Set theoretic aspects of the space of ultrafilters $\beta\mathbb{N}$

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Outline

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Introduction

Start with $\mathbb{N}$ the space of natural numbers with the discrete topology.

Definition

$\beta\mathbb{N}$ is the Čech-Stone compactification of $\mathbb{N}$. This is the compactification such that every $f : \mathbb{N} \to [0,1]$ has a unique continuous extension $\beta f : \beta\mathbb{N} \to [0,1]$.

$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{f} & [0,1] \\ \big\downarrow{\scriptstyle \mathrm{id}_{\mathbb{N}}} & & \big\| \\ \beta\mathbb{N} & \xrightarrow{\beta f} & [0,1] \end{array}$$

We will denote by $\mathbb{N}^*$ the Čech-Stone remainder $\beta\mathbb{N} \setminus \mathbb{N}$. $\beta\mathbb{N}$ and $\mathbb{N}^*$ are very interesting topological objects. Jan Van Mill calls them the **three headed monster**.

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The three heads of $\beta\mathbb{N}$.

- Under the Continuum Hypothesis CH it is **smiling and friendly**. Most questions have easy answers.
- The second head is the **ugly head of independence**. This head always tries to confuse you.
- The last and smallest is the ZFC **head** of $\beta\mathbb{N}$.

To illustrate this phenomenon we consider autohomeomorphisms of $\mathbb{N}^*$. Recall that the clopen algebra of $\mathbb{N}^*$ is $\mathcal{P}(\mathbb{N})/FIN$. We move back and forth between $\mathbb{N}^*$ and $\mathcal{P}(\mathbb{N})/FIN$ using Stone duality.

Outline

A characterization of $\mathbb{N}^*$

Under CH it is possible to give a nice combinatorial characterization of $\mathcal{P}(\mathbb{N})/FIN$. Given two elements a and b of a Boolean algebra $\mathcal{B}$ we say that a and b are **orthogonal** and write $a \perp b$ if $a \wedge b = 0$. We say that two subsets $\mathcal{F}$ and $\mathcal{G}$ of $\mathcal{B}$ are **orthogonal** if $a \perp b$, for every $a \in \mathcal{F}$ and $b \in \mathcal{G}$. We say that x **splits** $\mathcal{F}$ and $\mathcal{G}$ if $a \leq x$, for all $a \in \mathcal{F}$ and $x \perp b$, for all $b \in \mathcal{G}$.

Definition

We say that a Boolean algebra $\mathcal{B}$ satisfies **condition** H_ω if for every two countable orthogonal subsets $\mathcal{F}$ and $\mathcal{G}$ of $\mathcal{B}$ there is $x \in \mathcal{B}$ which splits $\mathcal{F}$ and $\mathcal{G}$.

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$\mathcal{P}(\mathbb{N})/FIN$ satisfies condition H_ω.

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$\mathcal{P}(\mathbb{N})/FIN$ satisfies condition H_ω.

There is a slightly stronger condition.

Definition

A Boolean algebra $\mathcal{B}$ satisfies **condition R_ω** if for any two orthogonal countable subsets $\mathcal{F}, \mathcal{G}$ of $\mathcal{B}$ and any countable $\mathcal{H} \subseteq \mathcal{B}$ such that for all finite $\mathcal{F}_0 \subseteq \mathcal{F}$ and $\mathcal{G}_0 \subseteq \mathcal{G}$ and $h \in \mathcal{H}$ we have $h \nleq \vee\mathcal{F}_0$ and $h \nleq \vee\mathcal{G}_0$ there exist $x \in \mathcal{B}$ which splits $\mathcal{F}$ and $\mathcal{G}$ and such that $0 < x \wedge h < x$, for all $h \in \mathcal{H}$.

Lemma

If a Boolean algebra $\mathcal{B}$ satisfies condition H_ω then it satisfies condition R_ω.

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$\mathcal{P}(\mathbb{N})/FIN$ satisfies condition R_ω.

Theorem

Assume CH. *Then any two Boolean algebras of cardinality at most* $\mathfrak{c}$ *satisfying condition* H_ω *are isomorphic.*

Proof.

Let $\mathcal{B}$ and $\mathcal{C}$ be two Boolean algebras of cardinality $\mathfrak{c}$ satisfying condition H_ω. List $\mathcal{B}$ as $\{b_\alpha : \alpha < \omega_1\}$ and $\mathcal{C}$ as $\{c_\alpha : \alpha < \omega_1\}$. W.l.o.g. $b_0 = 0$ and $c_0 = 0$. By induction build countable subalgebras $\mathcal{B}_\alpha$ and $\mathcal{C}_\alpha$ and isomorphisms $\sigma_\alpha : \mathcal{B}_\alpha \to \mathcal{C}_\alpha$ such that

1. $b_\alpha \in \mathcal{B}_\alpha$, $c_\alpha \in \mathcal{C}_\alpha$,
2. if $\alpha < \beta$ then $\mathcal{B}_\alpha \subseteq \mathcal{B}_\beta$ and $\mathcal{C}_\alpha \subseteq \mathcal{C}_\beta$, and $\sigma_\beta \restriction \mathcal{B}_\alpha = \sigma_\alpha$.

To do the inductive step use condition R_ω. □

This is the well-known Cantor's **back and forth argument**. There is a model theoretic explanation for this result: under CH $\mathcal{P}(\mathbb{N})/FIN$ is the unique saturated model of cardinality $\mathfrak{c}$ of the theory of atomless Boolean algebras.

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Let X be a topological space. A subset A of X is **C^*-embedded** in X if each map $f : A \to [0,1]$ can be extended to a map $\tilde{f} : X \to [0,1]$.

Definition

A space X is called an **F-space** if each cozero set in X is C^*-embedded in X.

Lemma

1. *X is an F-space iff βX is an F-space.*
2. *A normal space X is an an F-space iff any two disjoint open F_σ subsets of X have disjoint closures.*
3. *Each basically disconnected space is an F-space.*
4. *Any closed subspace of a normal F-space is again an F-space.*
5. *If an F-space satisfies the countable chain condition then it is extremely disconnected.*

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Lemma

Let X be a compact zero dimensional space. The following are equivalent:

1. *$CO(X)$ satisfies condition H_ω*
2. *X is an F-space and each nonempty G_δ subset of X has infinite interior.*

Corollary

Assume CH. *The following are equivalent for a topological space X:*

1. *$X \approx \mathbb{N}^*$*
2. *X is a compact, zero dimensional F-space of weight $\mathfrak{c}$ in which every nonempty G_δ set has infinite interior.*

Such a space is called a **Parovičenko space**.

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Theorem

Let X be a locally compact, σ-compact and noncompact space. Then X^* is an F-space and each nonempty G_δ in X^* has infinite interior.

Corollary

Let X be a zero-dimensional , locally compact, σ-compact and noncompact space of weight $\mathfrak{c}$. Then X^* and $\mathbb{N}^*$ are homeomorphic.

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Continuous images of $\mathbb{N}^*$

Theorem

Let $\mathcal{B}$ be a Boolean algebra of size at most $\aleph_1$. Then $\mathcal{B}$ is embedded into $\mathcal{P}(\mathbb{N})/FIN$.

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Each compact space of weight at most $\aleph_1$ is a continuous images of $\mathbb{N}^*$.

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Autohomeomorphisms of $\mathbb{N}^*$

π is an **almost permutation** of $\mathbb{N}$ if $D = \mathrm{dom}(\pi)$ and $R = \mathrm{ran}(\pi)$ and π is a bijection between D and R. Note that if π is an almost permutation of $\mathbb{N}$ then $\beta\pi \restriction \mathbb{N}^*$ is an autohomeomorphism of $\mathbb{N}^*$.

Question

Is any autohomeomorphism of $\mathbb{N}^*$ of this form?

Under CH the answer is **NO**.

Theorem

Assume CH. *Then* $\mathbb{N}^*$ *has exactly* $2^{\mathfrak{c}}$ *autohomeomorphisms.*

Proof.

By the characterization of $\mathbb{N}^*$ we have that $\mathbb{N}^* \approx (\mathbb{N} \times 2^{\mathfrak{c}})^*$. [Here $2^{\mathfrak{c}}$ denotes the Cantor cube of weight $\mathfrak{c}$.] $2^{\mathfrak{c}}$ is a topological group of cardinality $2^{\mathfrak{c}}$ and so has $2^{\mathfrak{c}}$ autohomeomorphisms. It follows that $\mathbb{N}^*$ also has $2^{\mathfrak{c}}$ homeomorphisms. □

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P-points and nonhomogeneity of $\mathbb{N}^*$

Since $\mathbb{N}$ is homogeneous it is natural to ask if $\mathbb{N}^*$ if homogenous as well. We show that under CH it is not. In fact, this result does not need CH.

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Lemma

$\mathbb{N}^*$ *cannot be covered by* $\aleph_1$ *nowhere dense sets.*

Proof.

Let $\{D_\alpha : \alpha < \omega_1\}$ be a family of $\aleph_1$ nowhere dense subsets of $\mathbb{N}^*$. Build a family $\{C_\alpha : \alpha < \omega_1\}$ of $\aleph_1$ clopen subsets of $\mathbb{N}^*$ such that:

1. $C_\alpha \cap D_\alpha = \varnothing$, for all α,
2. if $\alpha < \beta$ then $C_\beta \subseteq C_\alpha$.

At limit stages of the construction, use diagonalization, i.e. property H_ω. Then $\cap\{C_\alpha : \alpha < \omega_1\}$ is disjoint from $\bigcup\{D_\alpha : \alpha < \omega_1\}$. □

Corollary

Assume CH. *Then* $\mathbb{N}^*$ *contains* P*-points.*

Proof.

Let $\mathcal{A}$ be the family $\{\bar{U} \setminus U : U$ is an open F_σ subset of $\mathbb{N}^*\}$. By CH $|\mathcal{A}| = \aleph_1$. Then any point of $\mathbb{N}^* \setminus \bigcup \mathcal{A}$ is a P-point. □

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Theorem

Assume CH. *Let* $p, q \in \mathbb{N}^*$ *be* P*-points. Then there is an autohomemorphism* h *of* $\mathbb{N}^*$ *such that* $h(p) = q$.

Since being a P-point is a topological property and there are obviously points which are not P-points we have the following.

Theorem

Assume CH. *Then* $\mathbb{N}^*$ *is not homogenous.*

In fact this is true even without CH.

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If we do not assume CH many of the properties of $\beta\mathbb{N}$ and $\mathbb{N}^*$ may fail and some new properties emerge depending on the model of set theory we are working in.
First, we point out that the characterization of $\mathcal{P}(\mathbb{N})/FIN$ fails if CH does not hold.

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CH *is equivalent to the statement that all Boolean algebras of cardinality* $\mathfrak{c}$ *which satisfy condition* H_ω *are isomorphic.*

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Example (A Parovičenko space with a point of character $\aleph_1$)

We build a strictly decreasing sequence $\{C_\alpha : \alpha < \omega_1\}$ of clopen subsets of $\mathbb{N}^*$. Let $P = \bigcap\{C_\alpha : \alpha < \omega_1\}$. Consider the quotient space $S = \mathbb{N}^*/P$ obtained by collapsing P to a single point. One shows easily that S is an F-space. If we let $p = \{P\}$ then $\chi(p, S) = \aleph_1$.

Example (A Parovičenko space in which every point has character $\mathfrak{c}$)

Let $2^{\mathfrak{c}}$ be the Cantor cube of weight $\mathfrak{c}$. Consider the space $T = (\mathbb{N} \times 2^{\mathfrak{c}})^*$, the Čech-Stone remainder of $\mathbb{N} \times 2^{\mathfrak{c}}$. Since $\mathbb{N} \times 2^{\mathfrak{c}}$ is zero-dimensional, σ-compact space of weight $\mathfrak{c}$ it follows that T is a Parovičenko space. For $\alpha < \mathfrak{c}$ and $i \in \{0, 1\}$ let

$$K(\alpha, i) = \{x \in 2^{\mathfrak{c}} : x(\alpha) = i\}$$

and let $L(\alpha, i) = T \cap \overline{\mathbb{N} \times K(\alpha, i)}$. Let $\mathcal{L} = \{L(\alpha, i) : \alpha < \mathfrak{c}, i \in \{0, 1\}\}$. One can show that the intersection of any uncountable subfamily of $\mathcal{L}$ has empty interior. On the other hand any point of T belongs to $\mathfrak{c}$ many members of $\mathcal{L}$. It follows that any point of T has character $\mathfrak{c}$.

Obviously, the topological translation of the characterization of $\mathcal{P}(\omega)/FIN$ also fails of CH does not hold. Moreover, in special models of set theory one can say much more.

Theorem

It is relatively consistent with the standard axioms ZFC *of set theory that* $\mathbb{N}^*$ *is not homeomorphic to* $(\mathbb{N} \times 2^{\mathfrak{c}})^*$.

In fact, this holds in the model for Martin's Axiom (MA) plus the negation of CH.
Let $A(\omega)$ be the 1-point compactificaton of the integers, i.e. a converging sequence. The following result follows from some work of Shelah.

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It is relatively consistent with ZFC *that* $\mathbb{N}^*$ *and* $(\mathbb{N} \times A(\omega))^*$ *are not homeomorphic.*

Obviously, the topological translation of the characterization of $\mathcal{P}(\omega)/FIN$ also fails of CH does not hold. Moreover, in special models of set theory one can say much more.

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It is relatively consistent with the standard axioms ZFC *of set theory that* $\mathbb{N}^*$ *is not homeomorphic to* $(\mathbb{N} \times 2^{\mathfrak{c}})^*$.

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Continuous images of $\mathbb{N}^*$

Let $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$. We say that $\mathcal{F}$ has **the finite intersection property** if $\cap\mathcal{F}_0$ is infinite, for every finite $\mathcal{F}_0 \subseteq \mathcal{F}$.

Definition

$P(\mathfrak{c})$ is the statement that for every $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$ of size less than $\mathfrak{c}$, if $\mathcal{F}$ has the finite intersection property then there is an infinite $B \subseteq \mathbb{N}$ such that $B \subseteq_* A$, for all $A \in \mathcal{F}$.

Remark $P(\mathfrak{c})$ is a consequence of MA + $\neg$CH and so is consistent with $\neg$CH.

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Assume $P(\mathfrak{c})$. Then every compact space of weight less than $\mathfrak{c}$ is a continuous image of $\mathbb{N}^*$.

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How about compact spaces of weight $\mathfrak{c}$?

Theorem (Kunen)

It is relatively consistent with MA + ¬CH *that there is a Boolean algebra of size* $\mathfrak{c}$ *which does not embed into* $\mathcal{P}(\mathbb{N})/FIN$.

Let $\mathcal{M}$ be the measure algebra of $[0, 1]$, i.e. $\mathcal{B}/\mathcal{I}$, where $\mathcal{B}$ is the algebra of Borel subsets of $[0, 1]$ and $\mathcal{I}$ is the ideal of Lebesgue null sets.

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An autohomeomorphism of $\mathbb{N}^*$ is called **trivial** if it is of the form π^*, for some almost permutation π of $\mathbb{N}$. Notice that there are only $\mathfrak{c}$ trivial autohomeomorphisms of $\mathbb{N}^*$. Under CH there are $2^{\mathfrak{c}}$ autohomeomorphisms of $\mathbb{N}^*$ thus there are many nontrivial ones. However we have the following.

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P-points and nonhomogeneity of $\mathbb{N}^*$

We have seen that under CH there are P-points in $\mathbb{N}^*$. Since there are always non P-points, it follows that $\mathbb{N}^*$ is not homogeneous. Under $\neg$CH the situation is different.

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It is relatively consistent with ZFC that there are no P-points in $\mathbb{N}^*$.

However, one can still show that $\mathbb{N}^*$ is not homogenous without any additional assumptions.

Definition

A point $\mathcal{P} \in \mathbb{N}^*$ is called a **weak P-point** if $p \notin \bar{D}$, for any countable $D \subseteq \mathbb{N}^*$.

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Outline

What is wrong with CH?

We have seen that CH resolves essentially all questions about $\beta\mathbb{N}$ and $\mathbb{N}^*$. So, it is natural to ask.

Why not simply assume CH and forget about other models of set theory?

Answers

- Because under CH we miss some of the subtle issues involving $\mathbb{N}^*$.
- There are questions about other important mathematical structures which CH does not answer and we do not have an axiom stronger than CH which decides them in a coherent way.

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Gaps in $\mathcal{P}(\mathbb{N})/FIN$

A key notion in the study of $\mathbb{N}^*$ is that of a **gap**. Given $A, B \subseteq \mathbb{N}$ we say that A and B are **orthogonal** and write $A \perp B$ if $A \cap B$ is finite. We write $A \subseteq_* B$ if $A \setminus B$ is finite. Given two subfamilies $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{P}(\mathbb{N})$ we say that $(\mathcal{A}, \mathcal{B})$ is a **pre-gap** if $A \perp B$, for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

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If $\mathcal{A}$ and $\mathcal{B}$ are totally ordered by $\subseteq_*$ in order type κ and λ respectively we say that $(\mathcal{A}, \mathcal{B})$ is a **(κ, λ)-gap**.

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We saw that $\mathcal{P}(\mathbb{N})$ satisfies condition H_ω. This can be rephrased as the following.

Fact

There are no (ω, ω)-gaps in $\mathcal{P}(\mathbb{N})/FIN$.

If one works under ¬CH it is natural to generalize H_ω to larger cardinals. However, we have the following.

Theorem (Hausdorff)

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Open Coloring Axiom

In various models of set theory one can have a variety of other gaps in $\mathcal{P}(\mathbb{N})$. However, there is an axiom which is relatively consistent with ZFC + ¬CH and gives a coherent and fairly complete of $\mathbb{N}^*$.

Definition (Open Coloring Axiom)

Let X be a set of reals and

$$[X]^2 = K_0 \cup K_1$$

a coloring where K_0 is open in the product topology of $[X]^2$. Then one of the following holds:

1. there is an uncountable $H \subseteq X$ such that $[H]^2 \subseteq K_0$, or
2. we can write $X = \bigcup\{X_n : n < \omega\}$, with $[X_n]^2 \subseteq K_1$, for all n.

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Axioms of this form were studied in detail by Abraham, Rubin and Shelah. The current formulation is due to Todorčević.

One can prove this statement outright if X is Borel or analytic. The strength of OCA comes from allowing X to be arbitrary. It is easy to show that OCA implies $\neg$CH.

Theorem

If ZFC *is consistent then so is the theory* ZFC + MA + $\neg$CH + OCA.

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OCA implies that the only nontrivial gaps in $\mathcal{P}(\mathbb{N})/FIN$ are $(\aleph_1, \aleph_1)$-gaps of the type constructed by Hausdorff.

Theorem

Assume OCA. *If κ and λ are regular cardinals and there is a (κ, λ)-gap in $\mathcal{P}(\mathbb{N})/FIN$ then $\kappa = \lambda = \aleph_1$.*

Theorem (V.)

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Many more results on the structure of $\mathcal{P}(\mathbb{N})/\mathcal{I}$, for some analytic ideal $\mathcal{I}$ were obtained by Dow, Farah and other.

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Open problems

A map $f : \mathbb{N}^* \to \mathbb{N}^*$ is called **trivial** if there is $\pi : \mathbb{N} \to \beta\mathbb{N}$ such that $f = \pi^*$.

Question 1

Is it possible to construct a nontrivial map $f : \mathbb{N}^* \to \mathbb{N}^*$ without any additonal axioms?

Question 2

Is it possible to construct a nonseparable extremely disconnected image of $\mathbb{N}^*$ without using additional set-theoretic axioms?

A copy of $\mathbb{N}^*$ in a compact space is **nontrivial** if it is nowhere dense and not of the form $\bar{D} \setminus D$, for some countable set D.

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A copy of $\mathbb{N}^*$ in a compact space is **nontrivial** if it is nowhere dense and not of the form $\bar{D} \setminus D$, for some countable set D.

Question 3

Is it possible to construct a nontrivial copy of $\mathbb{N}^*$ inside itself without using additional set-theoretic axioms?

Open problems

A map $f : \mathbb{N}^* \to \mathbb{N}^*$ is called **trivial** if there is $\pi : \mathbb{N} \to \beta\mathbb{N}$ such that $f = \pi^*$.

Question 1

Is it possible to construct a nontrivial map $f : \mathbb{N}^* \to \mathbb{N}^*$ without any additonal axioms?

Question 2

Is it possible to construct a nonseparable extremely disconnected image of $\mathbb{N}^*$ without using additional set-theoretic axioms?

A copy of $\mathbb{N}^*$ in a compact space is **nontrivial** if it is nowhere dense and not of the form $\bar{D} \setminus D$, for some countable set D.

Question 3

Is it possible to construct a nontrivial copy of $\mathbb{N}^*$ inside itself without using additional set-theoretic axioms?