Existence of harmonic functions via Perron's method

Let $\Omega \subset \mathbb{R}^d$ be a bounded open set and let $\phi : \partial\Omega \to \mathbb{R}$ be a continuous function. In the next theorem we will use the Perron's method to prove the existence of a function $u : \overline{\Omega} \to \mathbb{R}$ solving the problem

$$\Delta u = 0 \quad \text{in} \quad \Omega\,, \qquad u = \phi \quad \text{on} \quad \partial\Omega\,.$$

The argument is self-contained and only makes use of the Poisson's formula for balls in $\mathbb{R}^d$. An essential element of the proof are the boundary barriers from Lemma 2.

In what follows, given an open set Ω and a boundary datum ϕ as above, we define the following family of superharmonic functions

$$\mathcal{A} := \Big\{ w : \overline{\Omega} \to \mathbb{R} \;:\; w \in C(\overline{\Omega})\,,\; w \ge \phi \text{ on } \partial\Omega\,,\; \Delta w \le 0 \text{ in } \Omega \Big\}, \tag{1}$$

where the inequality

$$\Delta w \le 0 \quad \text{in} \quad \Omega$$

is intended in viscosity sense.

We recall that "$\Delta w \le 0$ in Ω in viscosity sense" means that if a smooth function $P : \Omega \to \mathbb{R}$ touches w from below at some point $X \in \Omega$ (that is, $P \le w$ in Ω and $P(X) = w(X)$), then $\Delta P(X) \le 0$.

Teorema 1 (Existence of viscosity solutions via the Perron's method)**.** *Let Ω be a bounded open set admitting an exterior ball at every point on the boundary. Let $\phi : \partial\Omega \to \mathbb{R}$ be a continuous function and let $\mathcal{A}$ be the class from (1). Then, the function*

$$u : \overline{\Omega} \to \mathbb{R},$$

defined as

$$u(X) = \inf \Big\{ w(X) \;:\; w \in \mathcal{A} \Big\} \quad \textit{for every} \quad X \in \overline{\Omega},$$

has the following properties:

(a) $\Delta u = 0$ *in* Ω *in viscosity sense;*
(b) $u = \phi$ *on* $\partial\Omega$*;*
(c) u *is continuous on* $\overline{\Omega}$*.*

Proof. We proceed in two steps.

Step 1. Harmonicity of u: proof of (a).

We first notice that the functions w are bounded from below. Indeed, let $\underline{u} : \overline{\Omega} \to \mathbb{R}$ be a continuous function, which is smooth in Ω and such that

$$\underline{u} < \phi \quad \text{on} \quad \partial\Omega \qquad \text{and} \qquad \Delta \underline{u} > 0 \quad \text{in} \quad \Omega.$$

Then, for every $w \in \mathcal{A}$, we have that:

- w is superharmonic in viscosity sense;
- $\underline{u}$ is subharmonic and smooth;
- $w > \underline{u}$ on $\partial\Omega$.

This implies that

$$\underline{u} \le w \quad \text{in} \quad \overline{\Omega}.$$

We fix a ball

$$B_r \subset \Omega.$$

We will show that u is harmonic in B_r. We fix a dense countable set

$$Q \subset \overline{\Omega}.$$

and we select a sequence of functions $w_n \in \mathcal{A}$ such that:

$$u(q) = \lim_{n \to +\infty} w_n(q) \quad \text{for every} \quad q \in Q\,.$$

Moreover, by replacing w_n with $w_1 \wedge w_2 \wedge \cdots \wedge w_n \in \mathcal{A}$, we can also suppose that:

$$\text{the sequence of functions } w_n : \overline{\Omega} \to \mathbb{R} \text{ is decreasing.}$$

For every n we consider the function

$$h_n : \overline{B}_r \to \mathbb{R}$$

which is continuous on $\overline{B}_r$, harmonic in B_r and is such that

$$h_n = w_n \quad \text{on} \quad \partial B_r.$$

We notice that by the maximum principle

$$\underline{u} \le h_n \le w_n \quad \text{in} \quad \overline{B}_r.$$

We next claim that the function

$$v_n : \overline{\Omega} \to \mathbb{R}\ , \qquad v_n(x) = \begin{cases} h_n(x) & \text{if} \quad x \in \overline{B}_r\,, \\ w_n(x) & \text{if} \quad x \in \overline{\Omega} \setminus B_r\,, \end{cases}$$

is in $\mathcal{A}$. Indeed, let P be a polynomial touching v_n from below at some point $X_0 \in \Omega$. We consider two cases.

- If $X_0 \in B_r$, then P touches from below the harmonic function h_n at X_0 and thus $\Delta P(X_0) \le 0$.
- If $X_0 \in \Omega \setminus B_r$, then P touches from below also the function w_n at X_0, and so we get again $\Delta P(X_0) \le 0$.

This proves that $v_n \in \mathcal{A}$. We now notice that v_n is a monotone decreasing sequence of superharmonic functions such that

$$\underline{u} \le v_n \le w_1 \quad \text{for every} \quad n \ge 1.$$

In particular, the pointwise limit

$$v(x) := \lim_{n\to+\infty} v_n(x),$$

exists and is finite for every $x \in \overline{\Omega}$. Now, by the monotone convergence theorem, we have that

$$0 = \lim_{n\to+\infty} \int_{B_r} h_n(x)\Delta\psi(x)\,dx = \int_{B_r} v(x)\Delta\psi(x)\,dx \qquad \text{for every} \qquad \psi \in C^\infty_c(B_r),$$

$$0 \ge \lim_{n\to+\infty} \int_{B_r} v_n(x)\Delta\psi(x)\,dx = \int_{B_r} v(x)\Delta\psi(x)\,dx \qquad \text{for every} \qquad \psi \in C^\infty_c(\Omega)\,,\ \psi \ge 0.$$

This implies that v is harmonic in B_r and superharmonic in Ω.

Notice that, since $v_n \in \mathcal{A}$ we have that

$$u \le v_n \quad \text{in} \quad \overline{\Omega},$$

for every $n \ge 1$, so that

$$u \le v \quad \text{in} \quad \overline{\Omega}.$$

On the other hand, by the choice of the dense set Q, we have

$$u(q) = v(q) \quad \text{for all} \quad q \in Q.$$

Suppose that in some point $X \in B_r$ we have

$$u(X) < v(X),$$

and take a ball $B_\rho(X) \subset B_r$. Since the functions v_n are harmonic and bounded in $B_\rho(X)$ we have that (up to a subsequence) v_n converges uniformly to v in $B_{\rho/2}(X)$. But then one can find $q \in Q \cap B_{\rho/2}(X)$ such that

$$u(q) < v(q),$$

which is a contradiction.

Step 2. u agrees with ϕ at the boundary: proof of (b).

Let $X_0 \in \partial\Omega$ be fixed. By the definition of the class $\mathcal{A}$ we have that

$$\phi(X_0) \le w(X_0) \quad \text{for all} \quad w \in \mathcal{A},$$

By taking the infimum over all $w \in \mathcal{A}$, we get

$$\phi(X_0) = \inf_{w\in\mathcal{A}} w(X_0) \le u(X_0).$$

In order to show that

$$\phi(X_0) = u(X_0), \tag{2}$$

we argue by contradiction and we suppose that $u(X_\infty) - \phi(X_\infty) = \varepsilon > 0$. By Lemma 2 there is a competitor $\overline{w} \in \mathcal{A}$ such that

$$u(X_\infty) - \phi(X_\infty) \le \overline{w}(X_\infty) - \phi(X_\infty) < \varepsilon.$$

This is a contradiction and proves (2).

Step 3. Continuity of u up to the boundary: proof of (c).

Let $X_\infty \in \partial\Omega$ be fixed. It is sufficient to show that given a sequence $X_n \in \Omega$ converging to X_∞, we have

$$\lim_{n\to+\infty} u(X_n) = u(X_\infty).$$

We notice that, since all the functions $w \in \mathcal{A}$ are continuous on $\overline{\Omega}$ and $u = \inf_{\mathcal{A}} w$, it holds

$$\lim_{n\to+\infty} u(X_n) \le u(X_\infty).$$

Suppose by contradiction that

$$\lim_{n\to+\infty} u(X_n) < u(X_\infty).$$

By the previous point

$$\phi(X_\infty) = u(X_\infty).$$

so we have

$$\lim_{n\to+\infty} u(X_n) < \phi(X_\infty).$$

Without loss of generality we can assume that there is a positive constant $\varepsilon > 0$ such that

$$\phi(X_\infty) = 0 \qquad \text{while} \qquad u(X_n) < -\varepsilon \quad \text{for every} \quad n \ge 1.$$

Let $\underline{w}$ be a competitor constructed in Lemma 2. Then, for every $w \in \mathcal{A}$ we have

$$\underline{w}(X) \le w(X) \quad \text{for all} \quad X \in \overline{\Omega}.$$

As a consequence,

$$\underline{w}(X) \le u(X) \quad \text{for all} \quad X \in \overline{\Omega}.$$

So, in particular,

$$-\varepsilon < \underline{w}(X_\infty) - \phi(X_\infty) = \underline{w}(X_\infty) = \lim_{n\to+\infty} \underline{w}(X_n) \le \lim_{n\to+\infty} u(X_n),$$

which is a contradiction. Thus, we have proved that

$$\lim_{n\to+\infty} u(X_n) = \phi(X_\infty) = u(X_\infty),$$

which concludes the proof of (c) and of the theorem. □

Lemma 2 (Upper and lower barriers at boundary points admitting an exterior ball)**.** *Let Ω be a bounded open set in $\mathbb{R}^d$ and let $\phi : \partial\Omega \to \mathbb{R}$ be a continuous function. Suppose that $X_0 \in \overline{\Omega}$ admits an exterior ball B at X_0, that is, an open ball in $\mathbb{R}^d$ such that $\overline{\Omega} \cap \overline{B} = \{X_0\}$. Then, for every $\varepsilon > 0$ there are continuous functions*

$$\underline{u} : \overline{\Omega} \to \mathbb{R} \qquad \textit{and} \qquad \overline{u} : \overline{\Omega} \to \mathbb{R},$$

such that:

$$\begin{cases} \Delta \underline{u} > 0 & \textit{in} \quad \Omega; \\ \underline{u} \le \phi & \textit{on} \quad \partial\Omega; \\ 0 \le \phi(X_0) - \underline{u}(X_0) \le \varepsilon; \end{cases} \qquad \textit{and} \qquad \begin{cases} \Delta \overline{u} < 0 & \textit{in} \quad \Omega; \\ \phi \le \overline{u} & \textit{on} \quad \partial\Omega; \\ 0 \le \overline{u}(X_0) - \phi(X_0) \le \varepsilon. \end{cases} \tag{3}$$

Proof. Without loss of generality we can suppose that $\phi(X_0) = 0$. We proceed in several steps.

Step 1. Choice of an exterior tangent ball. Suppose that $B = B_R(Y_0)$ is the exterior ball at $X_0 \in \overline{\Omega}$,

$$\overline{\Omega} \cap \overline{B}_R(Y_0) = \{X_0\}.$$

Let $r > 0$ be such that

$$|\phi(X)| \le \varepsilon \qquad \text{for every} \qquad X \in \partial\Omega \cap B_r(X_0).$$

We now set

$$\rho := \frac{1}{4} \min\{R, r\},$$

and we consider the ball $B_\rho(Z_0)$, where

$$Z_0 = X_0 + \rho \frac{Y_0 - X_0}{|Y_0 - X_0|}\,,$$

which is an exterior ball at X_0 and is such that:

$$X_0 \in \partial B_\rho(Z_0)\,, \qquad B_\rho(Z_0) \subset \mathbb{R}^d \setminus \Omega\,, \qquad B_{2\rho}(Z_0) \subset B_r(X_0).$$

Step 2. A radially increasing superharmonic function in every dimension. Consider the function

$$g:(0,+\infty)\to(0,+\infty)\ ,\qquad g(r)=1-r^{-d},$$

and let

$$G:\mathbb{R}^d\setminus\{0\}\to\mathbb{R}$$

be defined as follows:

$$G(X)=1-|X|^{-d}=g(|X|).$$

We notice that by construction

$$G=0\quad\text{on}\quad\partial B_1$$

while we can compute the Laplacian of G in polar coordinates $r=|X|$ and $\theta=X/|X|$ as

$$\begin{aligned}\Delta G(X)&=\frac{1}{r^{d-1}}\partial_r\left[r^{d-1}\partial_r g(r)\right]\\&=-\frac{1}{r^{d-1}}\partial_r\left[r^{d-1}\partial_r\left[r^{-d}\right]\right]\\&=\frac{d}{r^{d-1}}\partial_r\left[r^{d-1}r^{-d-1}\right]\\&=\frac{d}{r^{d-1}}\partial_r\left[r^{-2}\right]\\&=\frac{-2d}{r^{d-4}},\end{aligned}$$

so in any dimension $d\geq 2$ we have

$$\Delta G(X)<0\quad\text{for}\quad X\in\mathbb{R}^d\setminus\{0\}.$$

Step 3. Construction of $\underline{u}$ and $\overline{u}$. Let now $C>0$ be a constant such that

$$C>\|\phi\|_{L^\infty(\partial\Omega)}$$

and let

$$\eta:\mathbb{R}^d\to\mathbb{R}$$

be defined as

$$\eta(X):=\begin{cases}\frac{C}{g(2)}g\left(\frac{|X-Z_0|}{\rho}\right) & \text{if}\quad |X-Z_0|\geq\rho,\\ 0 & \text{if}\quad |X-Z_0|\leq\rho.\end{cases}$$

Then η has the following properties

$$\begin{cases}\Delta\eta<0 & \text{in}\quad \mathbb{R}^d\setminus\overline{B}_\rho(Z_0),\\ \eta\equiv 0 & \text{in}\quad \overline{B}_\rho(Z_0),\\ \eta\geq C & \text{in}\quad \mathbb{R}^d\setminus B_{2\rho}(Z_0).\end{cases}$$

By construction, we have that

$$-\varepsilon-\eta(X)\leq\phi(X)\leq\varepsilon+\eta(X)\quad\text{for every}\quad X\in\partial\Omega.$$

Thus, the functions

$$\overline{u}(X):=\varepsilon+\eta(X)\qquad\text{and}\qquad\underline{u}(X):=-\varepsilon-\eta(X)$$

satisfy the conditions in (3). □