', t)x, \end{aligned}$$

so that $R(C(t)) \subseteq R(C(t'))$. As a consequence,

$$\overline{R(P_T^{1/2})} \subseteq \overline{\bigcup_{0 \le t < T} R(C(t))}.$$

□

Lemma 19. *Let $B(s)$ and $C(s)$ be defined by (26) and (41). Then for every $x, y \in H$, $s \in [0, T[$ and $t \in [s, T[$ we have*

$$\lim_{\varepsilon \to 0^+} \left| \langle C(t)x, P_T^{1/2}U(T, T-\varepsilon)\Phi(T-\varepsilon, s)y\rangle_H - \langle B(t)x, \Phi(t, s)y\rangle_H \right| = 0. \tag{45}$$

Proof. We fix $s \in [0, T[\,$, $t \in [s, T[$ and $x, y \in H$. Let us analyze the quantity to be estimated. To begin with, we have for $\varepsilon \in]0, T-t[$

$$\begin{aligned} &\left| \langle C(t)x, P_T^{1/2}U(T, T-\varepsilon)\Phi(T-\varepsilon, s)y\rangle_H - \langle B(t)x, \Phi(t, s)y\rangle_H \right| \\ &\le \left| \langle C(t)x, P_T^{1/2}U(T, T-\varepsilon)\Phi(T-\varepsilon, s)y\rangle_H - \langle Q(T-\varepsilon)\Phi(T-\varepsilon, t)x, \Phi(T-\varepsilon, s)y\rangle_H \right| \\ &\qquad + \left| \langle Q(T-\varepsilon)\Phi(T-\varepsilon, t)x, \Phi(T-\varepsilon, s)y\rangle_H - \langle B(t)x, \Phi(t, s)y\rangle_H \right|. \end{aligned} \tag{46}$$

In order to estimate the first term on the right-hand side of (46), we start from equation (21): we first replace there ε by δ and then choose $T-\varepsilon$ (with $0 < \delta < \varepsilon$) in place of s, $\Phi(T-\varepsilon, t)x$ in place of x and $\Phi(T-\varepsilon, s)y$ in place of y. Then we find, using (20),

$$\begin{aligned} &\langle Q(T-\varepsilon)\Phi(T-\varepsilon, t)x, \Phi(T-\varepsilon, s)y\rangle_H \\ &\quad = \langle Q(T-\delta)^{1/2}\Phi(T-\delta, t)x, Q(T-\delta)^{1/2}U(T-\delta, T-\varepsilon)\Phi(T-\varepsilon, s)y\rangle_H \\ &\qquad + \int_{T-\varepsilon}^{T-\delta} \langle M(r)\Phi(r, t)x, U(r, T-\varepsilon)\Phi(T-\varepsilon, s)y\rangle_H \, dr. \end{aligned} \tag{47}$$

As $\delta \to 0^+$, by strong convergence of $Q(T-\delta)^{1/2}U(T-\delta, T-\varepsilon)y$ to $P_T^{1/2}U(T, T-\varepsilon)y$ and by (43), (41), we deduce

$$
\begin{aligned}
&\left|\langle Q(T-\varepsilon)\Phi(T-\varepsilon,t)x, \Phi(T-\varepsilon,s)y\rangle_H - \langle C(t)x, P_T^{1/2}U(T,T-\varepsilon)\Phi(T-\varepsilon,s)y\rangle_H\right| \\
&\quad = \left|\int_{T-\varepsilon}^T \langle M(r)\Phi(r,t)x, U(r,T-\varepsilon)\Phi(T-\varepsilon,s)y\rangle_H \, dr\right| \\
&\quad = \left|\int_{T-\varepsilon}^T \langle M(r)\Phi(r,t)x, y_\varepsilon(r;s,y)\rangle_H \, dr\right| \\
&\quad \le K_0 \|\bar{y}(\cdot;t,x)\|_{L^2(T-\varepsilon,T;H)} \|y_\varepsilon(\cdot;s,y)\|_{L^2(s,T;H)}
\end{aligned}
\tag{48}
$$

where K_0 is a constant; since $y_\varepsilon(\cdot;s,y)$ is bounded in $L^2(s,T;H)$ by a constant K_y, we proceed as in the proof of Lemma 14, and we deduce

$$
\begin{aligned}
&\left|\int_{T-\varepsilon}^T \langle M(r)\Phi(r,t)x, U(r,T-\varepsilon)\Phi(T-\varepsilon,s)y\rangle_H \, dr\right| \le \\
&\qquad \le K_0 K_y \, \varepsilon^\alpha \|\bar{y}(\cdot;t,x)\|_{L^{\frac{2}{1-2\alpha}}(T-\varepsilon,T;H)} \le C K_y \varepsilon^\alpha \|x\|_H.
\end{aligned}
\tag{49}
$$

The second term in the right-hand side of (46) can be estimated by subtracting (27) from (22), having replaced in both equations s by t and y by $\Phi(t,s)y$: indeed, we have, using (20), (31) and (33),

$$
\begin{aligned}
0 = {} & \langle Q(T-\varepsilon)\Phi(T-\varepsilon,t)x, \Phi(T-\varepsilon,s)y\rangle_H - \langle B(t)x, \Phi(t,s)y\rangle_H - \\
& - \int_{T-\varepsilon}^T \langle M(r)\Phi(r,t)x, \Phi(r,s)y\rangle_H \, dr - \int_{T-\varepsilon}^T \langle N(r)\bar{u}(r;t,x), \bar{u}(r;s,y)\rangle_U \, dr.
\end{aligned}
$$

Hence, proceeding as before, we get

$$
\begin{aligned}
&\left|\langle Q(T-\varepsilon)\Phi(T-\varepsilon,t)x, \Phi(T-\varepsilon,s)y\rangle_H - \langle B(t)x, \Phi(t,s)y\rangle_H\right| \le \\
&\quad \le K\varepsilon^{2\alpha} \|\bar{y}(\cdot;t,x)\|_{L^{\frac{1}{1-2\alpha}}(T-\varepsilon,T;H)} \|\bar{y}(\cdot;s,y)\|_{L^{\frac{1}{1-2\alpha}}(T-\varepsilon,T;H)} + \\
&\qquad + K\|\bar{u}(\cdot;t,x)\|_{L^2(T-\varepsilon,T;U)} \|\bar{u}(\cdot;s,y)\|_{L^2(T-\varepsilon,T;U)} \\
&\quad \le K\|x\|_H \left(\varepsilon^{2\alpha}\|y\|_H + \omega_y(\varepsilon)\right),
\end{aligned}
\tag{50}
$$

where K is an absolute constant and $\omega_y(\varepsilon)$ decreases monotonically to 0 as $\varepsilon \to 0^+$. By (46), (48) and (50) we obtain the desired conclusion. □

The following statement is an important variant of the preceding lemma.

Lemma 20. *Let $C(s)$ be defined by (41). Then for every $s \in [0,T]$ we have*

$$
\lim_{\varepsilon \to 0^+} \langle C(T-\varepsilon)x, P_T^{1/2}U(T,T-\varepsilon)\Phi(T-\varepsilon,s)y\rangle_H = \langle P_T^{1/2}x, C(s)y\rangle_H \qquad \forall x, y \in H.
$$

Proof. We go back once again to (21). We replace ε by δ (with $0 < \delta < \varepsilon$), s by $T-\varepsilon$ and y by $\Phi(T-\varepsilon, s)y$: we get

$$\begin{aligned}
&\langle Q(T-\varepsilon)x, \Phi(T.\varepsilon, s)y\rangle_H \\
&= \langle Q(T-\delta)^{\frac{1}{2}}\Phi(T-\delta, T-\varepsilon)x, Q(T-\delta)^{\frac{1}{2}}U(T-\delta, T-\varepsilon)\Phi(T-\varepsilon, s)y\rangle_H \\
&\qquad + \int_{T-\varepsilon}^{T-\delta} \langle M(r)\Phi(r, T-\varepsilon)x, U(r, T-\varepsilon)\Phi(T-\varepsilon, s)y\rangle_H \, dr,
\end{aligned}$$

and by (43), strong convergence of $Q(T-\delta)^{\frac{1}{2}}U(T-\delta, T-\varepsilon)$ and (41), as $\delta \to 0^+$,

$$\begin{aligned}
&\langle Q(T-\varepsilon)^{\frac{1}{2}}x, Q(T-\varepsilon)^{\frac{1}{2}}\Phi(T-\varepsilon, s)y\rangle_H \\
&= \langle C(T-\varepsilon)x, P_T^{1/2}U(T, T-\varepsilon)\Phi(T-\varepsilon, s)y\rangle_H \\
&\qquad + \int_{T-\varepsilon}^{T} \langle M(r)\Phi(r, T-\varepsilon)x, U(r, T-\varepsilon)\Phi(T-\varepsilon, s)y\rangle_H \, dr.
\end{aligned}$$

The last integral tends to 0 as $\varepsilon \to 0^+$ by Lemma 14, arguing as in (49). Thus, using again strong convergence of $Q(\cdot)^{\frac{1}{2}}$ and (43), we conclude that

$$\begin{aligned}
&\lim_{\varepsilon \to 0^+} \langle C(T-\varepsilon)x, P_T^{1/2}U(T, T-\varepsilon)\Phi(T-\varepsilon, s)y\rangle_H \\
&= \lim_{\varepsilon \to 0^+} \langle Q(T-\varepsilon)^{\frac{1}{2}}x, Q(T-\varepsilon)^{\frac{1}{2}}\Phi(T-\varepsilon, s)y\rangle_H = \langle P_T^{1/2}x, C(s)y\rangle_H \,,
\end{aligned}$$

which is our claim. □

The following lemma is crucial, in order to obtain the key relation $\bar{u}(\cdot; s, x) \in D(L_{sT})$.

Lemma 21. *Let $\bar{u}(\cdot; s, x)$ and $C(s)$ be defined by (31) and (41). For every $x \in H$ and $s \in [0, T[$ we have $P_T^{1/2}C(s)x \in D(L_{sT}^*)$ and*

$$\bar{u}(\cdot, s; x) = -N(\cdot)^{-1}L_{sT}^* P_T^{1/2}C(s)x - N(\cdot)^{-1}L_s^*[M(\cdot)\Phi(\cdot, s)x]. \tag{51}$$

Proof. We start from the integral equation (21), which can be rewritten in strong form, with s replaced by t and x replaced by $\Phi(t, s)x$, as

$$Q(t)\Phi(t, s)x = U(T-\varepsilon, t)^*Q(T-\varepsilon)\Phi(T-\varepsilon, s)x + \int_t^{T-\varepsilon} U(r, t)^*M(r)\Phi(r, s)x \, dr.$$

We operate with $G(t)^*A(t)^*$ in both members: by (31) we obtain

$$\begin{aligned}
N(t)\bar{u}(t, s, x) &= G(t)^*A(t)^*Q(t)\Phi(t, s)x \\
&= G(t)^*A(t)^*U(T-\varepsilon, t)^*Q(T-\varepsilon)\Phi(T-\varepsilon, s)x \\
&\qquad + \int_t^{T-\varepsilon} G(t)^*A(t)^*U(r, t)^*M(r)\Phi(r, s)x \, dr.
\end{aligned} \tag{52}$$

It is easily seen that the second term in the last member of (52) is exactly the function $-L_s^*[\chi_{[s,T-\varepsilon]}M(\cdot)\Phi(\cdot,s)x]$ evaluated in t, which converges as $\varepsilon \to 0^*$ to $-L_s^*[M(\cdot)\Phi(\cdot,s)x]$ in $L^2(s,T;U)$: indeed, by [1, Lemma 4.4(iv)],

$$\int_s^T \left\| L_s^*[\chi_{[s,T-\varepsilon]}M(\cdot)\Phi(\cdot,s)x](t) - L_s^*[M(\cdot)\Phi(\cdot,s)x](t)\right\|_U^2 \, dt$$
$$= \int_s^T \left\| L_s^*[\chi_{[T-\varepsilon,T]}M(\cdot)\Phi(\cdot,s)x]](t)\right\|_U^2 \, dt \le \left[\int_{T-\varepsilon}^T \|M(r)\Phi(r,s)x\|_u^q \, dr\right]^{\frac{2}{q}},$$

with $q = \frac{2}{1-2\alpha}$, and the last quantity goes to 0 as $\varepsilon \to 0^+$.
By difference, the first term in the last member of (52) belongs to $L^2(s,T;U)$ and it converges to $G(\cdot)^*A(\cdot)^*U(T,\cdot)^*P_T^{1/2}C(s)x$ in $L^2(s,T;U)$. By definition (see (13)), this means $P_T^{1/2}C(s)x \in D(L_{sT}^*)$ and formula (51) follows. □

We now state the key result of this proof.

Proposition 22. *Let $\bar{u}(\cdot;s,x)$ and $C(s)$ be defined by (31) and (41). For $0 \le s < T$ and $x \in H$ we have $\bar{u}(\cdot;s,x) \in D(P_T^{1/2}L_{sT})$ and*

$$C(s)x = P_T^{1/2}U(T,s)x + P_T^{1/2}L_{sT}\bar{u}(\cdot;s,x).$$

Proof. First of all, we note that, by Lemma 21 and (13), it holds $P_T^{1/2}C(t)x \in D(L_{tT}^*) \subseteq D(L_{sT}^*)$ for every $s \in [0,t]$. Thus, recalling (34), we start from the limit in (45) and rewrite it as

$$\begin{aligned}\langle B(t)x, \Phi(t,s)y\rangle_H &= \lim_{\varepsilon\to 0^+} \langle C(t)x, P_T^{1/2}U(T,s)y + P_T^{1/2}L_{sT}u_\varepsilon(\cdot;s,x)\rangle_H \\ &= \langle C(t)x, P_T^{1/2}U(T,s)y\rangle_H + \lim_{\varepsilon\to 0^+} \langle L_{sT}^*P_T^{1/2}C(t)x, u_\varepsilon(\cdot;s,y)\rangle_{L^2(s,T;U)} \\ &= \langle C(t)x, P_T^{1/2}U(T,s)y\rangle_H + \langle L_{sT}^*P_T^{1/2}C(t)x, \bar{u}(\cdot;s,y)\rangle_{L^2(s,T;U)} \\ &= \langle x, C(t)^*P_T^{1/2}U(T,s)y + [L_{sT}^*P_T^{1/2}C(t)]^*\bar{u}(\cdot;s,y)\rangle_H .\end{aligned}$$

By Lemma 21, the operator $L_{sT}^*P_T^{1/2}C(t)$ is closed with domain H; hence it is bounded, and of course its adjoint $[L_{sT}^*P_T^{1/2}C(t)]^*$ is bounded in H, too. Since $x \in H$ is arbitrary and $B(t) = B(t)^*$, we deduce

$$B(t)\Phi(t,s)y = C(t)^*P_T^{1/2}U(T,s)y + [L_{sT}^*P_T^{1/2}C(t)]^*\bar{u}(\cdot;s,y).$$

We note that the operator $L_{sT}^*P_T^{1/2}$ is obviously closed and its domain

$$D(L_{sT}^*P_T^{1/2}) \supseteq \left[\bigcup_{s\le t<T} R(C(t))\right] \cup \ker P_T^{1/2}$$

is dense in H by Lemma 18(ii); moreover $C(t)$ is bounded. So we may apply [23, Theorem 13.2], obtaining $[L_{sT}^* P_T^{1/2} C(t)]^* = C(t)^* [L_{sT}^* P_T^{1/2}]^*$. Thus we may write

$$B(t)\Phi(t,s)y = C(t)^* \left[P_T^{1/2} U(T,s)y + [L_{sT}^* P_T^{1/2}]^* \bar{u}(\cdot; s, y) \right]; \tag{53}$$

In particular, we observe that

$$L^2(s;T;U) = D([L_{sT}^* P_T^{1/2} C(t)]^*) = D(C(t)^* [L_{sT}^* P_T^{1/2}]^*) \subseteq D([L_{sT}^* P_T^{1/2}]^*).$$

It is easy to verify that the operator $[L_{sT}^* P_T^{1/2}]^*$ is closed; since it has the whole space $L^2(s,T;U)$ as its domain, it must be bounded, i.e. $[L_{sT}^* P_T^{1/2}]^* \in \mathcal{L}(L^2(s,T;U), H)$. Moreover, since $P_T^{1/2} \in \mathcal{L}(H)$ and L_{sT} is densely defined, again by [23, Theorem 13.2] we have $[L_{sT}^* P_T^{1/2}]^* = [(P_T^{1/2} L_{sT})^*]^*$; so, $[L_{sT}^* P_T^{1/2}]^*$ is a bounded extension of the closed operator $P_T^{1/2} L_{sT}$. Using this information, we may write, as $\varepsilon \to 0^+$,

$$\begin{aligned} P_T^{1/2} L_{sT} u_\varepsilon = P_T^{1/2} U(T, T-\varepsilon)\Phi(T-\varepsilon, s)x &= P_T^{1/2} U(t,s)x + P_T^{1/2} L_{sT} u_\varepsilon(\cdot; s, x) \\ &= P_T^{1/2} U(t,s)x + [L_{sT}^* P_T^{1/2}]^* u_\varepsilon(\cdot; s, x) \\ &\to P_T^{1/2} U(t,s)x + [L_{sT}^* P_T^{1/2}]^* \bar{u}(\cdot; s, x). \end{aligned}$$

Thus,

$$u_\varepsilon(\cdot; s, x) \to \bar{u}(\cdot; s, x) \text{ in } L^2(s,T;U), \quad P_T^{1/2} L_{sT} u_\varepsilon(\cdot; s, x) \to [L_{sT}^* P_T^{1/2}]^* \bar{u}(\cdot; s, x) \text{ in } H;$$

by Hypothesis 7, $\bar{u}(\cdot; s; x) \in D(L_{sT})$ and $P_T^{1/2} L_{sT} \bar{u}(\cdot; s, x) = [L_{sT}^* P_T^{1/2}]^* \bar{u}(\cdot; s, x)$.
Set momentarily

$$E(s)x = \lim_{\varepsilon \to 0^+} P_T^{1/2} U(T, T-\varepsilon)\Phi(T-\varepsilon, s)x.$$

By (45) with $t = s$, using Lemma 20, we deduce for every $x \in H$

$$\langle P_T^{1/2} x, C(s)y \rangle_H = \lim_{\varepsilon \to 0^+} \langle C(T-\varepsilon)x, P_T^{1/2} U(T, T-\varepsilon)\Phi(T-\varepsilon, s)y \rangle_H = \langle P_T^{1/2} x, E(s)y \rangle_H .$$

Since $E(s)$ and $C(s)$ are bounded operators, by density we have

$$\langle z, C(s)y \rangle_H = \langle z, E(s)y \rangle_H \qquad \forall z \in \overline{R(P_T^{1/2})}.$$

As both $E(s)$ and $C(s)$ have range contained in $\overline{R(P_T^{1/2})}$, we deduce $E(s) = C(s)$, i.e.

$$P_T^{1/2} U(T, T-\varepsilon)\Phi(T-\varepsilon, s)x \to C(s)x \quad \text{in } H \quad \text{as } \varepsilon \to 0^+,$$

and the result follows. □

Corollary 23. *Let $C(s)$ and $B(s)$ be defined by (41) and (26). For every $s \in [0,T[$ we have the equality*

$$B(s) = C(s)^*C(s);$$

consequently

$$Q(T-\varepsilon)^{1/2}\Phi(T-\varepsilon,s)x \to C(s)x \quad \textit{in } H \quad \textit{as } \varepsilon \to 0^+.$$

Proof. Indeed, by Lemma 19 with $t = s$, we have for every $x, y \in H$,

$$\langle B(s)x, y\rangle_H = \lim_{\varepsilon\to 0^+} \langle C(s)x, P_T^{1/2} U(T, T-\varepsilon)\Phi(T-\varepsilon, s)y\rangle_H;$$

hence, by Proposition 22,

$$\langle B(s)x, y\rangle_H = \langle C(s)x, C(s)y\rangle_H \qquad \forall x, y \in H,$$

and this means $B(s) = C(s)^*C(s)$. In particular, we deduce (by Lemma 17) that $\Pi Q(T-\varepsilon)^{1/2}\Phi(T-\varepsilon,s)x \rightharpoonup C(s)x$ as $\varepsilon \to 0^+$, and

$$\begin{aligned}\lim_{\varepsilon\to 0^+} \|Q(T-\varepsilon)^{1/2}\Phi(T-\varepsilon,s)x\|_H^2 &= (B(s)x,x)_H = \|C(s)x\|_H^2 \\ &\le \liminf_{\varepsilon\to 0^+} \|\Pi Q(T-\varepsilon)^{1/2}\Phi(T-\varepsilon,s)x\|_H^2 .\end{aligned}$$

Hence, $\Pi Q(T-\varepsilon)^{1/2}\Phi(T-\varepsilon,s)x \rightharpoonup C(s)x$ and $\|\Pi Q(T-\varepsilon)^{1/2}\Phi(T-\varepsilon,s)x\|_H^2 \to \|C(s)x\|_H^2$ as $\varepsilon \to 0^*$: it foollows that

$$\|\Pi Q(T-\varepsilon)^{1/2}\Phi(T-\varepsilon,s)x - C(s)x\|_H \to 0, \qquad \|(I-\Pi)Q(T-\varepsilon)^{1/2}\Phi(T-\varepsilon,s)x\|_H^2 \to 0,$$

and finally $Q(T-\varepsilon)^{1/2}\Phi(T-\varepsilon,s)x \to C(s)x$ in H. □

We can now conclude the proof of Theorem 8 **(b)**. In fact, since $\bar{u}(\cdot; s, x) \in D(L_{sT})$, we have, recalling (25) (and writing $\bar{u}(\cdot; s, x) = \bar{u}$ and $\bar{y}(\cdot; s, x) = \bar{y}$)

$$\begin{aligned}\langle P(s)x, x\rangle_H &= J_s(\widehat{u}(\cdot; s, x)) \le J_s(\bar{u}(\cdot; s, x)) \\ &= \int_s^T \langle M(r)\bar{y}(r), \bar{y}(r)\rangle_H\, dr + \int_s^T \langle N(r)\bar{u}(r), \bar{u}(r)\rangle_U\, dr + \langle B(s)x, x\rangle_H \\ &= \langle Q(s)x, x\rangle_H .\end{aligned}$$

This proves **(b)** in (37).

Proof of Theorem 9. The argument is very easy: $P(\cdot)$ is the unique solution of the Riccati equation; hence $J_s(\widehat{u}) = (P(s)x,x)_H = (Q(s)x,x)_H = J_s(\bar{u})$, with $\widehat{u}$ given by (29) and $\bar{u}$ given by (31). As the optimal control is unique, we have $\widehat{u} = \bar{u}$, so that the optimal state $\widehat{y} = \widehat{\Phi}(\cdot, s)x$ coincides with $\bar{y} = \Phi(\cdot, s)x$, too. On the other hand, by Corollary 23, $t \mapsto Q(t)^{1/2}\Phi(t,s)x = P(t)^{1/2}\widehat{\Phi}(t,s)x$ is continuous in $[s,T]$ with $P_T^{1/2}\Phi(T,s)x = C(s)x = P_T^{1/2}[U(T,s)x + L_{sT}\bar{u}(\cdot; s, x)] = P_T^{1/2}[U(T,s)x + L_{sT}\widehat{u}(\cdot; s, x)] = P_T^{1/2}\widehat{\Phi}(T,s)x$. The proof is complete.

Declarations

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- The authors contributed equally to this work.

Appendix A Proof of Lemma 10

Let us split the terms on the right-hand side of equation (17). Recalling (8), we have $y(t) = U(t,s)x + [L_s u](t)$, with L_s given by (9), and hence

$$
\begin{aligned}
&-\int_s^{T-\varepsilon} \|M(r)^{1/2} y(r)\|_H^2 \, dr = -\int_s^{T-\varepsilon} \|M(r)^{1/2} U(r,s)x\|_H^2 \, dr \\
&-2\mathrm{Re}\int_s^{T-\varepsilon} \langle M(r)U(r,s)x, [L_s u](r)\rangle_H \, dr - \int_s^{T-\varepsilon} \|M(r)^{1/2}[L_s u](r)\|_H^2 \, dr \\
&=: M_1 + M_2 + M_3.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
&\int_s^{T-\varepsilon} \|N(r)^{1/2}u(r) - N(r)^{-1/2}G(r)^*A(r)^*Q(r)y(r)\|_U^2 \, dr \\
&= \int_s^{T-\varepsilon} \|N(r)^{1/2}u(r)\|_U^2 \, dr - 2\mathrm{Re}\int_s^{T-\varepsilon} \langle u(r), G(r)^*A(r)^*Q(r)y(r)\rangle_U \, dr \\
&+ \int_s^{T-\varepsilon} \|N(r)^{-1/2}G(r)^*A(r)^*Q(r)y(r)\|_U^2 \, dr,
\end{aligned}
$$

and using again (8) we write

$$
\begin{aligned}
&\int_s^{T-\varepsilon} \|N(r)^{1/2}u(r) - N(r)^{-1/2}G(r)^*A(r)^*Q(r)y(r)\|_U^2 \, dr \\
&= \int_s^{T-\varepsilon} \|N(r)^{1/2}u(r)\|_U^2 \, dr - 2\mathrm{Re}\int_s^{T-\varepsilon} \langle u(r), G(r)^*A(r)^*Q(r)U(r,s)x\rangle_U \, dr \\
&-2\mathrm{Re}\int_s^{T-\varepsilon} \langle u(r), G(r)^*A(r)^*Q(r)[L_s u](r)\rangle_U \, dr \\
&+ \int_s^{T-\varepsilon} \|N(r)^{-1/2}G(r)^*A(r)^*Q(r)U(r,s)x\|_U^2 \, dr \\
&+2\mathrm{Re}\int_s^{T-\varepsilon} \langle N(r)^{-1}G(r)^*A(r)^*Q(r)U(r,s)x, G(r)^*A(r)^*Q(r)[L_s u](r)\rangle_U \, dr \\
&+ \int_s^{T-\varepsilon} \|N(r)^{-1/2}G(r)^*A(r)^*Q(r)[L_s u](r)\|_U^2 \, dr
\end{aligned}
$$

$$=: N_1 + N_2 + N_3 + N_4 + N_5 + N_6 \,.$$

Hence in the right member of (17) the term N_1 cancels and we get

$$\begin{aligned} &-\int_s^{T-\varepsilon} \|M(r)^{1/2}y(r)\|_H^2\, dr - \int_s^{T-\varepsilon} \|N(r)^{1/2}u(r)\|_U^2\, dr + \\ &\quad + \int_s^{T-\varepsilon} \|N(r)^{1/2}u(r) - N(r)^{-1/2}G(r)^*A(r)^*Q(r)y(r)\|_U^2\, dr = \\ &\quad = M_1 + M_2 + M_3 + N_2 + N_3 + N_4 + N_5 + N_6 \,. \end{aligned} \tag{A1}$$

Now we note that, by (15) with $x = y$,

$$\begin{aligned} M_1 + N_4 &= -\int_s^{T-\varepsilon} \|M(r)^{1/2}U(r,s)x\|_H^2\, dr \\ &+ \int_s^{T-\varepsilon} \|N(r)^{-1/2}G(r)^*A(r)^*Q(r)U(r,s)x\|_U^2\, dr \\ &= -\langle Q(s)x, x\rangle_H + \langle Q(T-\varepsilon)U(T-\varepsilon,s)x, U(T-\varepsilon,s)x\rangle_H \,. \end{aligned}$$

Next, we set

$$L_s^{*\varepsilon} v](t) = -G(t)^*A(t)^* \int_t^{T-\varepsilon} U(\sigma,t)^* v(\sigma)\, d\sigma, \quad v \in L^2(s, T-\varepsilon; H); \tag{A2}$$

this is the adjoint of $L_s : L^2(s, T-\varepsilon; U) \to L^2(s, T-\varepsilon; H)$. For this operator we need the following property:

Lemma 24. *Let L_s, L_s^ε be given by (9), (A2). If $\Psi \in L^\infty(s, T-\varepsilon; \Sigma(H))$, and $v \in L^2(s, T-\varepsilon; U)$, then we have*

$$\begin{aligned} &\int_s^{T-\varepsilon} \langle [L_s^{*\varepsilon}\Psi(\cdot)L_s v](r), v(r)\rangle_U\, dr \\ &= -2\mathrm{Re} \int_s^{T-\varepsilon} \left\langle G(q)^*A(q)^* \left[\int_q^{T-\varepsilon} U(r,q)^*\Psi(r)U(r,q) dr\right] [L_s v](q), v(q) \right\rangle_U dq. \end{aligned}$$

Proof. Indeed we have

$$\begin{aligned} &\int_s^{T-\varepsilon} \langle [L_s^{*\varepsilon}\Psi(\cdot)L_s v](r), v(r)\rangle_U\, dr = \int_s^{T-\varepsilon} \langle \Psi(r)[L_s v](r), [L_s v](r)\rangle_H \\ &= \int_s^{T-\varepsilon} \int_s^r \int_s^r \langle \Psi(r)U(r,\sigma)A(\sigma)G(\sigma)v(\sigma), U(r,q)A(q)G(q)v(q)\rangle_H\, d\sigma dq dr. \end{aligned}$$

The integrand in the last term is a symmetric function $a(\sigma,q)$: we rewrite this term as

$$\begin{aligned}
&\int_s^{T-\varepsilon}\int_s^r\int_s^r a(\sigma,q)\,d\sigma dq dr\\
&\quad=\int_s^{T-\varepsilon}\int_s^r\left[\int_s^q a(\sigma,q)\,d\sigma+\int_q^r a(\sigma,q)\,d\sigma\right]dqdr\\
&\quad=\int_s^{T-\varepsilon}\left[\int_s^r\int_s^q a(\sigma,q)\,d\sigma dq+\int_s^r\int_s^\sigma a(\sigma,q)\,dqd\sigma\right]dr\\
&\quad=2\mathrm{Re}\int_s^{T-\varepsilon}\int_s^r\int_s^q a(\sigma,q)\,d\sigma dq dr,
\end{aligned}$$

so that

$$\begin{aligned}
&\int_s^{T-\varepsilon}\langle[L_s^{*\varepsilon}\Psi(\cdot)L_s v](r),v(r)\rangle_U\,dr\\
&=2\mathrm{Re}\int_s^{T-\varepsilon}\int_s^r\int_s^q\langle\Psi(r)U(r,\sigma)A(\sigma)G(\sigma)v(\sigma),U(r,q)A(q)G(q)v(q)\rangle_H\,d\sigma dq dr\\
&=2\mathrm{Re}\int_s^{T-\varepsilon}\int_q^{T-\varepsilon}\left\langle G(q)^*A(q)^*U(r,q)^*\Psi(r)U(r,q)\right.\\
&\qquad\left.\times\int_s^q U(q,\sigma)A(\sigma)G(\sigma)v(\sigma)\,d\sigma,v(q)\right\rangle_U drdq\\
&=-2\mathrm{Re}\int_s^{T-\varepsilon}\left\langle G(q)^*A(q)^*\left[\int_q^{T-\varepsilon}U(r,q)^*\Psi(r)U(r,q)\,dr\right][L_s v](q),v(q)\right\rangle_U dq.
\end{aligned}$$

□

Let us go back to (A1): we have

$$\begin{aligned}
&M_2+N_5+N_2=-2\mathrm{Re}\int_s^{T-\varepsilon}\langle M(r)U(r,s)x,[L_s u](r)\rangle_H\,dr\\
&\quad+2\mathrm{Re}\int_s^{T-\varepsilon}\langle N(r)^{-1}G(r)^*A(r)^*Q(r)U(r,s)x,G(r)^*A(r)^*Q(r)[L_s u](r)\rangle_U\,dr\\
&\quad-2\mathrm{Re}\int_s^{T-\varepsilon}\langle u(r),G(r)^*A(r)^*Q(r)U(r,s)x\rangle_U\,dr\\
&\;=-2\mathrm{Re}\int_s^{T-\varepsilon}\langle[L_s^{*\varepsilon}M(\cdot)U(\cdot,s)x](r),u(r)\rangle_U\,dr\\
&\quad+2\mathrm{Re}\int_s^{T-\varepsilon}\left\langle[L_s^{*\varepsilon}Q(\cdot)A(\cdot)G(\cdot)N(\cdot)^{-1}G(\cdot)^*A(\cdot)^*Q(\cdot)U(\cdot,s)x](r),u(r)\right\rangle_U\,dr\\
&\quad-2\mathrm{Re}\int_s^{T-\varepsilon}\langle G(r)^*A(r)^*Q(r)U(r,s)x,u(r)\rangle_U\,dr.
\end{aligned}$$

Using (A2) and (15) we find

$$M_2 + N_5 + N_2 = 2\mathrm{Re} \int_s^{T-\varepsilon} \left\langle G(r)^* A(r)^* \left[\int_r^{T-\varepsilon} \big[U(\sigma,r)^* M(\sigma) U(\sigma,r) \right.\right.$$

$$-U(\sigma,r)^* Q(\sigma) A(\sigma) G(\sigma) N(\sigma)^{-1} G(\sigma)^* A(\sigma)^* Q(\sigma) U(\sigma,r)\big]\, d\sigma$$

$$\left.\left. -Q(r) \right] U(r,s)x, u(r) \right\rangle_U dr$$

$$= -2\mathrm{Re} \int_s^{T-\varepsilon} \langle G(r)^* A(r)^* U(T-\varepsilon,r)^* Q(T-\varepsilon) U(T-\varepsilon,s)x, u(r)\rangle_U \, dr$$

$$= 2\mathrm{Re} \langle Q(T-\varepsilon) U(T-\varepsilon,s)x, [L_s u](T-\varepsilon)\rangle_H \, .$$

In addition,

$$M_3 + N_3 + N_6 = -\int_s^{T-\varepsilon} \|M(r)^{1/2} [L_s u](r)\|_H^2 \, dr$$

$$-2\mathrm{Re} \int_s^{T-\varepsilon} \langle u(r), G(r)^* A(r)^* Q(r) [L_s u](r)\rangle_U \, dr$$

$$+ \int_s^{T-\varepsilon} \|N(r)^{-1/2} G(r)^* A(r)^* Q(r) [L_s u](r)\|_U^2 \, dr$$

$$= -\int_s^{T-\varepsilon} \langle [L_s^{*\varepsilon} M(\cdot) L_s u(\cdot)](r), u(r)\rangle_U \, dr$$

$$-2\mathrm{Re} \int_s^{T-\varepsilon} \langle G(r)^* A(r)^* Q(r) [L_s u](r), u(r)\rangle_U \, dr$$

$$+ \int_s^{T-\varepsilon} \langle [L_s^{*\varepsilon} Q(\cdot) A(\cdot) G(\cdot) N(\cdot)^{-1} G(\cdot)^* A(\cdot)^* Q(\cdot) L_s u](r), u(r)\rangle_U \, dr.$$

Putting together the first and third term, and using lemma 24, we get

$$M_3 + N_3 + N_6$$

$$= -2\mathrm{Re} \int_s^{T-\varepsilon} \langle G(r)^* A(r)^* Q(r) [L_s u](r), u(r)\rangle_U \, dr$$

$$- \int_s^{T-\varepsilon} \langle L_s^{*\varepsilon} [M(\cdot) - Q(\cdot) A(\cdot) G(\cdot) N(\cdot)^{-1} G(\cdot)^* A(\cdot)^* Q(\cdot) L_s u](r), u(r)\rangle_U \, dr$$

$$= -2\mathrm{Re} \int_s^{T-\varepsilon} \langle G(r)^* A(r)^* Q(r) [L_s u](r), u(r)\rangle_U \, dr +$$

$$+2\mathrm{Re} \int_s^{T-\varepsilon} \left\langle G(r)^* A(r)^* \left[\int_r^{T-\varepsilon} U(\sigma,r)^* \Big[M(\sigma) \right.\right.$$

$$-Q(\sigma)A(\sigma)G(\sigma)N(\sigma)^{-1}G(\sigma)^*A(\sigma)^*Q(\sigma)\Big]U(\sigma,r)\,d\sigma\Big][L_s u](r),u(r)\Big\rangle_U dr.$$

Recalling again (15), we get

$$\begin{aligned}
&M_3+N_3+N_6\\
&= -2\mathrm{Re}\int_s^{T-\varepsilon}\langle G(r)^*A(r)^*U(T-\varepsilon,r)^*Q(T-\varepsilon)U(T-\varepsilon,r)[L_s u](r),u(r)\rangle_U\,dr\\
&= 2\mathrm{Re}\int_s^{T-\varepsilon}\int_s^r\langle G(r)^*A(r)^*U(T-\varepsilon,r)^*Q(T-\varepsilon)U(T-\varepsilon,\sigma)A(\sigma)G(\sigma)u(\sigma)\,d\sigma,\\
&\qquad\qquad u(r)\rangle_U\,dr\\
&= 2\mathrm{Re}\int_s^{T-\varepsilon}\int_s^r\langle Q(T-\varepsilon)U(T-\varepsilon,\sigma)A(\sigma)G(\sigma)u(\sigma),\\
&\qquad\qquad U(T-\varepsilon,r)A(r)G(r)u(r)\rangle_U\,d\sigma dr;
\end{aligned}$$

Since the last integrand is a symmetric function $b(\sigma,r)$, as in the proof of Lemma 24 we get

$$\begin{aligned}
&M_3+N_3+N_6 = 2\mathrm{Re}\int_s^{T-\varepsilon}\int_s^r b(\sigma,r)\,d\sigma dr\\
&= \int_s^{T-\varepsilon}\int_s^r b(\sigma,r)\,d\sigma dr + \int_s^{T-\varepsilon}\int_s^r b(r,\sigma)\,d\sigma dr\\
&= \int_s^{T-\varepsilon}\int_s^r b(\sigma,r)\,d\sigma dr + \int_s^{T-\varepsilon}\int_\sigma^{T-\varepsilon} b(r,\sigma)\,dr d\sigma = \int_s^{T-\varepsilon}\int_s^{T-\varepsilon} b(\sigma,r)\,d\sigma dr,
\end{aligned}$$

so that finally

$$\begin{aligned}
M_3+N_3+N_6 &= \int_s^{T-\varepsilon}\int_s^{T-\varepsilon}\langle Q(T-\varepsilon)U(T-\varepsilon,\sigma)A(\sigma)G(\sigma)u(\sigma),\\
&\qquad\qquad U(T-\varepsilon,r)A(r)G(r)u(r)\rangle_U\,d\sigma dr\\
&= \langle Q(T-\varepsilon)[L_s u](T-\varepsilon),[L_s u](T-\varepsilon)\rangle_H\,.
\end{aligned}$$

Summing up, and recalling (8) and (9), we have

$$\begin{aligned}
&M_1+M_2+M_3+N_2+N_3+N_4+N_5+N_6\\
&\quad= -\langle Q(s)x,x\rangle_H + \langle Q(T-\varepsilon)U(T-\varepsilon,s)x,U(T-\varepsilon,s)x\rangle_H\\
&\qquad +2\mathrm{Re}\langle Q(T-\varepsilon)U(T-\varepsilon,s)x,[L_s u](T-\varepsilon)\rangle_H\\
&\qquad +\langle Q(T-\varepsilon)[L_s u](T-\varepsilon),[L_s u](T-\varepsilon)\rangle_H\\
&\quad= -\langle Q(s)x,x\rangle_H + \langle Q(T-\varepsilon)y(T-\varepsilon),y(T-\varepsilon)\rangle_H\,,
\end{aligned}$$

which proves (17) and concludes the proof of Lemma 10. □

Appendix B Proof of Lemma 12

In order to prove (21), we start from the integral Riccati equation (15) and insert in place of $U(T-\varepsilon,s)x$ and $U(r,s)x$ their expressions in terms of $\Phi(T-\varepsilon,s)x$ and $\Phi(r,s)x$, given by (18). All calculations are legitimate, since there are singularities only at T. Setting for simplicity

$$K(q,\sigma) = U(q,\sigma)A(\sigma)G(\sigma)N(\sigma)^{-1}G(\sigma)^*A(\sigma)^*Q(\sigma), \tag{B3}$$

we get

$$\begin{aligned}
\langle Q(s)x,y\rangle_H &= \langle Q(T-\varepsilon)\Phi(T-\varepsilon,s)x, U(T-\varepsilon,s)y\rangle_H \\
&+ \left\langle Q(T-\varepsilon)\int_s^{T-\varepsilon} K(T-\varepsilon,\sigma)\Phi(\sigma,s)x\,d\sigma, U(T-\varepsilon,s)y\right\rangle_H \\
&+ \int_s^{T-\varepsilon} \langle M(r)\Phi(r,s)x, U(r,s)y\rangle_H\,dr \\
&+ \int_s^{T-\varepsilon} \left\langle M(r)\int_s^r K(r,\sigma)\Phi(\sigma,s)x\,d\sigma, U(r,s)y\right\rangle_H dr \\
&- \int_s^{T-\varepsilon} \left\langle N(r)^{-1}G(r)^*A(r)^*Q(r)\Phi(r,s)x, G(r)^*A(r)^*Q(r)U(r,s)y\right\rangle_H dr \\
&- \int_s^{T-\varepsilon} \left\langle N(r)^{-1}G(r)^*A(r)^*Q(r)\int_s^r K(r,\sigma)\Phi(\sigma,s)x\,d\sigma,\right. \\
&\qquad\qquad \left. G(r)^*A(r)^*Q(r)U(r,s)y\right\rangle_H dr \\
&= \langle Q(T-\varepsilon)\Phi(T-\varepsilon,s)x, U(T-\varepsilon,s)y\rangle_H + \int_s^{T-\varepsilon}\langle M(r)\Phi(r,s)x, U(r,s)y\rangle_H\,dr + \\
&\quad + I_1 + I_2 + I_3 + I_4\,,
\end{aligned}$$

where I_1, I_2, I_3, I_4 are the second term and the last three ones of the right member of the first equality. We observe now that if we set

$$g(r) := A(r)G(r)N(r)^{-1}G(r)^*A(r)^*Q(r)\Phi(r,s)x, \quad h(r) := U(r,s)y,$$

then, using (B3) and Fubini-Tonelli's Theorem, we can rewrite the sum of these terms as

$$
\begin{aligned}
I_1 + I_2 + I_3 + I_4 = \int_s^{T-\varepsilon} \Bigg[& \langle Q(T-\varepsilon)U(T-\varepsilon,\sigma)g(\sigma), U(T-\varepsilon,\sigma)h(\sigma)\rangle_H \\
& + \int_\sigma^{T-\varepsilon} \langle M(r)U(r,\sigma)g(\sigma), U(r,\sigma)h(\sigma)\rangle_H \, dr - \langle Q(\sigma)g(\sigma), h(\sigma)\rangle_H \\
& - \int_\sigma^{T-\varepsilon} \langle N(r)^{-1}G(r)^*A(r)^*Q(r)U(r,\sigma)g(\sigma), G(r)^*A(r)^*Q(r)U(r,\sigma)h(\sigma)\rangle_U \, dr \Bigg] d\sigma \\
= \int_s^{T-\varepsilon} 0 \, d\sigma = 0,
\end{aligned}
$$

where the penultimate equality follows by (15). This proves equation (21).
We now prove (22): in a quite similar way, we insert in (21), in place of $U(T-\varepsilon, s)x$ and $U(r,s)x$, their expressions in terms of $\Phi(T-\varepsilon, s)x$ and $\Phi(r,s)x$, given by (18): we have

$$
\begin{aligned}
\langle Q(s)x, y\rangle_H &= \langle Q(T-\varepsilon)\Phi(T-\varepsilon,s)x, \Phi(T-\varepsilon,s)y\rangle_H \\
&\quad + \left\langle Q(T-\varepsilon)\Phi(T-\varepsilon,s)x, \int_s^{T-\varepsilon} K(T-\varepsilon,\sigma)\Phi(\sigma,s)y \, d\sigma \right\rangle_H \\
&\quad + \int_s^{T-\varepsilon} \langle M(r)\Phi(r,s)x, \Phi(r,s)y\rangle_H \, dr \\
&\quad + \int_s^{T-\varepsilon} \left\langle M(r)\Phi(r,s)x, \int_s^r K(r,\sigma)\Phi(\sigma,s)y \right\rangle_H d\sigma \\
&= \langle Q(T-\varepsilon)\Phi(T-\varepsilon,s)x, \Phi(T-\varepsilon,s)y\rangle_H \\
&\quad + \int_s^{T-\varepsilon} \langle M(r)\Phi(r,s)x, \Phi(r,s)y\rangle_H \, dr + J_1 + J_2,
\end{aligned}
$$

where J_1 and J_2 are the second and fourth term of the right member of the first equality. Recalling (B3), we set

$$
g(\sigma) = \Phi(\sigma,s)x, \qquad h(\sigma) = A(\sigma)G(\sigma)N(\sigma)^{-1}G(\sigma)^*A(\sigma)^*Q(\sigma)\Phi(\sigma,s)y,
$$

and using Fubini-Tonelli's Theorem we can write

$$
\begin{aligned}
J_1 + J_2 = \int_s^{T-\varepsilon} \Bigg[& \langle Q(T-\varepsilon)\Phi(T-\varepsilon,\sigma)g(\sigma), U(T-\varepsilon,\sigma)h(\sigma)\rangle_H \\
& + \int_\sigma^{T-\varepsilon} \langle M(r)\Phi(r,\sigma)g(\sigma), U(r,\sigma)h(\sigma)\rangle_H \, dr \Bigg] d\sigma
\end{aligned}
$$

$$= \int_s^{T-\varepsilon} \langle Q(\sigma)g(\sigma), h(\sigma)\rangle_H \, d\sigma$$
$$= \int_s^{T-\varepsilon} \langle N(\sigma)^{-1}G(\sigma)^*A(\sigma)^*Q(\sigma)\Phi(\sigma,s)x, G(\sigma)^*A(\sigma)^*Q(\sigma)\Phi(\sigma,s)y\rangle_H \, d\sigma,$$

where in the penultimate equality we have used (21). This proves equation (22) and concludes the proof. □

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