Ideal limits of sequences of continuous functions and a game

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- Def. $J - \lim x_n = x$ if for each $\epsilon > 0$ $\{n : |x - x_n| > \epsilon\} \in J$
- Def. For $f, f_n \in \mathbb{R}^X$, $J - \lim f_n = f$ if for each $x \in X$, $J - \lim f_n(x) = f(x)$
- For $\mathcal{F} \subset \mathbb{R}^X$ let $J - \lim \mathcal{F}$ denotes all J-limits of sequences of functions from $\mathcal{F}$
- Let $f_n : P(\omega) \to P(\omega)$;

$$f_n(A) = \begin{cases} 0 & \text{if } n \in A \\ 1 & \text{if } n \notin A \end{cases}$$

Then $J - \lim f_n = \chi_J$ if J is maximal. So $J - \lim C(P(\omega))$ contains nonmeasurable functions if J maximal.

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- Thm. (Katetov) For each $\alpha < \omega_1$ there is a Borel ideal $\mathcal{N}^\alpha \subset P(\omega)$ with $\mathcal{N}^\alpha - \lim C(X) = B_\alpha(X)$.
  $\mathcal{N}^2 = FIN \times FIN = \{A \subset \omega \times \omega : \forall_n^\infty A_n \text{ is finite}\}$
- Thm. (Kostyrko, Salat, Wilczynski) Let I be the ideal of sets of density zero. Then $J - \lim(C(X)) = B_1(X)$ for Polish space X.

2 methods of proof.

Method 1 Show that J-limit of sequence of continuous functions has a point of continuity on each perfect set.

Method 2 if f_n is J-convergent (statistically convergent) and $\{f_n(x)\}_n$ is bounded for each x then $\frac{f_1+f_2+\cdots+f_n}{n}$ is pointwise convergent to the same limit.

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Theorem

Let X be a complete metric space. Assume that one of the following conditions holds

1. *J is an analytic P-ideal.*
2. *J is an F_σ-ideal.*
3. *$J = NWD(\mathbb{Q})$.*

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- $G(J)$ infinite game (Laflamme) where J is an ideal on the integers. Player I in the n[th] move plays an elements $C_n \in J$, player II plays a finite subsets of integers F_n with $F_n \cap C_n = \emptyset$. Player I wins when $\bigcup_n F_n \in J$.

Theorem

Let X be a complete metric space. Assume that player II has a winning strategy in $G(J)$. Then $J - \lim C(X) = B_1(X)$.

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Assume that player I has a winning strategy in $G(J)$. Then $B_2(X) \subset J - \lim C(X)$.

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Let J be a Borel ideal. For each complete metric space X, $J - \lim C(X) = B_1(X)$ iff J does not contain a copy of $FIN \times FIN$

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Assume CH. There is maximal P-ideal such that $(J - \lim C(\mathbb{R})) \cap B_{\omega_1}(\mathbb{R}) = B_1(\mathbb{R})$

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Assume that $G(J)$ *is determined for each analytic ideal* J*. Then for each ideal* J, $(J - \lim C(\mathbb{R})) \cap B_{\omega_1}(\mathbb{R}) = B_1(\mathbb{R})$ *iff* J *does not contain a copy of* $FIN \times FIN$

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