Harmonic functions with continuous boundary datum

Teorema 1. *Let Ω be a bounded open set in $\mathbb{R}^d$ satisfying the exterior ball condition. Let $g : \partial\Omega \to \mathbb{R}$ be a continuous function. Then, there is a unique continuous function*

$$u : \overline{\Omega} \to \mathbb{R}\ , \quad u \in C(\overline{\Omega}) \cap C^2(\Omega)\,,$$

such that

$$\Delta u = 0 \quad in \quad \Omega \qquad and \qquad u = g \quad on \quad \partial\Omega\,.$$

Proof. For every $\varepsilon > 0$ we define the function

$$g_\varepsilon(x) = \min\Big\{ g(y) + \frac{1}{\varepsilon}|x-y| \ :\ y \in \partial\Omega \Big\}.$$

Step 1. For every $x \in \partial\Omega$, we have that $g_\varepsilon(x) \le g(x)$. Indeed,

$$g_\varepsilon(x) = \inf\Big\{ g(y) + \frac{1}{\varepsilon}|x-y| \ :\ y \in \partial\Omega \Big\} \le g(x) + \frac{1}{\varepsilon}|x-x| = g(x).$$

Step 2. $g_\varepsilon \to g$ uniformly on $\partial\Omega$. Let $\varepsilon > 0$ be fixed. Since g is continuous, for every $x \in \partial\Omega$ there is a point $y_x \in \partial\Omega$ that realizes the inf in the right hand side of the definition of $g_\varepsilon(x)$, that is

$$g_\varepsilon(x) = g(y_x) + \frac{1}{\varepsilon}|y_x - x|.$$

Now, since

$$g(y_x) + \frac{1}{\varepsilon}|y_x - x| = g_\varepsilon(x) \le g(x),$$

we have that

$$|y_x - x| \le \varepsilon M,$$

where

$$M := \max_{\partial\Omega} g - \min_{\partial\Omega} g\,.$$

As a consequence, for every $x \in \partial\Omega$,

$$|g_\varepsilon(x) - g(x)| = g(x) - g_\varepsilon(x) \le g(x) - g(y_x) \le \omega\left(\varepsilon M\right),$$

where

$$\omega : (0,+\infty) \to (0,+\infty)$$

is the (uniform) modulus of continuity of g on $\partial\Omega$, that is:

$$\omega(r) := \sup\Big\{ |g(x) - g(y)| \ :\ x, y \in \partial\Omega,\ |x-y| \le r \Big\}.$$

Step 3. For every fixed $\varepsilon > 0$, the function $g_\varepsilon : \partial\Omega \to \mathbb{R}$ is Lipschitz continuous. In fact, given $x_1, x_2 \in \partial\Omega$ let $y_1, y_2 \in \partial\Omega$ be such that

$$g_\varepsilon(x_i) = g(y_i) + \frac{1}{\varepsilon}|y_i - x_i| = \min\Big\{ g(y) + \frac{1}{\varepsilon}|y - x_i| \ :\ y \in \partial\Omega \Big\},$$

for $i = 1, 2$. Then, we have

$$\begin{aligned}
g_\varepsilon(x_2) - g_\varepsilon(x_1) &= \Big(g(y_2) + \frac{1}{\varepsilon}|y_2 - x_2| \Big) - \Big(g(y_1) + \frac{1}{\varepsilon}|y_1 - x_1|^2 \Big) \\
&= \min\Big\{ g(y) + \frac{1}{\varepsilon}|y - x_2| \ :\ y \in \partial B_r \Big\} - \Big(g(y_1) + \frac{1}{\varepsilon}|y_1 - x_1| \Big) \\
&\le \Big(g(y_1) + \frac{1}{\varepsilon}|y_1 - x_2| \Big) - \Big(g(y_1) + \frac{1}{\varepsilon}|y_1 - x_1| \Big) \\
&\le \frac{1}{\varepsilon}\Big(|y_1 - x_2| - |y_1 - x_1| \Big) \\
&\le \frac{1}{\varepsilon}|x_2 - x_1|.
\end{aligned}$$

Step 4. Existence. For every $\varepsilon > 0$, there is a harmonic function

$$u_\varepsilon \in H^1(\Omega),$$

which solves

$$\Delta u_\varepsilon = 0 \quad \text{in} \quad \Omega\ , \qquad u_\varepsilon = g_\varepsilon \quad \text{su} \quad \partial\Omega,$$

in the sense that u_ε is the minimizer of the variational problem

$$\min\Big\{\int_\Omega |\nabla\psi|^2\,dx\ :\ \psi\in H^1(\Omega),\ \psi-\widetilde g_\varepsilon\in H^1_0(\Omega)\Big\},$$

$\widetilde g_\varepsilon \in H^1(\Omega)$ being any Lipschitz extension of g_ε to $\mathbb{R}^d$. Moreover, we know that the function

$$\widetilde u_\varepsilon:\overline\Omega\to\mathbb{R}\ ,\qquad \widetilde u_\varepsilon(x)=\begin{cases}u_\varepsilon(x) & \text{if} \quad x\in\Omega,\\ g_\varepsilon(x) & \text{if} \quad x\in\partial\Omega,\end{cases}$$

is continuous on $\overline\Omega$.

The sequence $g_\varepsilon:\partial\Omega\to\mathbb{R}$ is increasing towards g as $\varepsilon\to0$. As a consequence, by the strong maximum principle, also the family of functions $\widetilde u_\varepsilon:\overline\Omega\to\mathbb{R}$ is increasing with respect to the parameter ε. Moreover, since

$$\|\widetilde u_\varepsilon-\widetilde u_\delta\|_{L^\infty(\Omega)}\le\|g_\varepsilon-g_\delta\|_{L^\infty(\partial\Omega)},$$

we have that $\widetilde u_\varepsilon:\overline\Omega\to\mathbb{R}$ converges uniformly on $\overline\Omega$ to a continuous function

$$u:\overline\Omega\to\mathbb{R}$$

such that $u=g$ on $\partial\Omega$. Finally, since u satisfies the mean value property (being a uniform limit of functions satisfying the mean-value property), we have that u is harmonic (and smooth) in Ω.

Step 5. Uniqueness. Suppose that there are two continuous functions $u,v:\overline\Omega\to\mathbb{R}$ such that

$$\Delta u=0 \quad\text{in}\quad \Omega \qquad\text{and}\qquad u=g \quad\text{on}\quad \partial\Omega\,,$$
$$\Delta v=0 \quad\text{in}\quad \Omega \qquad\text{and}\qquad v=g \quad\text{on}\quad \partial\Omega\,.$$

For every $t>0$ consider the family of functions

$$v_t:\overline\Omega\to\mathbb{R}\ ,\qquad v_t(x)=v(x)+t.$$

Since u and $v$are continuous and bounded on $\overline\Omega$, for t large enough we have that

(1) $$v_t(x)\ge u(x)\quad\text{for every}\quad x\in\overline\Omega.$$

Let t_* be defined as

$$t_*=\inf\Big\{t\in\mathbb{R}\ :\ (1)\ \text{holds for } v_t\Big\}.$$

We notice that:

- $t_*\ge0$. In fact, if (1) holds for t, then

  $$t+g(x)=v_t(x)\ge u(x)=g(x)\quad\text{for every}\quad x\in\partial\Omega,$$

  so necessarily $t\ge0$.
- t_* is a minimum, that is,

(2) $$v_{t_*}(x)\ge u(x)\quad\text{for every}\quad x\in\overline\Omega.$$

- if $t<t_*$, then there is a point $x\in\overline\Omega$ such that

  $$v_{t_*}(x)<u(x).$$

In particular, the last point implies that if t_n is a sequence such that

$$t_n<t_ast\qquad\text{and}\qquad \lim_{n\to+\infty}t_n=t_*\,,$$

then there is a sequence of points $x_n\in\overline\Omega$ such that

$$t_n+v(x_n)=v_{t_n}(x_n)<u(x_n).$$

Since Ω is bounded, up to extracting a subsequence, we can find a point $x_*\in\overline\Omega$ such that $x_n\to x_*$ as $n\to+\infty$. Now, the continuity of u and v implies that

$$t_*+v(x_*)\le u(x_*).$$

which together with (2) implies that

$$t_* + v(x_*) = u(x_*).$$

We are now ready to prove that $t_* = 0$. We suppose by contradiction that $t_* > 0$ and consider two cases.

Case 1. $x_* \in \partial\Omega$. Since $v = u = g$ on $\partial\Omega$, we get

$$t_* + v(x_*) = t^* + g(x_*) > g(x_*) = u(x_*),$$

which is a contradiction.

Case 2. $x_* \in \Omega$. In this case we have that u and v_{t_*} are two harmonic functions such that

$$v_{t_*}(x_*) = u(x_*) \qquad \text{and} \qquad v_{t_*} \geq u \quad \text{in} \quad \Omega.$$

By the strong maximum principle, we get that

$$v_{t_*} \equiv u \quad \text{in} \quad \Omega_*\,,$$

where Ω_* is the connected component of Ω containing x_*. Since u and v_{t_*} are continuous up to the boundary, we get that

$$v_{t_*} \equiv u \quad \text{on} \quad \partial\Omega_*\,,$$

so there is a boundary point $y_* \in \partial\Omega_* \subset \partial\Omega$ such that

$$t_* + v(y_*) = u(y_*).$$

This is impossible by *Case 1.* Thus, we have a contradiction, so we get that

$$t_* = 0,$$

which implies that

$$v \geq u \quad \text{in} \quad \overline{\Omega}\,.$$

Analogously

$$u \geq v \quad \text{in} \quad \overline{\Omega}\,,$$

which proves that the solution u is unique. □

The strong maximum principle

Teorema 2. *Let Ω be a connected open set in $\mathbb{R}^d$. Suppose that $u : \Omega \to \mathbb{R}$ and $v : \Omega \to \mathbb{R}$ are two smooth harmonic functions in Ω such that*

$$u \geq v \quad \textit{in} \quad \Omega.$$

Then, one of the following holds:

(1) *$u(x) > v(x)$ for every $x \in \Omega$;*

(2) *$u \equiv v$ in Ω.*

Proof. Suppose that (1) does not hold. Then, there is a point $x_* \in \Omega$ such that $u(x_*) = v(x_*)$. Consider the coincidence set

$$\mathcal{C} := \{x \in \Omega \ : \ u(x) = v(x)\}.$$

We know that $x_* \in \mathcal{C}$, so $\mathcal{C}$ is non-empty. Let $y \in \mathcal{C}$. By the mean value property, we have that for every ball $B_r(y) \subset \Omega$ it holds

$$0 = u(y) - v(y) = \frac{1}{|B_r|}\int_{B_r(y)} \big(u(x) - v(x)\big)\,dx.$$

Since $u - v \geq 0$ this implies that

$$u \equiv v \quad \text{in} \quad B_r(y).$$

Thus, the coincidence set $\mathcal{C}$ is open. On the other hand, since u and v are continuous $\mathcal{C}$ is also relatively closed. Since Ω is connected, this implies that $\mathcal{C} = \Omega$. □

Harmonic measures

Let $\Omega \subset \mathbb{R}^d$ be a bounded open set satisfying the exterior ball condition and let the point $X_0 \in \Omega$ be fixed. We define the operator $\mathcal{L}_X$ on the space of continuous functions on $\partial\Omega$,

$$\mathcal{L}_X : C(\partial\Omega) \to \mathbb{R},$$

as follows. Given a function

$$g \in C(\partial\Omega),$$

we consider the harmonic extension $h \in C(\overline{\Omega}) \cap C^2(\Omega)$ solution to

$$\Delta h = 0 \quad \text{in} \quad \Omega\,, \qquad h = g \quad \text{on} \quad \partial\Omega\,,$$

and we define

$$\mathcal{L}_X(g) = h(X).$$

We notice that:

- the operator $\mathcal{L}_X : C(\partial\Omega) \to \mathbb{R}$ is linear, that is,
$$\mathcal{L}_X(g_1 + g_2) = \mathcal{L}_X(g_1) + \mathcal{L}_X(g_2) \quad \text{for every} \quad g_1, g_2 \in C(\partial\Omega),$$
and
$$\mathcal{L}_X(c\,g) = c\mathcal{L}_X(g) \quad \text{for every} \quad g \in C(\partial\Omega) \quad \text{and every} \quad c \in \mathbb{R}.$$
- the operator $\mathcal{L}_X : C(\partial\Omega) \to \mathbb{R}$ is monotone, that is, if
$$g_1, g_2 \in C(\partial\Omega) \quad \text{such that} \quad g_1 \le g_2 \quad \text{on} \quad \partial\Omega,$$
then, by the (weak) maximum principle, the harmonic extensions h_1 and h_2 satisfy
$$h_1 \le h_2 \quad \text{in} \quad \overline{\Omega},$$
and as a consequence
$$\mathcal{L}_X(g_1) = h_1(X) \le h_2(X) = \mathcal{L}_X(g_2).$$

By the Riesz representation theorem, there is a positive Borel measure on $\partial\Omega$ such that

$$\int_{\partial\Omega} g(y)\, d\mu_X(y) = \mathcal{L}_X(g) \quad \text{for all} \quad g \in C(\partial\Omega).$$

Moreover, μ_X is a probability measure. In fact, if we take the boundary datum

$$g \in C(\partial\Omega)\,, \quad g \equiv 1 \quad \text{on} \quad \partial\Omega\,,$$

then its harmonic extension is the constant function

$$h \in C(\overline{\Omega})\,, \quad h \equiv 1 \quad \text{on} \quad \overline{\Omega}\,,$$

which gives (by the definition of $\mathcal{L}_X$) that

$$\mathcal{L}_X(1) = h(X) = 1.$$

This implies:

$$\mu_X(\partial\Omega) = \int_{\partial\Omega} 1\, d\mu_X = \mathcal{L}_X(1) = 1.$$

Definizione 3. *The measure μ_X is called harmonic measure with pole at X.*

Osservazione 4. *In some special cases the harmonic measure can be computed explicitly. For instance, if Ω is the ball B_R, then the Poisson's formula implies that for every $X \in B_R$ the harmonic measure μ_X is absolutely continuous with respect to the surface measure , $d\mathcal{H}^{d-1}$ on ∂B_R:*

$$\mu_X = \rho_X\, d\mathcal{H}^{d-1},$$

and its density function ρ_X is

$$\rho_X : \partial B_R \to \mathbb{R}\,, \qquad \rho_X(Y) = \frac{R^2 - |X|^2}{d\omega_d R}\,\frac{1}{|X - Y|^d} \quad \textit{for all} \quad Y \in \partial B_R.$$

An exercise about a harmonic function with discontinuous boundary datum

Esercizio 5. *Let* Ω *be a bounded open set in* $\mathbb{R}^d$ *and let*

$$u_\varepsilon \in C^2(\Omega) \cap L^\infty(\Omega)\,, \quad \varepsilon > 0\,,$$

be a family of harmonic functions,

$$\Delta u_\varepsilon = 0 \quad in \quad \Omega\,,$$

monotone (decreasing or increasing) in ε *and bounded in* $L^\infty(\Omega)$*. Then, the pointwise limit*

$$u_0(X) = \lim_{\varepsilon \to 0} u_\varepsilon(X)\,,$$

exists, satisfies $u_0 \in C^2(\Omega) \cap L^\infty(\Omega)$*, and is a harmonic function in* Ω*.*

Osservazione 6. *In the next exercice, given two points* $X, Y \in \mathbb{R}^2$*, we use the notation:*

$$(X,Y) = \Big\{(1-t)X + tY \ : \ 0 < t < 1\Big\};$$

$$[X,Y] = \Big\{(1-t)X + tY \ : \ 0 \le t \le 1\Big\}.$$

Esercizio 7. *Let* Ω *be the square* $\Omega := (0,1) \times (0,1)$ *with vertices*

$$A = \begin{pmatrix}0\\0\end{pmatrix},\ B = \begin{pmatrix}1\\0\end{pmatrix},\ C = \begin{pmatrix}1\\1\end{pmatrix},\ and\ D = \begin{pmatrix}0\\1\end{pmatrix}.$$

Prove that there is a function

$$u : \overline{\Omega} \to \mathbb{R}$$

which is continuous on $\overline{\Omega} \setminus \Big\{A, B, C, D\Big\}$ *and which solves the PDE*

$$\begin{cases} \Delta u = 0 \quad in \quad \Omega, \\ u = 0 \quad on \quad (A,B) \quad and \quad (C,D), \\ u = 1 \quad on \quad (B,C) \quad and \quad (A,D). \end{cases}$$

Prove that, at the vertex $A = (0,0)$*,* u *can be written in polar coordinates as follows:*

$$u(r,\theta) = \frac{1}{\pi}\theta + O(r^2).$$

Precisely:

(a) *prove that*

$$h(r,\theta) = u(r,\theta) - \frac{1}{\pi}\theta\,,$$

is harmonic in $B_1 \cap Q$ *and has zero boundary datum on* $[A,B] \cup [A,D]$*;*

(b) *prove that there is a constant* $C > 0$ *such that*

$$|h(r,\theta)| \le Cr^2 \quad in \quad B_1 \cap Q.$$