A remark on the ultrapower cardinality and the continuum problem

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Outline

1. Regular ultrafilters
2. Two cardinal problem
3. Continuum displacement

Definition

An ultrafilter D over κ is (α, β)-regular if there is $E \subseteq D$ of cardinality β such that $\bigcap X = \emptyset$ for all $X \subseteq E$ of cardinality α.

- D is β-regular if it is (ω, β)-regular.
- D is regular if it is κ-regular.

Suppose that κ is an infinite cardinal, D is a regular ultrafilter over κ and that $\omega \leqslant \lambda \leqslant \kappa$. Then

$$|\prod_D \lambda| = 2^\kappa.$$

Let D be an ultrafilter over infinite cardinal κ. If D is uniform ($|X| = \kappa$ for all $X \in D$) and $\kappa^{<\kappa} = \kappa$, then

$$|\prod_D \kappa| = 2^\kappa.$$

Definition

Let $\mathcal{L}$ be a first order language and let U be a unary predicate symbol of $\mathcal{L}$. We say that an $\mathcal{L}$-theory T admits pair (κ, λ) if there is an $\mathcal{L}$-structure $\mathcal{M} = (M, U^{\mathcal{M}}, \dots)$ such that:

- $\mathcal{M} \models T$
- $|M| = \kappa$
- $|U^{\mathcal{M}}| = \lambda$.

Some examples:

- If T admits (κ, λ), then T admits (κ', λ'), where $\lambda \leqslant \lambda' \leqslant \kappa' \leqslant \kappa$.
- If T admits (κ, λ) and D is an ultrafilter over κ, then T admits $(|\prod_D \kappa|, |\prod_D \lambda|)$.
- There is a theory admitting (κ^+, κ) for all κ and not admitting any (κ^{++}, κ) for any κ.
- There is a theory admitting $(2^\kappa, \kappa)$ for all κ and not admitting any $((2^\kappa)^+, \kappa)$ for any κ.

Definition

Suppose that T admits (κ, λ). We say that (κ, λ) is:

- Left large gap (LLG), if T doesn't admit (κ^+, λ).
- Right large gap (RLG), if T doesn't admit (κ, λ^+).
- Large gap (LG), if it is both LLG and RLG.

Example

Suppose that $(\Lambda(\kappa), \kappa)$ is LLG for T for all κ, and that $(\kappa, \Gamma(\kappa))$ is RLG for T for all κ. Then, T admits $(|\prod_D \Lambda(\kappa)|, |\prod_D \kappa|)$ and $(|\prod_D \kappa|, |\prod_D \Gamma(\kappa)|,)$, so

$$|\prod_D \kappa| \leqslant |\prod_D \Lambda(\kappa)| \leqslant \Lambda(|\prod_D (\kappa)|)$$

and

$$\Gamma(|\prod_D \kappa|) \leqslant |\prod_D \Gamma(\kappa)| \leqslant |\prod_D \kappa|.$$

The continuum function is a cardinal function

$$\aleph_\alpha \mapsto 2^{\aleph_\alpha}, \quad \alpha \in \mathrm{On}.$$

Definition

The continuum displacement function is an ordinal function $f : \mathrm{On} \longrightarrow \mathrm{On}$ such that

$$2^{\aleph_\alpha} = \aleph_{\alpha+f(\alpha)}, \quad \alpha \in \mathrm{On}.$$

Example

Initial boundaries on the CP-displacement f:

- $2^{\aleph_\alpha} > \aleph_\alpha$, so $f(\alpha) \geqslant 1$.
- $2^{\aleph_\alpha} \leqslant \aleph_{2^{\aleph_\alpha}} = \aleph_{\alpha+2^{\aleph_\alpha}}$, so $f(\alpha) \leqslant 2^{\aleph_\alpha}$.

Example

Suppose that the CP-displacement f is constant, i.e.

$$(\forall \alpha \in \mathrm{On}) f(\alpha) = \beta$$

for some fixed $\beta \in \mathrm{On}$. Then $\beta < \omega$.

Theorem

Let $(\aleph_\xi(\lambda), \lambda)$ be LLG for theory T for all infinite cardinals λ. Fix some $\kappa \geqslant \omega$. Suppose that $\aleph_\xi(\kappa)^{<\aleph_\xi(\kappa)} = \aleph_\xi(\kappa)$ and that D is a uniform, nonregular ultrafilter over $\aleph_\xi(\kappa)$ "jumping" after κ, i.e.

$$|\prod_D \kappa| < |\prod_D \aleph_\xi(\kappa)|.$$

Let $\aleph_\alpha = |\prod_D \kappa|$ and $\aleph_\beta = \aleph_\xi(\kappa)$. Then,

$$\alpha < \beta + f(\beta) \leqslant \alpha + \xi \leqslant \alpha + \beta.$$

$$\begin{aligned} \aleph_\alpha &= |\prod_D \kappa| \\ &< |\prod_D \aleph_\beta| \\ &= 2^{\aleph_\beta} \\ &= \aleph_{\beta+f(\beta)} \end{aligned}$$

Hence,

$$\alpha < \beta + f(\beta).$$

$$\begin{aligned}
\aleph_{\beta+f(\beta)} &= |\prod_D \aleph_\xi(\kappa)| \\
&\leqslant \aleph_\xi(|\prod_D \kappa|) \\
&= \aleph_\xi(\aleph_\alpha) \\
&= \aleph_{\alpha+\xi} \\
&\leqslant \aleph_{\alpha+\beta}.
\end{aligned}$$

Thus,

$$\beta + f(\beta) \leqslant \alpha + \xi \leqslant \alpha + \beta.$$

Example

Let $\kappa = \omega$, $\xi = 17$. Then,

$$2^{\aleph_{17}} \leqslant \aleph_{\alpha+17}.$$

In addition, if $|\prod_D \omega| \leqslant \aleph_{17}$, then

$$2^{\aleph_{17}} \leqslant \aleph_{34}.$$

Example

If $2^{\aleph_{17}} = \aleph_{\omega+1}$, then there is no "jumping" ultrafilter over $\aleph_{17}$, i.e.

$$|\prod_D \omega| = |\prod_D \aleph_{17}|.$$