DECOMPOSABILITY OF ULTRAFILTERS, MODEL-THEORETICAL PRINCIPLES, AND COMPACTNESS OF TOPOLOGICAL SPACES

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Let μ, λ and κ be infinite cardinals.

Definition 1. An ultrafilter D over λ is said to be *μ-decomposable* if and only if there exists a function $f : \lambda \to \mu$ such that whenever $X \subseteq \mu$ and $|X| < \mu$ then $f^{-1}(X) \notin D$.

If a function f as above exists, it is called a *μ-decomposition* for D.

In other words, an ultrafilter D is μ-decomposable if and only if some quotient of D is uniform over μ.

It easy to see that a cardinal λ is measurable if and only if there exists some ultrafilter D uniform over λ such that D is not μ-decomposable, for every $\mu < \lambda$.

Thus, the existence of indecomposable ultrafilters can be seen as a weakening of measurability (they usually yield measurable cardinals in inner models, anyway).

Decomposable ultrafilters and their applications have been studied (sometimes under different terminology) by Silver, Kunen, Prikry, Cudnovskii, Ketonen, Magidor, Donder, Makowski, Shelah, among many others.

In particular, the following principle:

$A(\lambda, \mu)$ "*Every ultrafilter uniform over λ is μ-decomposable*"

has applications to appropriately defined compactness properties of logics extending first-order logic, and to compactness properties of products of topological spaces.

We shall introduce a variation on $A(\lambda, \mu)$ which furnishes stronger applications and involves more natural notions of compactness.

$A(\lambda, \mu)$ means that for every ultrafilter D uniform over λ there exists $f : \lambda \to \mu$ which is a μ-decomposition for D.

We can introduce a more refined notion.

Definition 2. $\lambda \overset{\kappa}{\Rightarrow} \mu$ means that there is a family F of functions from λ to μ such that $|F| = \kappa$ and for every ultrafilter D uniform over λ there exists $f \in F$ which is a μ-decomposition for D.

Clearly, if $\kappa \geq 2^\lambda$ then $\lambda \overset{\kappa}{\Rightarrow} \mu$ is equivalent to $A(\lambda, \mu)$.

For $\kappa < 2^\lambda$, $\lambda \overset{\kappa}{\Rightarrow} \mu$ is a notion connected with variations on weak compactness rather than measurability.

Just to give the flavour of the strength of this notion, if λ is the first weakly compact cardinal, then:

$A(\lambda, \omega)$ trivially holds,

while

$\lambda \overset{\lambda}{\Rightarrow} \omega$ fails.

Theorem 3. *Suppose that $\lambda \geq \mu$ are infinite regular cardinals, and $\kappa \geq \lambda$ is an infinite cardinal (the assumption λ and μ regular is just for convenience: a version of the result holds for arbitrary cardinals).*

The following conditions are equivalent.

(a) $\lambda \stackrel{\kappa}{\Rightarrow} \mu$ holds.

(b) (topological version) Whenever $(X_\beta)_{\beta<\kappa}$ is a family of topological spaces such that no X_β is $[\mu, \mu]$-compact, then $X = \prod_{\beta<\kappa} X_\beta$ is not $[\lambda, \lambda]$-compact.

(c) (alternative topological version) The topological space μ^κ is not $[\lambda, \lambda]$-compact, where μ is endowed with the topology whose open sets are the intervals $[0, \alpha)$ ($\alpha \leq \mu$), and μ^κ is endowed with the Tychonoff topology.

(d) (Ulam matrices-like version) There is a family $(B_{\alpha,\beta})_{\alpha<\mu,\beta<\kappa}$ of subsets of λ such that:

(i) For every $\beta < \kappa$, $\bigcup_{\alpha<\mu} B_{\alpha,\beta} = \lambda$;

(ii) For every $\beta < \kappa$ and $\alpha \leq \alpha' < \mu$, $B_{\alpha,\beta} \subseteq B_{\alpha',\beta}$;

(iii) For every function $g : \kappa \longrightarrow \mu$ there exists a finite subset $F \subseteq \kappa$ such that $|\bigcap_{\beta\in F} B_{g(\beta),\beta}| < \lambda$.

(e) (model-theoretical version) The model $\langle \lambda, <, \gamma \rangle_{\gamma < \lambda}$ *has an expansion* $\mathfrak{A}$ *in a language with at most* κ *new symbols such that whenever* $\mathfrak{B} \equiv \mathfrak{A}$ *and* $\mathfrak{B}$ *has an element* x *such that* $\mathfrak{B} \models \gamma < x$ *for every* $\gamma < \lambda$*, then* $\mathfrak{B}$ *has an element* y *such that* $\mathfrak{B} \models \alpha < y < \mu$ *for every* $\alpha < \mu$*.*

It is almost certain that there is a condition equivalent to the ones above involving compactness of logics extending first-order logic. This is true both for $\kappa \geq 2^\lambda$ and for $\kappa = \lambda$; I have not checked the intermediate cases.