Superfilters, Ramsey theory, and van der Waerden's Theorem

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Joint work with Nadav Samet (WIS $\rightarrow$ Google, Inc.)

Ultramath 2008

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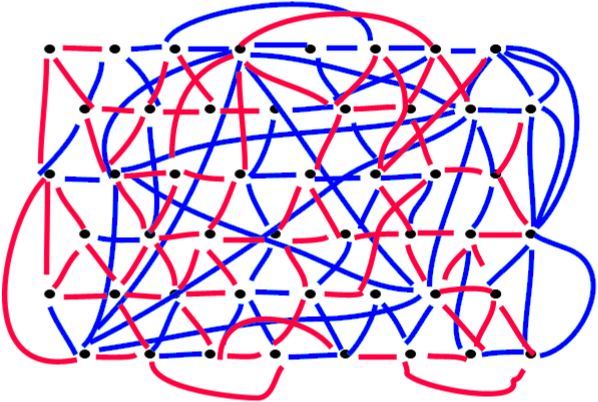

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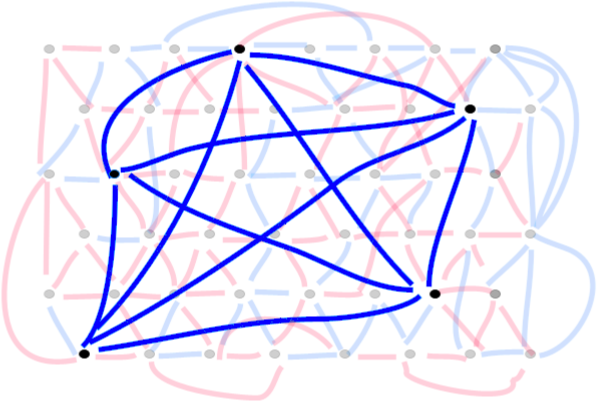

Then $\exists$ infinite complete monochromatic subgraph.

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Theorem

For suf's $\mathcal{S}$, TFAE:

1. *$\mathcal{S}$ is strongly Ramsey.*
2. *$\mathcal{S} \to (\mathcal{S})^n_k$ and $\mathcal{S}$ is shrinkable.*
3. *$\mathcal{S}$ is weakly Ramsey and shrinkable.*

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1. Ramsey Theorem. ($[\mathbb{N}]^\infty$ is strongly Ramsey.)
2. Booth-Kunen Theorem. (uf's are shrinkable.)

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$\mathsf{S}_1(\mathcal{S}, \mathcal{S})$: $\forall S_1, S_2, \dots \in \mathcal{S}$, $\exists s_n \in S_n$, $\{s_n : n \in \mathbb{N}\} \in \mathcal{S}$.

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$\mathcal{U}_1 \cup \mathcal{U}_2 \in \Omega \Rightarrow \mathcal{U}_1 \in \Omega$ or $\mathcal{U}_2 \in \Omega$.
$\therefore \mathcal{U}$ countable $\Rightarrow \{\mathcal{V} \subseteq \mathcal{U} : \mathcal{V} \in \Omega\}$ suf on $\mathcal{U}$!

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Generalization of Sch. Theorem and DM-K-M Conjecture:

Theorem

TFAE:

1. $\mathsf{S}_1(\mathcal{O}_{\mathcal{I}}, \mathcal{O}_{\mathcal{I}})$.
2. $\forall$ *disjoint* $\mathcal{U}_1, \mathcal{U}_2, \ldots \notin \mathcal{O}_{\mathcal{I}}$ *with* $\bigcup_n \mathcal{U}_n \in \mathcal{O}_{\mathcal{I}}$, $\exists \mathcal{V} \subseteq \bigcup_n \mathcal{U}_n$, $\mathcal{V} \in \mathcal{O}_{\mathcal{I}}$, $|\mathcal{V} \cap \mathcal{U}_n| \leq 1$ *for all* n.
3. $\mathcal{O}_{\mathcal{I}} \rightarrow (\mathcal{O}_{\mathcal{I}})^n_k$.

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Corollary

$\mathrm{AP} \to \lceil\mathrm{AP}\rceil^n_k$. *Implies Ramsey and van der Waerden!*

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