Existence of harmonic functions via Perron's method

Let $\Omega \subset \mathbb{R}^d$ be a bounded open set and let $\phi : \partial\Omega \to \mathbb{R}$ be a continuous function. In the next theorem we will use the Perron's method to prove the existence of a function $u : \overline{\Omega} \to \mathbb{R}$ solving the problem

$$\Delta u = 0 \quad \text{in} \quad \Omega\,, \qquad u = \phi \quad \text{on} \quad \partial\Omega\,.$$

The argument is self-contained and only makes use of the Poisson's formula for balls in $\mathbb{R}^d$. An essential element of the proof are the boundary barriers from Lemma 2.

In what follows, given an open set Ω and a boundary datum ϕ as above, we define the following family of superharmonic functions

$$\mathcal{A} := \Big\{ w : \overline{\Omega} \to \mathbb{R} \ : \ w \in C(\overline{\Omega})\,,\ w \geq \phi \text{ on } \partial\Omega\,,\ \Delta w \leq 0 \text{ in } \Omega \Big\}, \tag{1}$$

where the inequality

$$\Delta w \leq 0 \quad \text{in} \quad \Omega$$

is intended in viscosity sense.

We recall that $\Delta w \leq 0 \quad \text{in} \quad \Omega$ in viscosity sense means that if a smooth function $P : \Omega \to \mathbb{R}$ touches w from below at some point $X \in \Omega$ (that is, $P \geq w$ in Ω and $P(X) = w(X)$), then $\Delta P(X) \geq 0$.

Teorema 1 (Existence of viscosity solutions via the Perron's method)**.** *Let Ω be a bounded open set admitting an exterior ball at every point on the boundary. Let $\phi : \partial\Omega \to \mathbb{R}$ be a continuous function and let $\mathcal{A}$ be the class from (1). Then, the function*

$$u : \overline{\Omega} \to \mathbb{R},$$

defined as

$$u(X) = \inf \Big\{ w(X) \ : \ w \in \mathcal{A} \Big\} \quad \textit{for every} \quad X \in \overline{\Omega},$$

has the following properties:

(a) $\Delta u = 0$ *in* Ω *in viscosity sense;*
(b) $u = \phi$ *on* $\partial\Omega$*;*
(c) u *is continuous on* $\overline{\Omega}$.

Proof. We proceed in two steps.

Step 1. Harmonicity of u: proof of (a).

We first notice that the functions w are bounded from below. Indeed, let $\underline{u} : \overline{\Omega} \to \mathbb{R}$ be a continuous function, which is smooth in Ω and such that

$$\underline{u} < \phi \quad \text{on} \quad \partial\Omega \qquad \text{and} \qquad \Delta\underline{u} < 0 \quad \text{in} \quad \Omega.$$

Then, for every $w \in \mathcal{A}$, we have that:

- w is superharmonic in viscosity sense;
- $\underline{u}$ is subharmonic and smooth;
- $w > \underline{u}$ on $\partial\Omega$.

This implies that

$$\underline{u} \leq w \quad \text{in} \quad \overline{\Omega}.$$

We fix a ball

$$B_r \subset \Omega.$$

We will show that u is harmonic in B_r. We fix a dense countable set

$$Q \subset \partial B_r.$$

and we select a sequence of functions $w_n \in \mathcal{A}$ such that:

$$u(q) = \lim_{n\to+\infty} w_n(q) \quad \text{for every} \quad q \in Q\,.$$

Moreover, by replacing w_n with $w_1 \wedge w_2 \wedge \cdots \wedge w_n \in \mathcal{A}$, we can also suppose that:

$$\text{the sequence of functions } w_n : \overline{\Omega} \to \mathbb{R} \text{ is decreasing.}$$

For every n we consider the function

$$h_n : \overline{B}_r \to \mathbb{R}$$

which is continuous on $\overline{B}_r$, harmonic in B_r and is such that

$$h_n = w_n \quad \text{on} \quad \partial B_r.$$

We notice that by the maximum principle

$$\underline{u} \le h_n \le w_n \quad \text{in} \quad \overline{B}_r.$$

We next claim that the function

$$v_n : \overline{\Omega} \to \mathbb{R} \ , \qquad v_n(x) = \begin{cases} h_n(x) & \text{if} \quad x \in \overline{B}_r \ , \\ w_n(x) & \text{if} \quad x \in \overline{\Omega} \setminus B_r \ , x \end{cases}$$

is in $\mathcal{A}$. Indeed, suppose that a polynomial P is touching v_n from below in a point X_0:

- If $X_0 \in B_r$, then P touches from below the harmonic function h_n at X_0 and thus $\Delta P(X_0) \le 0$.
- If $X_0 \in \Omega \setminus B_r$, then P touches from below also the function w_n at X_0, and so we get again $\Delta P(X_0) \le 0$.

This proves that $v_n \in \mathcal{A}$. We now notice that h_n is a monotone sequence of harmonic functions such that

$$\underline{u} \le h_n \le w_1 \quad \text{for every} \quad n \ge 1.$$

In particular, the pointwise limit

$$h(x) = \lim_{n \to +\infty} h_n(x)$$

exists and is finite. Now, by the monotone convergence theorem, we have that

$$\lim_{n \to +\infty} \int_{B_r} h_n(x) \Delta\psi(x)\, dx = \int_{B_r} h(x) \Delta\psi(x)\, dx,$$

for every $\psi \in C_c^\infty(B_r)$. This implies that h is harmonic in B_r.

Notice that, since

$$u \le h_n \quad \text{in} \quad B_r,$$

for every $n \ge 1$, by the definition of u, we have that

$$u \le h \quad \text{in} \quad B_r.$$

Suppose that in some point $X \in B_r$ we have

$$u(X) < h(X).$$

By the definition of u, there is a function $w \in \mathcal{A}$ such that

$$w(X) < h(X).$$

Since $w(X) < h(X) \le h_n(X)$, we have that

$$w(X) < h_n(X).$$

But then, by the maximum principle, there is a boundary point $Y \in \partial B_r$ such that

$$w(Y) < h_n(Y).$$

By the continuity of w and h_n, and by the density of Q, we can find $q \in Q$ such that

$$w(q) < h_n(q),$$

but this is impossible by the choice of w_n. Thus

$$h \equiv u \quad \text{in} \quad B_r,$$

and so u is harmonic in Ω.

Step 2. u agrees with ϕ at the boundary: proof of (b).

Let $X_0 \in \partial\Omega$ be fixed. By the definition of the class $\mathcal{A}$ we have that

$$\phi(X_0) \le w(X_0) \quad \text{for all} \quad w \in \mathcal{A},$$

By taking the infimum over all $w \in \mathcal{A}$, we get

$$\phi(X_0) = \inf_{w \in \mathcal{A}} w(X_0) \le u(X_0).$$

In order to show that

$$\phi(X_0) = u(X_0), \tag{2}$$

we argue by contradiction and we suppose that $u(X_\infty) - \phi(X_\infty) = \varepsilon > 0$. By Lemma 2 there is a competitor $\overline{w} \in \mathcal{A}$ such that

$$u(X_\infty) - \phi(X_\infty) \le \overline{w}(X_\infty) - \phi(X_\infty) < \varepsilon.$$

This is a contradiction and proves (2).

Step 3. Continuity of u up to the boundary: proof of (c).

Let $X_\infty \in \partial\Omega$ be fixed. It is sufficient to show that given a sequence $X_n \in \Omega$ converging to X_∞, we have

$$\lim_{n\to+\infty} u(X_n) = u(X_\infty).$$

We notice that, since all the functions $w \in \mathcal{A}$ are continuous on $\overline{\Omega}$ and $u = \inf_{\mathcal{A}} w$, it holds

$$\lim_{n\to+\infty} u(X_n) \le u(X_\infty).$$

Suppose by contradiction that

$$\lim_{n\to+\infty} u(X_n) < u(X_\infty).$$

By the previous point

$$\phi(X_\infty) = u(X_\infty).$$

so we have

$$\lim_{n\to+\infty} u(X_n) < \phi(X_\infty).$$

Without loss of generality we can assume that there is a positive constant $\varepsilon > 0$ such that

$$\phi(X_\infty) = 0 \qquad \text{while} \qquad u(X_n) < -\varepsilon \quad \text{for every} \quad n \ge 1.$$

Let $\underline{w}$ be a competitor constructed in Lemma 2. Then, for every $w \in \mathcal{A}$ we have

$$\underline{w}(X) \le w(X) \quad \text{for all} \quad X \in \overline{\Omega}.$$

As a consequence,

$$\underline{w}(X) \le u(X) \quad \text{for all} \quad X \in \overline{\Omega}.$$

So, in particular,

$$-\varepsilon < \underline{w}(X_\infty) - \phi(X_\infty) = \underline{w}(X_\infty) = \lim_{n\to+\infty} \underline{w}(X_n) \le \lim_{n\to+\infty} u(X_n),$$

which is a contradiction. Thus, we have proved that

$$\lim_{n\to+\infty} u(X_n) = \phi(X_\infty) = u(X_\infty),$$

which concludes the proof of (c) and of the theorem. □

Lemma 2 (Upper and lower barriers at boundary points admitting an exterior ball)**.** *Let Ω be a bounded open set in $\mathbb{R}^d$ and let $\phi : \partial\Omega \to \mathbb{R}$ be a continuous function. Suppose that $X_0 \in \overline{\Omega}$ admits an exterior ball B at X_0, that is, an open ball in $\mathbb{R}^d$ such that $\overline{\Omega} \cap \overline{B} = \{X_0\}$. Then, for every $\varepsilon > 0$ there are continuous functions*

$$\underline{u} : \overline{\Omega} \to \mathbb{R} \qquad \text{and} \qquad \overline{u} : \overline{\Omega} \to \mathbb{R},$$

such that:

$$\begin{cases} \Delta \underline{u} > 0 & \text{in} \quad \Omega; \\ \underline{u} \le \phi & \text{on} \quad \partial\Omega; \\ 0 \le \phi(X_0) - \underline{u}(X_0) \le \varepsilon; \end{cases} \qquad \text{and} \qquad \begin{cases} \Delta \overline{u} < 0 & \text{in} \quad \Omega; \\ \phi \le \overline{u} & \text{on} \quad \partial\Omega; \\ 0 \le \overline{u}(X_0) - \phi(X_0) \le \varepsilon. \end{cases} \tag{3}$$

Proof. Without loss of generality we can suppose that $\phi(X_0) = 0$. We proceed in several steps.

Step 1. Choice of an exterior tangent ball. Suppose that $B = B_R(Y_0)$ is the exterior ball at $X_0 \in \overline{\Omega}$,

$$\overline{\Omega} \cap \overline{B}_R(Y_0) = \{X_0\}.$$

Let $r > 0$ be such that

$$|\phi(X)| \le \varepsilon \qquad \text{for every} \qquad X \in \partial\Omega \cap B_r(X_0).$$

We now set

$$\rho := \frac{1}{4}\min\{R, r\},$$

and we consider the ball $B_\rho(Z_0)$, where

$$Z_0 = X_0 + \rho\frac{Y_0 - X_0}{|Y_0 - X_0|}\,,$$

which is an exterior ball at X_0 and is such that:

$$X_0 \in \partial B_\rho(Z_0)\ , \qquad B_\rho(Z_0) \subset \mathbb{R}^d \setminus \Omega\ , \qquad B_{2\rho}(Z_0) \subset B_r(X_0).$$

Step 2. A radially increasing superharmonic function in every dimension. Consider the function

$$g : (0, +\infty) \to (0, +\infty)\ , \qquad g(r) = 1 - r^{-d},$$

and let

$$G : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$$

be defined as follows:

$$G(X) = 1 - |X|^{-d} = g(|X|).$$

We notice that by construction

$$G = 0 \quad \text{on} \quad \partial B_1$$

while we can compute the Laplacian of G in polar coordinates $r = |X|$ and $\theta = X/|X|$ as

$$\begin{aligned}
\Delta G(X) &= \frac{1}{r^{d-1}}\partial_r\left[r^{d-1}\partial_r g(r)\right]\\
&= -\frac{1}{r^{d-1}}\partial_r\left[r^{d-1}\partial_r\left[r^{-d}\right]\right]\\
&= \frac{d}{r^{d-1}}\partial_r\left[r^{d-1}r^{-d-1}\right]\\
&= \frac{d}{r^{d-1}}\partial_r\left[r^{-2}\right]\\
&= \frac{-2d}{r^{d-4}},
\end{aligned}$$

so in any dimension $d \geq 2$ we have

$$\Delta G(X) < 0 \quad \text{for} \quad X \in \mathbb{R}^d \setminus \{0\}.$$

Step 3. Construction of $\underline{u}$ and $\overline{u}$. Let now $C > 0$ be a constant such that

$$C > \|\phi\|_{L^\infty(\partial\Omega)}$$

and let

$$\eta : \mathbb{R}^d \to \mathbb{R}$$

be defined as

$$\eta(X) := \begin{cases} \frac{C}{g(2)}\, g\left(\frac{|X - Z_0|}{\rho}\right) & \text{if} \quad |X - Z_0| \geq \rho,\\ 0 & \text{if} \quad |X - Z_0| \leq \rho.\end{cases}$$

Then η has the following properties

$$\begin{cases} \Delta\eta < 0 & \text{in} \quad \mathbb{R}^d \setminus \overline{B}_\rho(Z_0),\\ \eta \equiv 0 & \text{in} \quad \overline{B}_\rho(Z_0),\\ \eta \geq C & \text{in} \quad \mathbb{R}^d \setminus B_{2\rho}(Z_0).\end{cases}$$

By construction, we have that

$$-\varepsilon - \eta(X) \leq \phi(X) \leq \varepsilon + \eta(X) \quad \text{for every} \quad X \in \partial\Omega.$$

Thus, the functions

$$\overline{u}(X) := \varepsilon + \eta(X) \qquad \text{and} \qquad \underline{u}(X) := -\varepsilon - \eta(X)$$

satisfy the conditions in (3). □